INTRODUCTION TO ALGEBRAIC K-THEORY

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Abstract. These are the lecture notes of the seminar *Introduction to Algebraic K-theory*, held in Regensburg in the winter semester 2026–2027.

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1. Introduction

2. Categories, functors and equivalences

We follow Cnossen's exposition in [Cno26, Sections 1.1–1.5 and 1.8]. Foundational treatments are given by quasi-categories [Joy02, Lur09], complete Segal spaces [Rez01], and synthetic category theory [RS17, CCNW26].

2.1. **The language of categories.** We start by assuming the existence of the objects we are interested in.

Axiom 2.1.1. We assume the following.

(1) There are categories $\mathcal{C}$ and functors between them. Every category has an identity functor, and functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit a composite $GF : \mathcal{C} \to \mathcal{E}$.
(2) For categories $\mathcal{C}, \mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, we may speak of natural equivalences $\alpha : F \simeq G$, with identities, inverses, and composition. There are equivalences between natural equivalences, and further equivalences in every higher dimension, with the same operations.
(3) Composition is associative and unital up to specified equivalences. For a functor $F : \mathcal{C} \to \mathcal{D}$, these include $\mathrm{id}_{\mathcal{D}} F \simeq F \simeq F \mathrm{id}_{\mathcal{C}}$; for composable functors F, G, H, they include $(HG)F \simeq H(GF)$. For functors $F, F' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$, natural equivalences $\alpha : F \simeq F'$ and $\beta : G \simeq G'$ admit a horizontal composite $\beta * \alpha : GF \simeq G'F'$. These operations satisfy the corresponding compatibility and coherence conditions in every dimension.

Definition 2.1.2. A commutative square of functors consists of its four functors together with a specified natural equivalence between the two composites. A commutative triangle of functors is defined similarly.

One of the most important concepts is equivalence of categories:

Definition 2.1.3. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural equivalences $GF \simeq \mathrm{id}_{\mathcal{C}}$ and $FG \simeq \mathrm{id}_{\mathcal{D}}$. We call G an *inverse* of F and we say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

Equivalences have the usual properties:

Remark 2.1.4. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. Equivalences are closed under composition and satisfy the two-out-of-three property. Indeed, if F and GF are equivalences, then $G \simeq (GF)F^{-1}$ is an equivalence. The other cases follow similarly.

The following axiom embeds ordinary category theory in our naive approach; its inclusion in the quasi-category model is given by the nerve construction [Lur09, Section 1.1.2].

Axiom 2.1.5. Every ordinary category determines a category. Ordinary functors and natural isomorphisms determine functors and natural equivalences, compatibly with identities and composition. Every functor between categories arising from ordinary categories comes from an ordinary functor.

In particular:

Notation 2.1.6. For an integer $n \geq 0$, let $[n]$ be the ordinary category associated with the ordered set $0 < \cdots < n$.

We can use the simplices $[n]$ to construct the basic objects of category theory:

Definition 2.1.7. Let $\mathcal{C}$ be a category.

(1) An *object* $x \in \mathcal{C}$ is a functor $x : [0] \to \mathcal{C}$.
(2) A *morphism* $f : x \to y$ is a functor $f : [1] \to \mathcal{C}$, together with identifications of its endpoints with the objects $x, y \in \mathcal{C}$.

(3) A *commutative triangle* in $\mathcal{C}$ is a functor $[2] \to \mathcal{C}$.

And so on.

Example 2.1.8. Let $\mathcal{C}$ be a category and let $x \in \mathcal{C}$ be an object. The composite $[1] \to [0] \xrightarrow{x} \mathcal{C}$ defines the *identity morphism* id_x.

Remark 2.1.9. A commutative triangle records a candidate composite of its two consecutive edges. The definition alone imposes no existence or uniqueness condition on such a composite. Composition requires an additional axiom.

2.2. **Basic constructions.** A fundamental principle of category theory is that constructions should be characterized by universal properties invariant under equivalence. Products and coproducts are examples:

Axiom 2.2.1. Let I be a set and let $\mathcal{C}_i$ be a category for each $i \in I$. There is a *product* $\prod_{i \in I} \mathcal{C}_i$, equipped with projections, such that specifying a functor from a category $\mathcal{T}$ into the product is equivalent to specifying functors $\mathcal{T} \to \mathcal{C}_i$ for every i.

Dually, there is a *coproduct* $\coprod_{i \in I} \mathcal{C}_i$, equipped with inclusions, such that specifying a functor from this coproduct into $\mathcal{T}$ is equivalent to specifying functors $\mathcal{C}_i \to \mathcal{T}$ for every i.

For the empty indexing set, we obtain the following conventions.

Notation 2.2.2. The empty product is equivalent to $[0]$, and the empty coproduct is the ordinary empty category. We write $*$ for the terminal category and $\varnothing$ for the empty category.

Using the terminality of the point category and the equivalence $[0] \simeq *$, we obtain constant functors:

Definition 2.2.3. Let $\mathcal{C}, \mathcal{D}$ be categories and let $x \in \mathcal{D}$ be an object. The *constant functor* with value x is the composite $\mathcal{C} \to * \xrightarrow{x} \mathcal{D}$. A category $\mathcal{C}$ is *contractible* if its canonical functor $\mathcal{C} \to *$ is an equivalence.

We now axiomatize pullbacks:

Axiom 2.2.4. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. There is a category $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, equipped with projections to $\mathcal{C}$ and $\mathcal{D}$ and a natural equivalence between their composites to $\mathcal{E}$. For every category $\mathcal{T}$, specifying a functor $\mathcal{T} \to \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is equivalent to specifying functors $u : \mathcal{T} \to \mathcal{C}$ and $v : \mathcal{T} \to \mathcal{D}$, together with a natural equivalence $Fu \simeq Gv$. We refer to these data as the *pullback* of $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. A commutative square of categories is a *pullback square* if its comparison functor to the pullback is an equivalence.

Since objects are functors from the terminal category, we obtain the following description.

Remark 2.2.5. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. An object of their pullback consists of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$, and a specified identification $F(c) \simeq G(d)$. The identification is part of the object.

Fibres are special cases of pullbacks.

Definition 2.2.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}, \mathcal{D}$, and let $y \in \mathcal{D}$ be an object. The *fibre* of F at y is $\mathcal{C} \times_{\mathcal{D}} \{y\}$.

We then have the following universality of coproducts:

Axiom 2.2.7. For a family of categories $\mathcal{C}_i$ indexed by a set, coproduct inclusions are disjoint and coproducts are stable under pullback. Every functor from a category $\mathcal{T}$ into $\coprod_i \mathcal{C}_i$ decomposes $\mathcal{T}$ as the coproduct of its pullbacks along the summand inclusions. A set I, regarded as an ordinary discrete category, is the coproduct of I copies of $*$.

Functor categories organize functors and natural transformations between two categories into a category.

Axiom 2.2.8. Given two categories $\mathcal{C}$ and $\mathcal{D}$, there exists a *functor category* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. It comes equipped with a functor $\mathrm{ev}:\mathrm{Fun}(\mathcal{C},\mathcal{D})\times\mathcal{C}\to\mathcal{D}$, called the *evaluation functor*. For another category $\mathcal{E}$ and a functor $g:\mathcal{E}\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$, we may then define its *uncurrying* $g^u:\mathcal{E}\times\mathcal{C}\to\mathcal{D}$ as the composite

$$\mathcal{E}\times\mathcal{C}\xrightarrow{g\times\mathrm{id}_{\mathcal{C}}}\mathrm{Fun}(\mathcal{C},\mathcal{D})\times\mathcal{C}\xrightarrow{\mathrm{ev}}\mathcal{D}.$$

We require that every functor $f:\mathcal{E}\times\mathcal{C}\to\mathcal{D}$ gives rise to a functor $f_c:\mathcal{E}\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$, called the *currying* of f, equipped with an identification $f=(f_c)^u$. Moreover, for two functors $g,h:\mathcal{E}\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$, uncurrying induces an equivalence $(g=h)\to(g^u=h^u)$.

The functor category gives a definition of natural transformations:

Definition 2.2.9. Let $\mathcal{C},\mathcal{D}$ be categories and let $F,G:\mathcal{C}\to\mathcal{D}$ be functors. A *natural transformation* $\alpha:F\to G$ is a morphism from F to G in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$.

The functor category allows us to apply the formal rules of Axiom 2.1.1 to functors and natural transformations.

Remark 2.2.10. Let $\mathcal{C},\mathcal{D}$ be categories. Taking $\mathcal{I}=*$ in Axiom 2.2.8 identifies objects of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ with functors $\mathcal{C}\to\mathcal{D}$. Taking $\mathcal{I}=[1]$ identifies natural transformations $\alpha:F\to G$ between functors $F,G:\mathcal{C}\to\mathcal{D}$ with functors $[1]\times\mathcal{C}\to\mathcal{D}$ whose endpoint restrictions are F and G. Precomposition and postcomposition define functors between functor categories, acting on both functors and natural transformations. Precomposition with the endpoint inclusions $0,1:*\to[1]$ defines the *source evaluation* $\mathrm{ev}_0:\mathrm{Fun}([1],\mathcal{C})\to\mathcal{C}$ and the *target evaluation* $\mathrm{ev}_1:\mathrm{Fun}([1],\mathcal{C})\to\mathcal{C}$, using $\mathrm{Fun}(*,\mathcal{C})\simeq\mathcal{C}$. On a morphism $f:x\to y$ between objects $x,y\in\mathcal{C}$, they take the values $\mathrm{ev}_0(f)=x$ and $\mathrm{ev}_1(f)=y$.

It furthermore allows us to express the universal properties of Axiom 2.2.1, Axiom 2.2.4 and Axiom 2.2.7 in terms of functor categories.

Proposition 2.2.11 ([Cno26, Proposition 1.3.24])**.** Let $\mathcal{I}$ be a category, let I be a set, and let $\mathcal{C}_i$ be a category for each $i\in I$. The canonical comparison functors induce natural equivalences

$$\mathrm{Fun}\left(\mathcal{I},\prod_{i\in I}\mathcal{C}_i\right)\simeq\prod_{i\in I}\mathrm{Fun}(\mathcal{I},\mathcal{C}_i),$$
$$\mathrm{Fun}\left(\coprod_{i\in I}\mathcal{C}_i,\mathcal{I}\right)\simeq\prod_{i\in I}\mathrm{Fun}(\mathcal{C}_i,\mathcal{I}).$$

For categories $\mathcal{C},\mathcal{D},\mathcal{E}$ and functors $F:\mathcal{C}\to\mathcal{E}$ and $G:\mathcal{D}\to\mathcal{E}$, there is also a natural equivalence

$$\mathrm{Fun}(\mathcal{I},\mathcal{C}\times_{\mathcal{E}}\mathcal{D})\simeq\mathrm{Fun}(\mathcal{I},\mathcal{C})\times_{\mathrm{Fun}(\mathcal{I},\mathcal{E})}\mathrm{Fun}(\mathcal{I},\mathcal{D}).$$

Currying induces a natural equivalence $\mathrm{Fun}(\mathcal{I},\mathrm{Fun}(\mathcal{C},\mathcal{D}))\simeq\mathrm{Fun}(\mathcal{I}\times\mathcal{C},\mathcal{D})$.

Proof. Uncurry the evaluation functors, apply the corresponding universal property, and curry again. This constructs inverse comparison functors; the uniqueness clauses supply the natural equivalences between their composites and the identities. For coproducts, Axiom 2.2.7 identifies the product of a parameter category with a coproduct with the coproduct of the corresponding products. □

Definition 2.2.12. A commutative square of categories is a *pushout square* if applying $\mathrm{Fun}(-,\mathcal{I})$ gives a pullback square for every category $\mathcal{I}$.

Axiom 2.2.13. For categories $\mathcal{C},\mathcal{D},\mathcal{E}$ and functors $\mathcal{E}\to\mathcal{C}$ and $\mathcal{E}\to\mathcal{D}$, there exists a pushout $\mathcal{C}\amalg_{\mathcal{E}}\mathcal{D}$.

Equivalently:

Lemma 2.2.14. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{T}$ be categories and let $F : \mathcal{E} \to \mathcal{C}$ and $G : \mathcal{E} \to \mathcal{D}$ be functors. Let $i_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \sqcup_{\mathcal{E}} \mathcal{D}$ and $i_{\mathcal{D}} : \mathcal{D} \to \mathcal{C} \sqcup_{\mathcal{E}} \mathcal{D}$ be the pushout inclusions. Restriction induces a natural equivalence of categories

$$(i_{\mathcal{C}}^*, i_{\mathcal{D}}^*) : \mathrm{Fun}(\mathcal{C} \sqcup_{\mathcal{E}} \mathcal{D}, \mathcal{T}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{T}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{T})} \mathrm{Fun}(\mathcal{D}, \mathcal{T}),$$

where the maps to $\mathrm{Fun}(\mathcal{E}, \mathcal{T})$ are precomposition with F and G.

Proof. By Definition 2.2.12, applying $\mathrm{Fun}(-, \mathcal{T})$ to the pushout square gives a pullback square. Its comparison functor to the pullback is precisely the restriction functor in the statement, so it is an equivalence. □

Lemma 2.2.15. Consider two adjacent commutative squares of categories.

(1) If the right square is a pullback, then the left square is a pullback if and only if the outer rectangle is a pullback.
(2) Dually, if the left square is a pushout, then the right square is a pushout if and only if the outer rectangle is a pushout.

We will refer to these rules as *pullback* and *pushout cancellation*.

Proof. Fix a category $\mathcal{T}$. The universal property in Axiom 2.2.4 identifies a functor from $\mathcal{T}$ into the upper-left vertex of the right pullback square with compatible functors into its upper-right and lower-left vertices. Substituting this description into the universal-property condition for the left square gives exactly the condition for the outer rectangle. The substitution is reversible and respects natural equivalences and all higher identifications, proving the pullback part.

For pushouts, apply $\mathrm{Fun}(-, \mathcal{T})$ for an arbitrary category $\mathcal{T}$. This reverses the order of the squares, and the left pushout becomes the right pullback by Definition 2.2.12. The first part of the proof identifies the pullback conditions for the other square and the outer rectangle. Requiring these conditions for every $\mathcal{T}$ gives the claim. □

2.3. **Commutative diagrams, composition, and animae.** We return to the problem of defining composition raised in Remark 2.1.9.

Notation. Write $\ulcorner$ for the ordinary subposet of $[1]\times[1]$ on the vertices $\{(0,0),(0,1),(1,0)\}$, the *span indexing category*. Write $\lrcorner$ for the ordinary subposet on the vertices $\{(0,1),(1,0),(1,1)\}$, the *cospan indexing category*.

Then:

Axiom 2.3.1. The following squares are pushouts, with $n \geq 2$ in the first square:

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [1] \\ {\scriptstyle n-1}\downarrow & & \downarrow{\scriptstyle \{n-1,n\}} \\ [n-1] & \hookrightarrow & [n] \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{0} & [1] \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \{0\}\times[1]} \\ [1] & \xrightarrow[[1]\times\{0\}]{} & \ulcorner \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{1} & [1] \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle [1]\times\{1\}} \\ [1] & \xrightarrow[\{1\}\times[1]]{} & \lrcorner \end{array} \qquad \begin{array}{ccc} [1] & \xrightarrow{02} & [2] \\ {\scriptstyle 02}\downarrow & & \downarrow \\ [2] & \longrightarrow & [1]\times[1] \end{array}$$

The maps are the indicated inclusions; in the last square, the right and bottom maps select the chains $(0,0) \to (0,1) \to (1,1)$ and $(0,0) \to (1,0) \to (1,1)$, respectively.

Definition 2.3.2. Let $\mathcal{C}$ be a category. A *commutative square* in $\mathcal{C}$ is a functor $[1]\times[1] \to \mathcal{C}$. A *span* in $\mathcal{C}$ is a functor $\ulcorner \to \mathcal{C}$, and a *cospan* in $\mathcal{C}$ is a functor $\lrcorner \to \mathcal{C}$.

By Axiom 2.3.1, commutative squares can be described as glued triangles:

Remark 2.3.3. The last square of Axiom 2.3.1, together with Lemma 2.2.14, identifies a commutative square with two commutative triangles identified along their common diagonal. Notice also that the four boundary arrows alone do not specify its commutativity.

The following *Segal property* follows from Axiom 2.3.1; see [Rez01, Section 4.1].

Proposition 2.3.4. Let $\mathcal{C}$ be a category. Let $i_{01}, i_{12} : [1] \to [2]$ be the inclusions of the consecutive edges. Restriction induces an equivalence

$$(i_{01}^*, i_{12}^*) : \mathrm{Fun}([2], \mathcal{C}) \to \mathrm{Fun}([1], \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}([1], \mathcal{C}),$$

where the first factor maps to $\mathcal{C}$ by target evaluation and the second by source evaluation.

Proof. Apply the universal property of the pushout $[2] \simeq [1] \amalg_{[0]} [1]$ in Axiom 2.3.1. □

Finally:

Construction 2.3.5. Let $\mathcal{C}$ be a category and let $f : x \to y$ and $g : y \to z$ be morphisms between objects $x, y, z \in \mathcal{C}$. The equivalence in Proposition 2.3.4 supplies a commutative triangle extending f and g. Its remaining edge is the *composite* $gf : x \to z$.

The functors $[2] \to [1] \to \mathcal{C}$ obtained from f by repeating either endpoint give the unit identifications $f \,\mathrm{id}_x \simeq f$ and $\mathrm{id}_y \, f \simeq f$. For a further morphism $h : z \to w$, with $w \in \mathcal{C}$ an object, Axiom 2.3.1 supplies an extension of the string f, g, h to $[3]$. Its triangular faces identify both $h(gf)$ and $(hg)f$ with the edge from the initial to the final vertex. The analogous extensions to $[n]$ supply the higher compatibility among these identifications.

Composition allows us to define equivalences in a category:

Definition 2.3.6. Let $\mathcal{C}$ be a category. A morphism $f : x \to y$, between objects $x, y \in \mathcal{C}$, is an *equivalence* if there is a morphism $g : y \to x$ and triangles identifying gf with id_x and fg with id_y. A category in which every morphism is an equivalence is an *anima*, or an ∞-groupoid.

Animae provide a categorical model for homotopy types.

Remark 2.3.7. A topological space determines an ∞-groupoid whose objects are its points, whose morphisms are paths, and whose higher identifications are homotopies and their higher analogues. Every morphism is invertible, with inverse represented by the reversed path. Conversely, every anima can be represented by a topological space, considered up to weak homotopy equivalence. This correspondence is known as the *homotopy hypothesis*. In the quasi-category model, this correspondence rests on Joyal's characterization of ∞-groupoids as Kan complexes [Joy02, Corollary 1.4], together with the classical equivalence between Kan complexes and homotopy types.

The groupoid core in Axiom 2.3.8 is the largest anima contained in a category. For the model-theoretic constructions see [Joy02, Corollary 1.5] and [Lur09, Proposition 1.2.5.3].

Axiom 2.3.8. For every category $\mathcal{C}$, there is an anima $\mathcal{C}^{\simeq}$ and a functor $\gamma_{\mathcal{C}} : \mathcal{C}^{\simeq} \to \mathcal{C}$ universal among functors from animae into $\mathcal{C}$. We will refer to this anima as the *groupoid core* of $\mathcal{C}$. Moreover, the two vertices induce an equivalence $* \amalg * \simeq [1]^{\simeq}$.

There is another anima that we can construct:

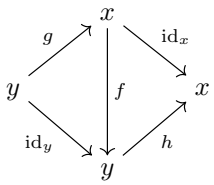
Definition 2.3.9. Let $\mathcal{C}$ be a category and let $x, y \in \mathcal{C}$ be objects. The category $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is the fibre at (x, y) of source and target evaluation $\mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$. We refer to $\mathrm{Hom}_{\mathcal{C}}(x, y)$ as the *anima of morphisms* from x to y.

Although we call this the anima of morphisms, we have not yet shown that it is an anima: the fibre construction in Definition 2.3.9 by itself only supplies a category of arrows with specified endpoints. This requires the *Rezk axiom*, Axiom 2.3.11.

Construction 2.3.10. Let $\mathcal{C}$ be a category. Form the category

$$\mathrm{Iso}(\mathcal{C}) = \mathrm{Fun}([3], \mathcal{C}) \times_{\mathrm{Fun}([1], \mathcal{C})^2} (\mathcal{C} \times \mathcal{C}),$$

where the maps are restriction to the edges $02, 13$ and $(x, y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)$ for objects $x, y \in \mathcal{C}$. Its objects record a morphism $f : x \to y$, morphisms $g, h : y \to x$, and the two commutative triangles



expressing $fg \simeq \mathrm{id}_y$ and $hf \simeq \mathrm{id}_x$. These are the faces 012 and 123 of the $[3]$-diagram, sharing the edge 12 labelled by f. By the Segal pushouts in Axiom 2.3.1, these two triangles determine the $[3]$-diagram used above. Identity diagrams give a functor $\mathcal{C} \to \mathrm{Iso}(\mathcal{C})$.

The Rezk axiom, Axiom 2.3.11, is the completeness condition of [Rez01, Section 6].

Axiom 2.3.11. For every category $\mathcal{C}$, the identity-diagram functor $\mathcal{C} \to \mathrm{Iso}(\mathcal{C})$ is an equivalence.

Then:

Proposition 2.3.12 ([Cno26, Proposition 1.4.25 and Corollaries 1.4.27–1.4.28])**.** Let $\mathcal{C}, \mathcal{D}$ be categories.

(1) An equivalence $x \to y$ between objects $x, y \in \mathcal{C}$ identifies them as functors $* \to \mathcal{C}$.
(2) Natural equivalences between functors $\mathcal{C} \to \mathcal{D}$ in the sense of Axiom 2.1.1 agree with invertible natural transformations.
(3) For objects $x, y \in \mathcal{C}$, the category $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an anima.

Proof. For the first claim, an equivalence $f : x \to y$ in $\mathcal{C}$, together with inverse data, determines an object $\widetilde{f} \in \mathrm{Iso}(\mathcal{C})$. By Axiom 2.3.11, there is an object $z \in \mathcal{C}$ and a natural equivalence between the identity diagram at z and $\widetilde{f}$. Evaluating at the source and target of the middle arrow gives natural equivalences $z \simeq x$ and $z \simeq y$, hence $x \simeq y$ as functors $* \to \mathcal{C}$.

For functors $F, G : \mathcal{C} \to \mathcal{D}$, the category encoding natural equivalences $F \simeq G$ is the fibre at (F, G) of the diagonal of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. This diagonal factors through the identity-diagram functor to $\mathrm{Iso}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$, followed by source and target evaluation. The first functor is an equivalence by Axiom 2.3.11, now applied to the functor category. Taking fibres therefore identifies natural equivalences with natural transformations equipped with left and right inverse data, proving the second claim.

By currying and Definition 2.3.9, a functor $[1] \to \mathrm{Hom}_{\mathcal{C}}(x, y)$ is a commutative square in $\mathcal{C}$ whose two sides at the fixed endpoints are the identity arrows of x and y. Decompose the square along its diagonal using Axiom 2.3.1. Collapsing the identity side in each triangle replaces that triangle by $[1]$, by the Segal pushout $[2] \simeq [1] \sqcup_{[0]} [1]$. The resulting intervals are glued along their common diagonal, giving $[1]$ again. By Proposition 2.2.11, this calculation identifies the constant-diagram functor $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Fun}([1], \mathrm{Hom}_{\mathcal{C}}(x, y))$ as an equivalence. Thus every arrow of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is naturally equivalent to an identity arrow. By Proposition 2.3.12(2), this natural equivalence is an invertible natural transformation, whose components identify the given arrow with an identity; hence that arrow is invertible. Therefore $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an anima. □

Remark 2.3.13. Let $\mathcal{C}$ be a category and let $f : x \to y$ be a morphism between objects $x, y \in \mathcal{C}$. In $\mathrm{Iso}(\mathcal{C})$, we choose morphisms $g, h : y \to x$ separately, together with identifications $\alpha : fg \simeq \mathrm{id}_y$ and $\beta : hf \simeq \mathrm{id}_x$. Could we instead use a single inverse $g : y \to x$, with identifications $\alpha : fg \simeq \mathrm{id}_y$ and $\beta : gf \simeq \mathrm{id}_x$? Choosing these identifications independently retains additional information. Indeed, when $f = \mathrm{id}_x$, both α and β identify g with id_x. Following α backwards and then β gives a loop in $\mathrm{Hom}_{\mathcal{C}}(x, x)$ based at id_x, which need not be trivial.

We finish this section with the construction of homotopy categories.

Notation 2.3.14. Let X be an anima. For objects $x, y \in X$, write $x \simeq y$ if there exists a morphism $x \to y$ in X. Since every morphism in an anima is invertible, this is an equivalence relation: identities give reflexivity, inverses give symmetry, and composition gives transitivity. The equivalence class $[x]$ of an object $x \in X$ is its *connected component*. We write $\pi_0 X$ for the set of these equivalence classes; it records the components while forgetting the particular morphisms and all higher identifications. We regard a set as an anima via its ordinary category with only identity morphisms. The assignment $x \mapsto [x]$ defines the canonical functor $X \to \pi_0 X$, which sends every morphism to an identity, since its endpoints represent the same class. An anima X is *discrete* if the canonical map $X \to \pi_0 X$ is an equivalence.

Definition 2.3.15. A category $\mathcal{C}$ is a 1-*category* if the anima $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is discrete for every pair of objects $x, y \in \mathcal{C}$.

We can then construct homotopy categories (see [Cno26, Example 1.8.14]):

Proposition 2.3.16. Let $\mathcal{C}$ be a category. There exist a 1-category $h\mathcal{C}$ and a functor $q : \mathcal{C} \to h\mathcal{C}$ such that, for every 1-category $\mathcal{D}$, restriction induces an equivalence

$$q^* : \mathrm{Fun}(h\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

We call $h\mathcal{C}$ the *homotopy category* of $\mathcal{C}$. The functor q induces a bijection on equivalence classes of objects and natural identifications $\mathrm{Hom}_{h\mathcal{C}}(qx, qy) \simeq \pi_0 \mathrm{Hom}_{\mathcal{C}}(x, y)$ for objects $x, y \in \mathcal{C}$.

The relation between 1-categories and the ordinary categories of Axiom 2.1.5 is as follows.

Remark 2.3.17. Every ordinary category, regarded as a category, is a 1-category: its mapping animae are the discrete animae associated with its ordinary sets of morphisms. Conversely, every 1-category is equivalent to a category arising from an ordinary category. In particular, for an ordinary category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, the mapping anima $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is equivalent to the usual morphism set regarded as a discrete anima.

Remark 2.3.18. For a category $\mathcal{C}$, a morphism $f : x \to y$ between objects $x, y \in \mathcal{C}$ is an equivalence precisely when its image under the canonical functor $q : \mathcal{C} \to h\mathcal{C}$ of Proposition 2.3.16 is an isomorphism.

2.4. **Further constructions.** We now construct the opposite category:

Axiom 2.4.1. Every category $\mathcal{C}$ has an *opposite* category $\mathcal{C}^{\mathrm{op}}$. For categories $\mathcal{C}, \mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, compatibly with composition and natural equivalences. There are natural equivalences $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C}$, and the construction agrees with the ordinary opposite-category construction.

Mapping animae are reversed in the opposite category.

Remark 2.4.2. For a category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, reversing arrows gives $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) \simeq \mathrm{Hom}_{\mathcal{C}}(y, x)$.

Overcategories and undercategories are fibres of the evaluation functors [Cno26, Section 21.3].

Definition 2.4.3. Let $\mathcal{C}$ be a category and let $x \in \mathcal{C}$ be an object. The *overcategory* $\mathcal{C}_{/x}$ is the fibre of target evaluation $\mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ at x, and the *undercategory* $\mathcal{C}_{x/}$ is the fibre of source evaluation at x, in the sense of Definition 2.2.6. Their objects are morphisms into x and out of x, respectively. Source and target evaluation, respectively, give their projections to $\mathcal{C}$.

Subcategories allow us to restrict the objects and morphisms of a category while retaining the higher identifications between the morphisms they contain.

Definition 2.4.4. Let $\mathcal{A}, \mathcal{C}$ be categories. A functor $i : \mathcal{A} \to \mathcal{C}$ is a *monomorphism* if the diagonal $\mathcal{A} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{A}$ is an equivalence. The monomorphism i is *full* if it is fully faithful, so that for all objects $x, y \in \mathcal{A}$ the induced map

$$\mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(ix, iy)$$

is an equivalence. It is *wide* if every object of $\mathcal{C}$ is equivalent to the image of an object of $\mathcal{A}$.

Axiom 2.4.5. Let $\mathcal{C}$ be a category and let A be an anima equipped with a monomorphism $A \hookrightarrow \mathrm{Fun}([1], \mathcal{C})^{\simeq}$. We regard A as a collection of morphisms of $\mathcal{C}$, invariant under equivalence of arrows. Suppose that A contains the identities of the endpoints of every morphism it contains and is closed under composition. Then there is a monomorphism $i : \mathcal{A} \to \mathcal{C}$ whose induced map on arrow cores identifies $\mathrm{Fun}([1], \mathcal{A})^{\simeq}$ with A. For every category $\mathcal{T}$, postcomposition induces a functor

$$i_* : \mathrm{Fun}(\mathcal{T}, \mathcal{A})^{\simeq} \to \mathrm{Fun}(\mathcal{T}, \mathcal{C})^{\simeq}$$

which identifies its source with the union of components consisting of functors that carry every morphism of $\mathcal{T}$ into A.

Construction 2.4.6. Let $\mathcal{C}$ be a category and let A be an anima equipped with a monomorphism $A \hookrightarrow \mathcal{C}^{\simeq}$, specifying an equivalence-closed collection of objects. The collection of all arrows between these objects is

$$\mathrm{Fun}([1], \mathcal{C})^{\simeq} \times_{\mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}} (A \times A),$$

where the map from the arrow core is source and target evaluation. Identities and composites have endpoints in A, so Axiom 2.4.5 supplies a subcategory $\mathcal{A} \hookrightarrow \mathcal{C}$. This is the *full subcategory spanned by* A: its core identifies with A, and it retains the entire mapping anima between every pair of its objects [Cno26, Definition 1.5.10 and Proposition 1.5.11]; see also [Lur09, Section 1.2.11] for the quasi-category construction. For every category $\mathcal{T}$, postcomposition identifies $\mathrm{Fun}(\mathcal{T}, \mathcal{A})$ with the full subcategory of $\mathrm{Fun}(\mathcal{T}, \mathcal{C})$ spanned by the functors taking their objects into A.

Remark 2.4.7. Let $i : \mathcal{A} \to \mathcal{C}$ be a monomorphism between categories $\mathcal{A}, \mathcal{C}$. For objects $x, y \in \mathcal{A}$, the mapping anima $\mathrm{Hom}_{\mathcal{A}}(x, y)$ identifies with a union of connected components of $\mathrm{Hom}_{\mathcal{C}}(ix, iy)$. Thus a subcategory may exclude morphisms, but it retains all homotopies and higher identifications between the morphisms it includes. With this convention, it also contains every equivalence in $\mathcal{C}$ between its objects: such an arrow is equivalent in the arrow category to an identity arrow. A full subcategory includes every component of each of these mapping animae; a wide subcategory includes every object but may exclude components of the mapping animae.

Let $\mathcal{T}$ be a category and let $F, G : \mathcal{T} \to \mathcal{A}$ be functors. A natural transformation $\alpha : iF \to iG$ lifts to a natural transformation $F \to G$ precisely when each component α_t, for an object $t \in \mathcal{T}$, belongs to $\mathcal{A}$. Indeed, apply Axiom 2.4.5 to its uncurrying $[1] \times \mathcal{T} \to \mathcal{C}$; the other arrows are composites of components of α with arrows supplied by F and G. For a full subcategory this condition is automatic, whereas for a subcategory that is not full it is an additional condition on the transformation.

Example 2.4.8. Let $\mathcal{C}$ be a category. The groupoid core $\mathcal{C}^{\simeq} \to \mathcal{C}$ is a wide subcategory retaining exactly the equivalences. It is full precisely when $\mathcal{C}$ is an anima. The ordinary category of finite sets and injective maps is a wide subcategory of the category of finite sets and all maps. It is not full: the map from a two-element set to a one-element set is not injective. In contrast, the subcategory spanned by finite sets of cardinality at most two is full when it contains all maps between those sets.

2.5. **Recognition of equivalences.**

Definition 2.5.1. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(1) We say that F is *fully faithful* if the induced map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ is an equivalence for every pair of objects $x, y \in \mathcal{C}$.
(2) We say that F is *essentially surjective* if every object $d \in \mathcal{D}$ is equivalent to $F(c)$ for some object $c \in \mathcal{C}$.

For model-theoretic foundations of the recognition criteria, see [Lur09, Section 1.2.10] and [Rez01, Section 7].

Theorem 2.5.2. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.

Remark 2.5.3. Full faithfulness is a condition on mapping animae and hence an equivalence between the ordinary homotopy categories alone does not imply an equivalence of categories.

Equivalences of functors can be detected pointwise; a synthetic proof is given in [RS17, Proposition 10.3].

Axiom 2.5.4. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be functors. A natural transformation $\alpha : F \to G$ is a natural equivalence if and only if α_x is an equivalence for every object $x \in \mathcal{C}$.

2.6. **The universal cocartesian fibration.** The axioms introduced so far also hold for classical 1-categories, interpreting natural equivalences as natural isomorphisms. The following axiom distinguishes the two theories by requiring a category Cat of categories whose mapping animae are the cores of functor categories.

Axiom 2.6.1. There are categories Cat, An and $\mathrm{Cat}_\bullet$, a fully faithful inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$, and a functor $\pi_{\mathrm{univ}} : \mathrm{Cat}_\bullet \to \mathrm{Cat}$ called the *universal cocartesian fibration*. An object $c \in \mathrm{Cat}$ determines a category $\mathcal{C}$ by the pullback square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow[c]{} & \mathrm{Cat}. \end{array}$$

Categories arising in this way are called *small categories*; the objects of An correspond to the small animae. For objects $c, d \in \mathrm{Cat}$ corresponding to small categories $\mathcal{C}, \mathcal{D}$, there is a natural equivalence $\mathrm{Hom}_{\mathrm{Cat}}(c, d) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$, compatible with identities and composition. We identify an object of Cat with its corresponding small category and write $\mathcal{C} \in \mathrm{Cat}$. The categories An and Cat admit small limits and colimits, meaning those indexed by small categories, and the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$ preserves both. The initial and terminal categories are small, and small categories are closed under products, coproducts, pullbacks, pushouts, functor categories, subcategories, and localizations.

Notation 2.6.2. We call An and Cat the *universe of animae* and the *universe of categories*. Smallness refers to the universal cocartesian fibration of Axiom 2.6.1. We allow larger categories $\widehat{\mathrm{An}}$ and $\widehat{\mathrm{Cat}}$ satisfying the same axioms, together with a fully faithful inclusion $\mathrm{Cat} \hookrightarrow \widehat{\mathrm{Cat}}$ restricting to $\mathrm{An} \hookrightarrow \widehat{\mathrm{An}}$, such that $\mathrm{An}, \mathrm{Cat} \in \widehat{\mathrm{Cat}}$ are small relative to this larger universe. Categories small relative to $\widehat{\mathrm{Cat}}$ are called *large categories* relative to the original universe; in particular, every small category is also large in this sense. All categorical constructions are understood relative to these categories and are compatible with enlargement. We enlarge the categories when necessary, retaining the notation Cat and An unless the distinction matters.

Remark 2.6.3. For constructions of An and Cat in the quasi-category model, see [Lur09, Definitions 1.2.16.1 and 3.0.0.1].

The classification of fibrations uses the following terminology.

Definition 2.6.4. Let $E, B \in \mathrm{Cat}$ be categories and let $p : E \to B$ be a functor.

(1) A morphism $e : x \to y$ between objects $x, y \in E$ is *p-cocartesian* if, for every object $z \in E$, the square of animae

$$\begin{array}{ccc} \mathrm{Hom}_E(y, z) & \xrightarrow{-\circ e} & \mathrm{Hom}_E(x, z) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_B(p(y), p(z)) & \xrightarrow[-\circ p(e)]{} & \mathrm{Hom}_B(p(x), p(z)) \end{array}$$

is a pullback.

(2) The functor p is a *cocartesian fibration* if, for every object $x \in E$, every object $b \in B$ and every morphism $u : p(x) \to b$, there is a p-cocartesian morphism $e : x \to y$ lifting u. Here a lift includes an identification $p(y) \simeq b$ under which $p(e)$ identifies with u.
(3) A morphism in E is *p-cartesian* if its opposite is p^{op}-cocartesian.
(4) The functor p is a *cartesian fibration* if $p^{\mathrm{op}} : E^{\mathrm{op}} \to B^{\mathrm{op}}$ is a cocartesian fibration.

Finally, a *left fibration* is a cocartesian fibration whose fibres are animae, and a *right fibration* is a cartesian fibration whose fibres are animae.

Write $\mathrm{Cocart}(B)$, $\mathrm{Cart}(B)$ and $\mathrm{LFib}(B) \subseteq \mathrm{Cocart}(B)$, $\mathrm{RFib}(B) \subseteq \mathrm{Cart}(B)$ for the categories of cocartesian and cartesian fibrations over B with small fibres and their full subcategories of left and right fibrations, respectively. Morphisms in $\mathrm{Cocart}(B)$ and $\mathrm{Cart}(B)$ are functors over B preserving cocartesian and cartesian morphisms, respectively.

The universal cocartesian fibration gives the following classification, which we use without proof; see [Cno26, Chapter 23] and [Lur09, Section 3.2].

Theorem 2.6.5 (Straightening and unstraightening)**.** Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. There are equivalences of categories

$$\mathrm{Str}^{\mathrm{cc}} : \mathrm{Cocart}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}) \qquad \text{and} \qquad \mathrm{Str}^{\mathrm{ct}} : \mathrm{Cart}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$$

whose inverse equivalences are given by pullback of π_{univ}. These equivalences are natural in $\mathcal{C}$: pullback of fibrations corresponds to restriction of functors. They restrict to natural equivalences

$$\mathrm{Str}^{\mathrm{cc}} : \mathrm{LFib}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{An}) \qquad \text{and} \qquad \mathrm{Str}^{\mathrm{ct}} : \mathrm{RFib}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}).$$

The value of a straightened functor at an object $c \in \mathcal{C}$ is the fibre over c, and its action on arrows is transport in the fibration. The passage from a fibration to its classifying functor is called *straightening*; the reverse passage is called *unstraightening*.

The groupoid core has the following mapping property.

Proposition 2.6.6. Let $\mathcal{C} \in \mathrm{Cat}$ be a category and let $X \in \mathrm{An}$ be an anima. Composition with $\gamma_{\mathcal{C}}$ induces an equivalence $\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \simeq \mathrm{Fun}(X, \mathcal{C})^{\simeq}$.

Proof. A functor from X takes its arrows to equivalences and hence factors through the core by Axiom 2.3.8. Natural equivalences between such functors, together with their higher identifications, factor in the same way. The functor category with target $\mathcal{C}^{\simeq}$ is itself an anima by Axiom 2.5.4. These factorizations give the asserted equivalence of animae. □

This result expresses that the core is *right adjoint* to the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$.

3. Adjunctions and universal constructions

Presheaves and the Yoneda embedding provide the basis for a synthetic treatment of adjunctions, limits, colimits, and Kan extensions.

3.1. Presheaves and the Yoneda lemma.

Mapping animae depend functorially on both variables. Compared to ordinary category theory, the functoriality of $\mathrm{Hom}_{\mathcal{C}}$ is an extremely non-trivial result (see [Cno26, Construction 23.4.2] for the synthetic version constructed via straightening and [Lur09, Section 5.1.3] for the quasi-category construction).

Theorem 3.1.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. There is a functor

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$$

whose value at objects $x, y \in \mathcal{C}$ is their mapping anima.

Proof. Let $s, t : \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ be source and target evaluation. The functor t is a cocartesian fibration, and

$$(s, t) : \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

preserves cocartesian arrows over $\mathcal{C}$, where the target is viewed over $\mathcal{C}$ by its second projection [Cno26, Lemma 23.4.1]. Straightening therefore gives a functor

$$\mathcal{C} \to \mathrm{Cat}_{/\mathcal{C}}, \qquad y \mapsto (\mathcal{C}_{/y} \to \mathcal{C}).$$

Here $\mathcal{C}_{/y}$ is the overcategory of Definition 2.4.3, and the indicated projection is induced by s. Each projection $\mathcal{C}_{/y} \to \mathcal{C}$ is a right fibration [Cno26, Construction 23.4.2], with fibre $\mathrm{Hom}_{\mathcal{C}}(x, y)$ at an object $x \in \mathcal{C}$ by Definition 2.3.9. The preceding functor thus takes values in right fibrations over $\mathcal{C}$. Applying the right-fibration case of Theorem 2.6.5 gives a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$. Uncurrying gives the required Hom functor. Its action on morphisms is precomposition in the first variable and postcomposition in the second. □

Then:

Definition 3.1.2. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. Its *category of presheaves* is

$$\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \in \widehat{\mathrm{Cat}}.$$

Currying Theorem 3.1.1 defines the *Yoneda functor*, which sends each object $x \in \mathcal{C}$ to its represented presheaf

$$よ_{\mathcal{C}} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}), \qquad x \mapsto \mathrm{Hom}_{\mathcal{C}}(-, x).$$

A presheaf $F \in \mathrm{PSh}(\mathcal{C})$ is *representable* if it is equivalent to $よ_{\mathcal{C}}(x)$ for some object $x \in \mathcal{C}$.

We use the Yoneda lemma in the form of [Cno26, Theorem 1.8.5]. An earlier synthetic proof is [RS17, Theorem 9.1], applied to the opposite category.

Theorem 3.1.3. Let $\mathcal{C} \in \mathrm{Cat}$ be a category, let $x \in \mathcal{C}$ be an object, and let $F \in \mathrm{PSh}(\mathcal{C})$ be a presheaf. Evaluation at the identity of x induces an equivalence

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(よ_{\mathcal{C}}(x), F) \simeq F(x),$$

natural in both x and F. In particular, the Yoneda functor $よ_{\mathcal{C}}$ is fully faithful.

Proof. We use the first part of the statement without proof. For full faithfulness, let $x, y \in \mathcal{C}$ be objects and apply Theorem 3.1.3 to the presheaf $F = よ_{\mathcal{C}}(y) \in \mathrm{PSh}(\mathcal{C})$. It gives a natural equivalence $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(よ_{\mathcal{C}}(x), よ_{\mathcal{C}}(y)) \simeq \mathrm{Hom}_{\mathcal{C}}(x, y)$. Under this equivalence, the map induced by the Yoneda functor is the identity. □

3.2. **Adjunctions.** An adjunction identifies two ways of describing the same morphism: one uses a functor on its source, and the other uses a functor on its target. Kan introduced adjoint functors in [Kan58, Definition 3.1]. For the exposition used here, see [Cno26, Section 21.1]; for foundational treatments, see [Lur09, Section 5.2.2] and [RS17, Section 11].

Definition 3.2.1. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ consists of a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, R(-))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$. We write $L \dashv R$ to indicate that such an isomorphism has been given. We call L the *left adjoint* and R the *right adjoint*.

Remark 3.2.2. An adjunction induces an adjunction at the level of homotopy categories.

Equivalences are examples of adjunctions, and adjunctions compose:

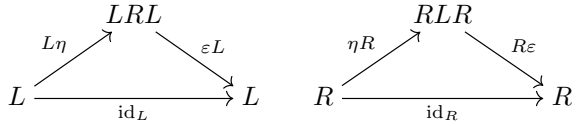
Example 3.2.3. For categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, an equivalence $L : \mathcal{C} \to \mathcal{D}$ has an inverse which is both a left and a right adjoint. If $\mathcal{E} \in \mathrm{Cat}$ is another category and $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ and $L' : \mathcal{D} \rightleftarrows \mathcal{E} : R'$ are adjunctions, then $L'L \dashv RR'$. The second assertion follows, for objects $x \in \mathcal{C}$ and $z \in \mathcal{E}$, from the successive equivalences

$$\mathrm{Hom}_{\mathcal{E}}(L'Lx, z) \simeq \mathrm{Hom}_{\mathcal{D}}(Lx, R'z) \simeq \mathrm{Hom}_{\mathcal{C}}(x, RR'z).$$

Adjunctions can equivalently be characterized in terms of units, counits, and triangle identities (see [Cno26, Proposition 21.1.2 and its footnote]).

Proposition 3.2.4. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. The following conditions are equivalent.

(1) There is an adjunction $L \dashv R$.
(2) There is a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to RL$, called the *unit*, such that for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$ the composite
$$\mathrm{Hom}_{\mathcal{D}}(Lx, y) \xrightarrow{R} \mathrm{Hom}_{\mathcal{C}}(RLx, Ry) \xrightarrow{-\circ\eta_x} \mathrm{Hom}_{\mathcal{C}}(x, Ry)$$
is an equivalence.
(3) There is a natural transformation $\varepsilon : LR \to \mathrm{id}_{\mathcal{D}}$, called the *counit*, such that for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$ the composite
$$\mathrm{Hom}_{\mathcal{C}}(x, Ry) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}(Lx, LRy) \xrightarrow{\varepsilon_y\circ-} \mathrm{Hom}_{\mathcal{D}}(Lx, y)$$
is an equivalence.
(4) There are natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to RL$ and $\varepsilon : LR \to \mathrm{id}_{\mathcal{D}}$ satisfying the following commutative triangles, called the *triangle identities*, in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$, respectively:



Moreover, specifying the adjunction in (1), the unit in (2), or the counit in (3) gives equivalent data.

Proof. Given the natural equivalence $\Phi_{x,y} : \mathrm{Hom}_{\mathcal{D}}(Lx, y) \simeq \mathrm{Hom}_{\mathcal{C}}(x, Ry)$, set $\eta_x = \Phi_{x,Lx}(\mathrm{id}_{Lx})$. The Yoneda lemma identifies $\Phi_{x,y}$ with $f \mapsto R(f) \circ \eta_x$, and naturality in x identifies these maps with a natural transformation η. This gives the equivalence of data in (1) and (2). Dually, setting $\varepsilon_y = \Phi^{-1}_{Ry,y}(\mathrm{id}_{Ry})$ gives the equivalence with (3), with inverse mapping formula $g \mapsto \varepsilon_y \circ L(g)$. Applying these mutually inverse formulas to the identity morphisms gives $\varepsilon_{Lx} \circ L(\eta_x) \simeq \mathrm{id}_{Lx}$ and $R(\varepsilon_y) \circ \eta_{Ry} \simeq \mathrm{id}_{Ry}$, naturally in x and y, proving (4). Conversely, suppose (4) holds. For morphisms $f : Lx \to y$ and $g : x \to Ry$, naturality and the triangle identities give

$$\varepsilon_y \circ LR(f) \circ L(\eta_x) \simeq f \circ \varepsilon_{Lx} \circ L(\eta_x) \simeq f,$$
$$R(\varepsilon_y) \circ RL(g) \circ \eta_x \simeq R(\varepsilon_y) \circ \eta_{Ry} \circ g \simeq g.$$

These are natural homotopies, so the composites in (2) and (3) are inverse equivalences of mapping animae and define an adjunction. □

Adjoints can be constructed object by object through the following universal properties [Cno26, Definition 21.1.3].

Definition 3.2.5. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $y \in \mathcal{D}$ be an object.

(1) A *right adjoint object* to y under F is an object $z \in \mathcal{C}$ equipped with a morphism $\varepsilon : Fz \to y$ such that, for every object $x \in \mathcal{C}$, the map
$$\mathrm{Hom}_{\mathcal{C}}(x, z) \to \mathrm{Hom}_{\mathcal{D}}(Fx, y), \qquad f \mapsto \varepsilon \circ F(f),$$

is an equivalence.

(2) Dually, a *left adjoint object* to y under F is an object $z \in \mathcal{C}$ equipped with a morphism $\eta : y \to Fz$ such that, for every object $x \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(y, Fx), \qquad f \mapsto F(f) \circ \eta,$$

is an equivalence.

These universal properties supply the functoriality, so that no separate choices of maps between adjoint objects are needed (see [Cno26, Lemma 21.1.4]).

Lemma 3.2.6. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathcal{D}' \in \mathrm{Cat}$ be a full subcategory of $\mathcal{D}$.

(1) If every object $y \in \mathcal{D}'$ admits a right adjoint object under F, these objects assemble into a functor $G : \mathcal{D}' \to \mathcal{C}$ with a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(F(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, G(-))$ on $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}'$.

(2) Dually, if every object $y \in \mathcal{D}'$ admits a left adjoint object under F, these objects assemble into a functor $L : \mathcal{D}' \to \mathcal{C}$ with a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{D}}(-, F(-))$ on $(\mathcal{D}')^{\mathrm{op}} \times \mathcal{C}$.

Proof. The functor $\mathcal{D}' \to \mathrm{PSh}(\mathcal{C})$ sending y to $\mathrm{Hom}_{\mathcal{D}}(F(-), y)$ takes values in representable presheaves by hypothesis. The fully faithful Yoneda embedding $\yoneda_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ identifies $\mathcal{C}$ with the full subcategory of representables, so this functor factors through $\yoneda_{\mathcal{C}}$ as G. The factorization gives the stated natural isomorphism. The left adjoint assertion follows by passing to opposite categories. □

Remark 3.2.7. This also implies that adjoints, when they exist, are unique up to equivalence compatible with their adjunction data; see [CCNW26, Proposition 5.4.5].

Taking $\mathcal{D}' = \mathcal{D}$ gives the following objectwise criterion for adjoints [Cno26, Corollary 21.1.5].

Corollary 3.2.8. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if every object $y \in \mathcal{D}$ admits a right adjoint object under F. Dually, F admits a left adjoint if and only if every object $y \in \mathcal{D}$ admits a left adjoint object under F.

Proof. An adjunction supplies the required objects together with its counit or unit by Proposition 3.2.4. Conversely, Lemma 3.2.6 supplies the functor and natural isomorphism defining the adjunction. □

3.3. **Limits and colimits.**

Definition 3.3.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. An object $0 \in \mathcal{C}$ is *initial* if $\mathrm{Hom}_{\mathcal{C}}(0, x)$ is contractible for every object $x \in \mathcal{C}$. An object $1 \in \mathcal{C}$ is *terminal* if $\mathrm{Hom}_{\mathcal{C}}(x, 1)$ is contractible for every object $x \in \mathcal{C}$.

Remark 3.3.2. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. By Definition 3.2.1 and Definition 3.3.1, the functor $\mathcal{C} \to [0]$ has a left adjoint precisely when $\mathcal{C}$ has an initial object, and a right adjoint precisely when it has a terminal object.

The definitions of cones and universal cones follow [Cno26, Definition 1.7.1]; see also [Lur09, Section 1.2.13] for the quasi-category formulation.

Definition 3.3.3. Let $I \in \mathrm{Cat}$ be a small category and let $\mathcal{C} \in \mathrm{Cat}$ be a category. Restriction along $I \to *$ defines the *constant-diagram functor* $\mathrm{const}_I : \mathcal{C} \simeq \mathrm{Fun}(*, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$. Let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram.

(1) A *cone* over D is an object $x \in \mathcal{C}$ equipped with a natural transformation $\lambda : \mathrm{const}_I(x) \to D$.

(2) It is a *limit cone* if, for every object $y \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(y, x) \to \mathrm{Hom}_{\mathrm{Fun}(I, \mathcal{C})}(\mathrm{const}_I(y), D), \qquad f \mapsto \lambda \circ \mathrm{const}_I(f),$$

is an equivalence. The object x is called the *limit* of D and is denoted by $\lim_I D$.

(3) Dually, a *cocone* under D is an object $x \in \mathcal{C}$ equipped with a natural transformation $\mu : D \to \mathrm{const}_I(x)$.
(4) It is a *colimit cocone* if, for every object $y \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(D, \mathrm{const}_I(y)), \qquad g \mapsto \mathrm{const}_I(g) \circ \mu,$$

is an equivalence. The object x is called the *colimit* of D and is denoted by $\mathrm{colim}_I D$.

Finally, the category $\mathcal{C}$ admits all I-indexed limits, respectively colimits, if every diagram $D \in \mathrm{Fun}(I, \mathcal{C})$ admits a limit cone, respectively a colimit cocone.

Limits and colimits are adjoint objects for the constant-diagram functor [Cno26, Lemma 21.2.1 and Corollary 21.2.2].

Proposition 3.3.4. Let $I \in \mathrm{Cat}$ be a small category, let $\mathcal{C} \in \mathrm{Cat}$ be a category, and let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram. Then:

(1) A limit cone over D is precisely a right adjoint object to D under const_I.
(2) A colimit cocone under D is precisely a left adjoint object to D under const_I.

In particular, $\mathcal{C}$ admits all I-indexed limits if and only if const_I admits a right adjoint $\lim_I : \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$. Dually, $\mathcal{C}$ admits all I-indexed colimits if and only if const_I admits a left adjoint $\mathrm{colim}_I : \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$.

Proof. The defining mapping equivalences in Definition 3.3.3 are exactly those of Definition 3.2.5 for the functor const_I. The existence and functoriality assertions follow from Corollary 3.2.8. □

The following recognition criterion expresses universality through limits of animae. We use it without proof; see [Cno26, Proposition 23.6.7] and, for the quasi-category foundations, [Lur09, Propositions 5.1.3.2 and 5.1.2.2].

Proposition 3.3.5. Let $I \in \mathrm{Cat}$ be a small category, let $\mathcal{C} \in \mathrm{Cat}$ be a category, let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram, and let $x \in \mathcal{C}$ be an object. Then:

(1) A cone $\lambda : \mathrm{const}_I(x) \to D$ is a limit cone if and only if, for every object $y \in \mathcal{C}$, the map induced by its components is an equivalence $\mathrm{Hom}_{\mathcal{C}}(y, x) \to \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(y, D(i))$.
(2) Dually, a cocone $\mu : D \to \mathrm{const}_I(x)$ is a colimit cocone if and only if, for every object $y \in \mathcal{C}$, the induced map is an equivalence $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(D(i), y)$.

Both maps are natural in y, and the limits on the right are taken in An.

The mapping criterion in Proposition 3.3.5 gives the following examples.

Example 3.3.6. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. For the empty diagram, Definition 3.3.3 reduces to Definition 3.3.1: its limit is a terminal object, and its colimit is an initial object. For the discrete category with two objects, limits are products and colimits are coproducts. Let $x, y \in \mathcal{C}$ be objects whose product and coproduct exist. For every object $t \in \mathcal{C}$, their projections and inclusions induce equivalences

$$\mathrm{Hom}_{\mathcal{C}}(t, x \times y) \simeq \mathrm{Hom}_{\mathcal{C}}(t, x) \times \mathrm{Hom}_{\mathcal{C}}(t, y),$$
$$\mathrm{Hom}_{\mathcal{C}}(x \sqcup y, t) \simeq \mathrm{Hom}_{\mathcal{C}}(x, t) \times \mathrm{Hom}_{\mathcal{C}}(y, t).$$

For objects $w, z \in \mathcal{C}$, a pullback is the limit of a diagram $x \to z \leftarrow y$, and a pushout is the colimit of a diagram $x \leftarrow w \to y$. When they exist, we denote them by $x \times_z y$ and $x \sqcup_w y$, respectively. By Proposition 3.3.5, their universal cones and cocones induce equivalences

$$\mathrm{Hom}_{\mathcal{C}}(t, x \times_z y) \simeq \mathrm{Hom}_{\mathcal{C}}(t, x) \times_{\mathrm{Hom}_{\mathcal{C}}(t,z)} \mathrm{Hom}_{\mathcal{C}}(t, y),$$
$$\mathrm{Hom}_{\mathcal{C}}(x \sqcup_w y, t) \simeq \mathrm{Hom}_{\mathcal{C}}(x, t) \times_{\mathrm{Hom}_{\mathcal{C}}(w,t)} \mathrm{Hom}_{\mathcal{C}}(y, t)$$

for every object $t \in \mathcal{C}$. The fibre products on the right are taken in An.

We now study preservation of (co)limits.

Definition 3.3.7. Let $I \in \mathrm{Cat}$ be a small category, let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram and let $x \in \mathcal{C}$ be an object. A cone $\lambda : \mathrm{const}_I(x) \to D$ in the sense of Definition 3.3.3 induces the cone

$$\mathrm{const}_I(Fx) \simeq F\,\mathrm{const}_I(x) \xrightarrow{F\lambda} FD.$$

Dually, a cocone $\mu : D \to \mathrm{const}_I(x)$ induces the cocone

$$FD \xrightarrow{F\mu} F\,\mathrm{const}_I(x) \simeq \mathrm{const}_I(Fx).$$

We say that F *preserves* a limit cone or colimit cocone if its image is again a limit cone or colimit cocone, respectively. The functor F preserves I-indexed limits, respectively colimits, if it preserves every such limit cone, respectively colimit cocone.

Preservation can be tested on the canonical comparison morphisms [Cno26, Lemma 21.2.5].

Proposition 3.3.8. Let $I \in \mathrm{Cat}$ be a small category, let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram.

(1) Suppose that D and FD admit limits. The image of a chosen limit cone over D induces a comparison morphism
$$F(\lim_I D) \to \lim_I (FD).$$
The functor F preserves this limit cone if and only if this morphism is an equivalence.

(2) Suppose that D and FD admit colimits. The image of a chosen colimit cocone under D induces a comparison morphism
$$\mathrm{colim}_I (FD) \to F(\mathrm{colim}_I D).$$
The functor F preserves this colimit cocone if and only if this morphism is an equivalence.

Proof. The universal property in Definition 3.3.3, applied to the image cone of Definition 3.3.7, gives the first comparison, compatible with the cones. If this comparison is an equivalence, the image cone is a limit cone. Conversely, if the image cone is a limit cone, the universal properties of the two limit cones give an inverse comparison. The colimit assertion is dual. □

Definition 3.3.9. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. We say that $\mathcal{C}$ admits *finite limits* if it admits a terminal object and pullbacks, and that it admits *finite colimits* if it admits an initial object and pushouts. For a category $\mathcal{D} \in \mathrm{Cat}$, a functor $F : \mathcal{C} \to \mathcal{D}$ preserving finite limits is called *left exact*, and one preserving finite colimits is called *right exact*.

Notation 3.3.10. We write $\mathrm{Cat}^{\mathrm{lex}}$ for the category of small categories admitting finite limits and left exact functors. Dually, $\mathrm{Cat}^{\mathrm{rex}}$ is the category of small categories admitting finite colimits and right exact functors. These are subcategories of Cat with restrictions on both objects and morphisms.

Preservation of finite limits and colimits can be checked on the following constructions.

Proposition 3.3.11. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(1) If $\mathcal{C}$ and $\mathcal{D}$ admit finite limits, then F is left exact if and only if it preserves terminal objects and pullbacks.

(2) Dually, if $\mathcal{C}$ and $\mathcal{D}$ admit finite colimits, then F is right exact if and only if it preserves initial objects and pushouts.

Proof. See [Lur09, Corollary 4.4.2.4 and Corollary 4.4.2.5] and their duals. □

Left adjoints preserve colimits, and right adjoints preserve limits.

Proposition 3.3.12. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction. Then L preserves every colimit cocone in $\mathcal{C}$, and R preserves every limit cone in $\mathcal{D}$.

Proof. Let $I \in \mathrm{Cat}$ be a small category, let $D \in \mathrm{Fun}(I, \mathcal{C})$ be a diagram, and let $\mu : D \to \mathrm{const}_I(x)$ be a colimit cocone with $x \in \mathcal{C}$. For every object $y \in \mathcal{D}$, the adjunction and Proposition 3.3.5 give natural equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(Lx, y) &\simeq \mathrm{Hom}_{\mathcal{C}}(x, Ry) \\ &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(D(i), Ry) \\ &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(LD(i), y). \end{aligned}$$

Their composite is the map induced by the image cocone $L\mu$. Thus Proposition 3.3.5 shows that $L\mu$ is a colimit cocone. The assertion for R follows dually. □

Limits and colimits in functor categories are computed pointwise:

Proposition 3.3.13. Let $A, I \in \mathrm{Cat}$ be small categories and let $\mathcal{D} \in \mathrm{Cat}$ be a category. Let $D \in \mathrm{Fun}(I, \mathrm{Fun}(A, \mathcal{D}))$ be a diagram and let $X \in \mathrm{Fun}(A, \mathcal{D})$ be an object.

(1) If $\mathcal{D}$ admits I-indexed limits, then $\mathrm{Fun}(A, \mathcal{D})$ admits I-indexed limits. A cone $\lambda : \mathrm{const}_I(X) \to D$ is a limit cone if and only if, for every object $a \in A$, its evaluation
$$\lambda(a) : \mathrm{const}_I(X(a)) \to D(-)(a)$$
is a limit cone in $\mathcal{D}$.

(2) If $\mathcal{D}$ admits I-indexed colimits, then $\mathrm{Fun}(A, \mathcal{D})$ admits I-indexed colimits. A cocone $\mu : D \to \mathrm{const}_I(X)$ is a colimit cocone if and only if its evaluation at every object $a \in A$ is a colimit cocone in $\mathcal{D}$.

Proof. Suppose that $\mathcal{D}$ admits I-indexed limits. By Proposition 3.3.4, there is an adjunction $\mathrm{const}_I \dashv \lim_I$. Postcomposition induces an adjunction on functor categories, with unit, counit and triangle identities inherited from this adjunction, so Proposition 3.2.4 applies; see [Cno26, Lemma 21.1.6]. Under the equivalence
$$\mathrm{Fun}(A, \mathrm{Fun}(I, \mathcal{D})) \simeq \mathrm{Fun}(I, \mathrm{Fun}(A, \mathcal{D})),$$
its left adjoint is the constant-diagram functor for $\mathrm{Fun}(A, \mathcal{D})$. Its right adjoint therefore constructs limits, with
$$(\lim_I D)(a) \simeq \lim_{i \in I} D(i)(a)$$
for every object $a \in A$. For a cone $\lambda : \mathrm{const}_I(X) \to D$, compare it with this constructed limit cone. The comparison $X \to \lim_I D$ is an equivalence if and only if its component at every object $a \in A$ is an equivalence, by Axiom 2.5.4. By the universal properties in Definition 3.3.3, these conditions say precisely that λ, respectively each $\lambda(a)$, is a limit cone. The colimit assertion follows dually. □

Corollary 3.3.14. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Then $\mathrm{PSh}(\mathcal{C})$ admits all small limits and colimits, and a cone or cocone is universal if and only if its evaluation at every object of $\mathcal{C}$ is universal.

Proof. Apply Proposition 3.3.13 to $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$, using the existence of small limits and colimits in An from Axiom 2.6.1. □

3.4. **Kan extensions.** Kan extensions are adjoint objects for restriction of diagrams.

Definition 3.4.1. Let $A, B \in \mathrm{Cat}$ be small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a category, and let $u : A \to B$ be a functor. Restriction along u defines the functor
$$u^* : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}), \qquad E \mapsto E \circ u.$$
Let $D \in \mathrm{Fun}(A, \mathcal{C})$ be a diagram.

(1) A *left Kan extension* of D along u is a diagram $u_!D \in \mathrm{Fun}(B, \mathcal{C})$ equipped with a natural transformation $\eta : D \to u^* u_! D$ such that, for every diagram $E \in \mathrm{Fun}(B, \mathcal{C})$, the map
$$\mathrm{Hom}_{\mathrm{Fun}(B,\mathcal{C})}(u_!D, E) \to \mathrm{Hom}_{\mathrm{Fun}(A,\mathcal{C})}(D, u^*E), \qquad \alpha \mapsto u^*(\alpha) \circ \eta,$$
is an equivalence.

(2) A *right Kan extension* of D along u is a diagram $u_*D \in \mathrm{Fun}(B, \mathcal{C})$ equipped with a natural transformation $\varepsilon : u^* u_* D \to D$ such that, for every diagram $E \in \mathrm{Fun}(B, \mathcal{C})$, the map
$$\mathrm{Hom}_{\mathrm{Fun}(B,\mathcal{C})}(E, u_*D) \to \mathrm{Hom}_{\mathrm{Fun}(A,\mathcal{C})}(u^*E, D), \qquad \beta \mapsto \varepsilon \circ u^*(\beta),$$
is an equivalence.

These are the universal mapping properties of [Cno26, Definition 21.4.1].

Proposition 3.4.2. Let $A, B \in \mathrm{Cat}$ be small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a category, and let $u : A \to B$ be a functor. Then:

(1) A left Kan extension of a diagram $D \in \mathrm{Fun}(A, \mathcal{C})$ is a left adjoint object to D under u^*.
(2) A right Kan extension of a diagram $D \in \mathrm{Fun}(A, \mathcal{C})$ is a right adjoint object to D under u^*.

Consequently, every diagram in $\mathrm{Fun}(A, \mathcal{C})$ admits a left Kan extension along u if and only if u^* admits a left adjoint $u_!$. Dually, every such diagram admits a right Kan extension along u if and only if u^* admits a right adjoint u_*. When both exist, the adjunctions are $u_! \dashv u^* \dashv u_*$.

Proof. The maps in Definition 3.4.1 are exactly those in Definition 3.2.5 for the restriction functor u^*. The existence and functoriality assertions follow from Corollary 3.2.8. □

Example 3.4.3. Let $A \in \mathrm{Cat}$ be a small category and let $\mathcal{C} \in \mathrm{Cat}$ be a category. For the functor $u : A \to *$, restriction is the constant-diagram functor of Definition 3.3.3. Thus a left Kan extension of a diagram $D \in \mathrm{Fun}(A, \mathcal{C})$ along u is a colimit cocone, and a right Kan extension is a limit cone, by Proposition 3.3.4.

The pointwise construction of Kan extensions uses the following pullbacks.

Construction 3.4.4. Let $A, B \in \mathrm{Cat}$ be categories, let $u : A \to B$ be a functor, and let $b \in B$ be an object. Using the overcategory and undercategory of Definition 2.4.3, form the pullbacks $A \times_B B_{/b}$ and $A \times_B B_{b/}$. The pullback $A \times_B B_{/b}$ has objects $(a, u(a) \to b)$, with $a \in A$ an object. The pullback $A \times_B B_{b/}$ has objects $(a, b \to u(a))$. Their morphisms include the commutative triangles expressing compatibility with the specified arrows.

Then:

Theorem 3.4.5. Let $A, B \in \mathrm{Cat}$ be small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a category, let $u : A \to B$ be a functor, and let $D \in \mathrm{Fun}(A, \mathcal{C})$ be a diagram.

(1) Suppose that, for every object $b \in B$, the diagram $A \times_B B_{/b} \to A \xrightarrow{D} \mathcal{C}$ admits a colimit. These colimits assemble into a diagram $u_!D \in \mathrm{Fun}(B, \mathcal{C})$ with
$$(u_!D)(b) \simeq \mathrm{colim}_{(a, u(a) \to b) \in A \times_B B_{/b}} D(a).$$
The maps $D(a) \to (u_!D)(u(a))$ associated with $(a, \mathrm{id}_{u(a)})$ assemble into a natural transformation $\eta : D \to u^* u_! D$ exhibiting a left Kan extension.

(2) Suppose that, for every object $b \in B$, the diagram $A \times_B B_{b/} \to A \xrightarrow{D} \mathcal{C}$ admits a limit. These limits assemble into a diagram $u_*D \in \mathrm{Fun}(B, \mathcal{C})$ with
$$(u_*D)(b) \simeq \lim_{(a, b \to u(a)) \in A \times_B B_{b/}} D(a).$$
The projections $(u_*D)(u(a)) \to D(a)$ associated with $(a, \mathrm{id}_{u(a)})$ assemble into a natural transformation $\varepsilon : u^* u_* D \to D$ exhibiting a right Kan extension.

Proof. See [Cno26, Theorem 21.4.3] and [Lur09, Sections 4.3.2–4.3.3, especially Proposition 4.3.3.7]. □

We conclude with an example:

Example 3.4.6. Let $\mathcal{C} \in \mathrm{Cat}$ be a category admitting pushouts. Let $u : \ulcorner \hookrightarrow [1] \times [1]$ be the inclusion of the span indexing category from Section 2.3. A diagram $D \in \mathrm{Fun}(\ulcorner, \mathcal{C})$ is a span $x \leftarrow w \rightarrow y$ of objects $w, x, y \in \mathcal{C}$. At an object $a \in \ulcorner$, the indexing category in Theorem 3.4.5 has terminal object $(a, \mathrm{id}_{u(a)})$, so $(u_! D)(u(a)) \simeq D(a)$. At $(1,1)$ the indexing category is $\ulcorner$ itself, so left Kan extension fills the span by the pushout square

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \amalg_w y. \end{array}$$

Dually, for a category $\mathcal{C} \in \mathrm{Cat}$ admitting pullbacks, right Kan extension along the inclusion $\lrcorner \hookrightarrow [1]\times[1]$ of the cospan indexing category from Section 2.3 fills a cospan by a pullback square.

3.5. **Presheaves as free cocompletions.** The Yoneda lemma in Theorem 3.1.3 describes maps from a representable presheaf. The density theorem describes an arbitrary presheaf as a colimit of representables.

Proposition 3.5.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category and let $P \in \mathrm{PSh}(\mathcal{C})$ be a presheaf. The category

$$\mathcal{C}_{/P} = \mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/P},$$

where the first map is $\text{よ}_{\mathcal{C}}$, is small. Its objects are pairs (c, α) with $c \in \mathcal{C}$ an object and $\alpha : \text{よ}_{\mathcal{C}}(c) \to P$ a morphism. The tautological cocone induces an equivalence

$$\operatorname*{colim}_{(c,\alpha)\in\mathcal{C}_{/P}} \text{よ}_{\mathcal{C}}(c) \to P.$$

Proof. See [Lur09, Lemmas 5.1.5.2–5.1.5.3]. □

Then:

Theorem 3.5.2. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category and let $\mathcal{D} \in \widehat{\mathrm{Cat}}$ be a category admitting all small colimits, with smallness relative to Cat as in Notation 2.6.2. Restriction along Yoneda induces an equivalence

$$\text{よ}^*_{\mathcal{C}} : \mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^{\mathrm{L}}$ is the full subcategory of functors preserving all small colimits.

Proof. Let $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a functor. By Proposition 3.5.1, each category $\mathcal{C}_{/P}$ is small. The pointwise construction of Theorem 3.4.5, applied in the enlarged category, therefore gives a left Kan extension $\overline{F} : \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ with

$$\overline{F}(P) = \operatorname*{colim}_{(c,\alpha)\in\mathcal{C}_{/P}} F(c)$$

for every presheaf $P \in \mathrm{PSh}(\mathcal{C})$. Only small colimits in $\mathcal{D}$ are needed, even though $\mathrm{PSh}(\mathcal{C})$ is large.

We verify colimit preservation using mapping animae in the enlarged universe. Let $j : \mathrm{PSh}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})$ be induced by the inclusion $\mathrm{An} \hookrightarrow \widehat{\mathrm{An}}$. This functor preserves small colimits: these are computed pointwise by Proposition 3.3.13, and the inclusion of animae preserves them by the enlargement convention of Notation 2.6.2. For an object $d \in \mathcal{D}$, the functor $H_d = \mathrm{Hom}_{\mathcal{D}}(F(-), d) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})$ is a presheaf valued in the larger category of animae. The mapping criterion of Proposition 3.3.5, density and Theorem 3.1.3 in the enlarged universe give natural equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(\overline{F}(P), d) &\simeq \lim_{(c,\alpha)\in\mathcal{C}^{\mathrm{op}}_{/P}} \mathrm{Hom}_{\mathcal{D}}(F(c), d) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})}(jP, H_d). \end{aligned}$$

In the second equivalence, jP is the colimit of the representables $j\text{よ}_{\mathcal{C}}(c)$ indexed by $\mathcal{C}_{/P}$, since j preserves the density cocone. Let $I \in \mathrm{Cat}$ be a small category and let $D \in \mathrm{Fun}(I, \mathrm{PSh}(\mathcal{C}))$ be a diagram. Since j preserves its colimit, these equivalences imply

$$\mathrm{Hom}_{\mathcal{D}}\big(\overline{F}(\mathrm{colim}_I D), d\big) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(\overline{F}(D(i)), d).$$

This is the map induced by the image colimit cocone, so Proposition 3.3.5 shows that $\overline{F}$ preserves small colimits.

By Proposition 3.4.2, left Kan extension defines a left adjoint $(\text{よ}_{\mathcal{C}})_!$ to restriction, and the preceding argument shows that it takes values in colimit-preserving functors. For an object $c \in \mathcal{C}$, the pair (c, id) is terminal in $\mathcal{C}_{/\text{よ}_{\mathcal{C}}(c)}$ by full faithfulness in Theorem 3.1.3. Thus the unit $F \to \text{よ}^*_{\mathcal{C}}\overline{F}$ is an equivalence. For a colimit-preserving functor $G \in \mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D})$, the counit $(\text{よ}_{\mathcal{C}})_!\text{よ}^*_{\mathcal{C}}G \to G$ is an equivalence on representables. Both functors preserve small colimits, so Proposition 3.5.1 makes the counit an equivalence on every presheaf. The unit and counit exhibit the asserted equivalence of functor categories. □

For $\mathcal{C} = *$, this gives $\mathrm{PSh}(*) \simeq \mathrm{An}$ and identifies colimit-preserving functors out of animae with their value at the point.

3.6. **Localizations, groupoid completion and reflective subcategories.** Localization makes a specified collection of morphisms invertible.

Definition 3.6.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category and let W be a small collection of its morphisms. A *localization* of $\mathcal{C}$ at W consists of a category $\mathcal{C}[W^{-1}] \in \mathrm{Cat}$ and a functor $q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which sends the morphisms in W to equivalences and such that, for every category $\mathcal{D} \in \mathrm{Cat}$, restriction induces an equivalence

$$q^* : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}_W(\mathcal{C}, \mathcal{D}).$$

Here the right side is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of functors that send W to equivalences.

We assume the existence of localizations:

Axiom 3.6.2. For every small category $\mathcal{C} \in \mathrm{Cat}$ and every small collection W of morphisms in $\mathcal{C}$, there exists a localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ in the sense of Definition 3.6.1.

Remark 3.6.3. For categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, a functor $F : \mathcal{C} \to \mathcal{D}$ sending a collection W of morphisms in $\mathcal{C}$ into a collection W' of morphisms in $\mathcal{D}$ induces a functor $\mathcal{C}[W^{-1}] \to \mathcal{D}[(W')^{-1}]$ by Definition 3.6.1.

There is an explicit way of constructing localizations when the given functor has a fully faithful right adjoint.

Definition 3.6.4. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories. A full subcategory $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is *reflective* if its inclusion admits a left adjoint $L : \mathcal{C} \to \mathcal{D}$. We call L a *reflective localization* and call the objects in the essential image of i *local objects*. For a collection S of morphisms of $\mathcal{C}$, an object $z \in \mathcal{C}$ is *S-local* if every morphism $s : x \to y$ in S induces an equivalence $\mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$. We write $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ for the full subcategory of S-local objects.

Remark 3.6.5. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $i : \mathcal{D} \hookrightarrow \mathcal{C}$ be a reflective subcategory as in Definition 3.6.4, with left adjoint L. Write $\eta : \mathrm{id}_{\mathcal{C}} \to iL$ and $\varepsilon : Li \to \mathrm{id}_{\mathcal{D}}$ for the unit and counit of Proposition 3.2.4. The counit $Li \to \mathrm{id}_{\mathcal{D}}$ is an equivalence, and this property characterizes the full faithfulness of the right adjoint; see [CCNW26, Proposition 6.1.7]. For objects $d \in \mathcal{D}$ and $x \in \mathcal{C}$, the unit $\eta_x : x \to iLx$ induces, by full faithfulness of i and Proposition 3.2.4, equivalences

$$\mathrm{Hom}_{\mathcal{C}}(iLx, i(d)) \simeq \mathrm{Hom}_{\mathcal{D}}(Lx, d) \simeq \mathrm{Hom}_{\mathcal{C}}(x, i(d)).$$

Thus η_x is the universal morphism from x to a local object. An object $x \in \mathcal{C}$ is local in the sense of Definition 3.6.4 if and only if η_x is an equivalence. Indeed, if $x \simeq i(d)$ for an object $d \in \mathcal{D}$, the triangle identity $i(\varepsilon_d) \circ \eta_{i(d)} \simeq \mathrm{id}_{i(d)}$ from Proposition 3.2.4 makes η_x an equivalence. Conversely, an equivalence $\eta_x : x \to iLx$ exhibits x as an object in the essential image of i.

The following proposition justifies the term reflective localization.

Proposition 3.6.6. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be categories and let $L : \mathcal{C} \rightleftarrows \mathcal{D} : i$ be an adjunction with fully faithful right adjoint. Let W be the collection of morphisms f of $\mathcal{C}$ for which $L(f)$ is an equivalence. Then L is a localization of $\mathcal{C}$ at W in the sense of Definition 3.6.1.

Proof. Fix a category $\mathcal{E} \in \mathrm{Cat}$. Restriction along L defines a functor

$$L^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_W(\mathcal{C}, \mathcal{E}).$$

The functor L^* takes values in the indicated full subcategory by the definition of W. An inverse is given by restriction $i^* : \mathrm{Fun}_W(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})$. For a functor $G \in \mathrm{Fun}(\mathcal{D}, \mathcal{E})$, the counit gives $GLi \simeq G$. For a functor $F \in \mathrm{Fun}_W(\mathcal{C}, \mathcal{E})$, the unit gives $F \to FiL$. The triangle identity $\varepsilon_{Lx} \circ L(\eta_x) \simeq \mathrm{id}_{Lx}$ from Proposition 3.2.4 and the invertibility of the counit imply that $L(\eta_x)$ is an equivalence for every object $x \in \mathcal{C}$. Thus η_x lies in W, so that $F(\eta_x)$ is an equivalence. By Axiom 2.5.4, these unit and counit transformations are natural equivalences. They exhibit L^* and i^* as inverse equivalences of functor categories, verifying Definition 3.6.1. □

Definition 3.6.7. The *geometric realization* $\|\mathcal{C}\| \in \mathrm{Cat}$ of a small category $\mathcal{C} \in \mathrm{Cat}$ is its localization at all morphisms in the sense of Definition 3.6.1. It is also called the *groupoid completion* of $\mathcal{C}$.

The localization exists by Axiom 3.6.2. The following result identifies its target as an anima.

Proposition 3.6.8. For every small category $\mathcal{C} \in \mathrm{Cat}$, its groupoid completion from Definition 3.6.7 is an anima $\|\mathcal{C}\| \in \mathrm{An}$.

Proof. See [CCNW26, Propositions 3.4.4–3.4.5]. □

Corollary 3.6.9. Let $i : \mathrm{An} \hookrightarrow \mathrm{Cat}$ be the inclusion of animae into categories, viewed in the enlarged category of Notation 2.6.2. There are adjunctions

$$\|-\| \dashv i \dashv (-)^{\simeq},$$

where $\|-\| : \mathrm{Cat} \to \mathrm{An}$ is groupoid completion and $(-)^{\simeq} : \mathrm{Cat} \to \mathrm{An}$ is the groupoid core. In particular, An is a reflective subcategory of Cat, and groupoid completion is a localization at the collection of functors $F : \mathcal{C} \to \mathcal{D}$ between small categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ for which $\|F\|$ is an equivalence. The local objects are precisely the animae.

Proof. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category and let $X \in \mathrm{An}$ be an anima. Every functor $\mathcal{C} \to X$ inverts all morphisms, so the localization universal property gives a natural equivalence

$$\mathrm{Fun}(\|\mathcal{C}\|, X) \simeq \mathrm{Fun}(\mathcal{C}, X).$$

Both functor categories are animae, since every natural transformation with values in an anima is invertible. Taking their cores therefore identifies this equivalence with $\mathrm{Hom}_{\mathrm{An}}(\|\mathcal{C}\|, X) \simeq \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, X)$. The objectwise criterion of Corollary 3.2.8 therefore gives $\|-\| \dashv i$. The adjunction $i \dashv (-)^{\simeq}$ follows from Proposition 2.6.6. Since i is fully faithful, An is a reflective subcategory of Cat with the stated local objects. Applying Proposition 3.6.6 in the enlarged category proves the localization assertion. □

This adjunction has the opposite orientation from the adjunction defining the groupoid core in Proposition 2.6.6. By Axiom 2.3.8, the core retains only the equivalences already present in $\mathcal{C}$, while groupoid completion makes all its morphisms invertible.

Example 3.6.10. The core $[1]^{\simeq}$ is the discrete anima with two points, whereas $\|[1]\|$ is contractible. Indeed, functors from $[1]$ to an anima $X \in \mathrm{An}$ form its path anima, which is equivalent to X by evaluation at either endpoint. By Definition 3.6.1 and Proposition 3.6.8, $\|[1]\|$ therefore has the same mapping property as the terminal anima, so Theorem 3.1.3 identifies it with $*$.

4. Presentability

4.1. **Completion under certain colimits.** Colimit completion freely adjoins specified colimits and can require given cocones to become colimit cocones. We now introduce a useful variant.

Notation 4.1.1. Let $\mathcal{K}$ be a collection of small categories. Colimits indexed by members of $\mathcal{K}$ are called $\mathcal{K}$-*colimits*. We write $\mathrm{Cat}^{\mathcal{K}\text{-colim}}$ for the category of small categories admitting $\mathcal{K}$-colimits and functors preserving them, and $\widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ for the corresponding category of large categories in the sense of Notation 2.6.2. In both cases, the indexing categories in $\mathcal{K}$ remain small relative to Cat. For categories $\mathcal{D}, \mathcal{E} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, we write $\mathrm{Fun}^{\mathcal{K}}(\mathcal{D}, \mathcal{E})$ for the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ spanned by functors preserving $\mathcal{K}$-colimits. For a small category $\mathcal{C} \in \mathrm{Cat}$, let $\mathcal{R}$ be a small family of cocones $(c_i)_{i\in I} \to c$ in $\mathcal{C}$, with $I \in \mathcal{K}$ a small category, $c_\bullet \in \mathrm{Fun}(I, \mathcal{C})$ a diagram, and $c \in \mathcal{C}$ an object. These cocones need not be colimit cocones in $\mathcal{C}$. For a category $\mathcal{D} \in \widehat{\mathrm{Cat}}$, write $\mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of functors carrying every cocone in $\mathcal{R}$ to a colimit cocone.

The construction uses the following presheaf-localization theorem.

Lemma 4.1.2. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category and let S be a small family of morphisms in $\mathrm{PSh}(\mathcal{C})$. The full subcategory $S^{-1}\mathrm{PSh}(\mathcal{C}) \in \widehat{\mathrm{Cat}}$ of S-local presheaves is reflective, with reflector $L : \mathrm{PSh}(\mathcal{C}) \to S^{-1}\mathrm{PSh}(\mathcal{C})$. Its small colimits are obtained by applying L to colimits in $\mathrm{PSh}(\mathcal{C})$. Moreover, for a presheaf $P \in \mathrm{PSh}(\mathcal{C})$ and an S-local presheaf $Q \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})$, the unit induces an equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})}(LP, Q) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})}(P, Q).$$

Proof. Existence of the reflector is the presheaf case of [Lur09, Proposition 5.5.4.15], which we use without proof. The colimit formula follows from the adjunction and the colimit mapping property. For the last assertion, the functor $\mathrm{Hom}(-, Q) : \mathrm{PSh}(\mathcal{C}) \to \widehat{\mathrm{An}}^{\mathrm{op}}$ preserves small colimits and inverts S. By the localization universal property for colimit-preserving functors, [Lur09, Proposition 5.5.4.20], it factors through L and therefore inverts each unit $P \to LP$. Mapping animae are taken in the indicated enlarged presheaf category; pointwise colimits agree with those in $\mathrm{PSh}(\mathcal{C})$. □

Construction 4.1.3. Let $\mathcal{K}$ be a collection of small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a small category, and let $\mathcal{R}$ be a small family of cocones in $\mathcal{C}$ indexed by members of $\mathcal{K}$. For each cocone $(c_i)_{i\in I} \to c$ in $\mathcal{R}$, form the comparison morphism

$$\mathrm{colim}_{i\in I} \yoneda_{\mathcal{C}}(c_i) \to \yoneda_{\mathcal{C}}(c)$$

in $\mathrm{PSh}(\mathcal{C})$, and let S be the family of these morphisms. Let $L : \mathrm{PSh}(\mathcal{C}) \to S^{-1}\mathrm{PSh}(\mathcal{C})$ be the reflector of Lemma 4.1.2. Define $\mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C}) \in \widehat{\mathrm{Cat}}$ to be the smallest full subcategory of $S^{-1}\mathrm{PSh}(\mathcal{C})$ containing the objects $L\yoneda_{\mathcal{C}}(c)$ for objects $c \in \mathcal{C}$ and closed under $\mathcal{K}$-colimits. Its canonical functor is $j : \mathcal{C} \to \mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C})$, induced by $L\yoneda_{\mathcal{C}}$. Since L preserves colimits and inverts S, the functor j carries the cocones in $\mathcal{R}$ to colimit cocones. When $\mathcal{R} = \varnothing$, the reflector is the identity and we write $\mathrm{PSh}_{\mathcal{K}}(\mathcal{C}) = \mathrm{PSh}_{\mathcal{K},\varnothing}(\mathcal{C})$. Thus $\mathrm{PSh}_{\mathcal{K}}(\mathcal{C})$ is the full subcategory of presheaves generated by the representables under $\mathcal{K}$-colimits, and j is the factorization of the Yoneda embedding.

Theorem 4.1.4 ([Lur09, Proposition 5.3.6.2])**.** Let $\mathcal{K}$ be a collection of small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a small category, and let $\mathcal{R}$ be a small family of cocones in $\mathcal{C}$ indexed by members of $\mathcal{K}$. For every category $\mathcal{D} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ admitting $\mathcal{K}$-colimits, restriction along the canonical functor j is an equivalence

$$j^* : \mathrm{Fun}^{\mathcal{K}}(\mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D}).$$

Its inverse is given by left Kan extension along j. If every cocone in $\mathcal{R}$ is already a colimit cocone in $\mathcal{C}$, then j is fully faithful. In particular, for $\mathcal{R} = \varnothing$, restriction along Yoneda is an equivalence

$$j^* : \mathrm{Fun}^{\mathcal{K}}(\mathrm{PSh}_{\mathcal{K}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Fix a functor $F \in \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$. For an object $d \in \mathcal{D}$, the presheaf $\mathrm{Hom}_{\mathcal{D}}(F(-), d) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})$ is S-local: for each prescribed cocone $(c_i)_{i \in I} \to c$, the colimit mapping property gives

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(F(c_i), d).$$

For a local presheaf $P \in S^{-1}\mathrm{PSh}(\mathcal{C})$, seek an object $\overline{F}(P) \in \mathcal{D}$ characterized by natural equivalences

$$\mathrm{Hom}_{\mathcal{D}}(\overline{F}(P), d) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})}\big(P, \mathrm{Hom}_{\mathcal{D}}(F(-), d)\big), \qquad d \in \mathcal{D}.$$

By Lemma 3.2.6, these objects form a functor wherever they exist. For $P = j(c) = L\text{ɣ}_{\mathcal{C}}(c)$ with $c \in \mathcal{C}$ an object, Lemma 4.1.2 and Yoneda identify the right-hand side with $\mathrm{Hom}_{\mathcal{D}}(F(c), d)$, so we may take $\overline{F}(j(c)) = F(c)$.

Let $I \in \mathcal{K}$ be a small category and let $P_\bullet \in \mathrm{Fun}(I, S^{-1}\mathrm{PSh}(\mathcal{C}))$ be a diagram for which these representing objects exist. Its colimit in local presheaves is $L(\mathrm{colim}_{i \in I} P_i)$, where the inner colimit is taken in presheaves. Locality and the colimit mapping property give

$$\begin{aligned}\mathrm{Hom}\big(L(\mathrm{colim}_{i \in I} P_i), \mathrm{Hom}_{\mathcal{D}}(F(-), d)\big) &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(\overline{F}(P_i), d) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathrm{colim}_{i \in I} \overline{F}(P_i), d\big).\end{aligned}$$

The first mapping anima is taken in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{An}})$. Hence the local presheaves on which $\overline{F}$ exists are closed under $\mathcal{K}$-colimits, and $\overline{F}$ preserves them. They therefore contain $\mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C})$.

For an object $P \in \mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C})$, the adjunction defining L identifies maps $j(c) \to P$ with maps $\text{ɣ}_{\mathcal{C}}(c) \to P$. Thus the category of objects of $\mathcal{C}$ mapping to P along j is its category of elements $\mathcal{C}_{/P} \in \mathrm{Cat}$ as a presheaf. Presheaf density, Proposition 3.5.1, identifies the defining mapping property of $\overline{F}(P)$ with that of $\mathrm{colim}_{(c,\alpha) \in \mathcal{C}_{/P}} F(c)$. This colimit consequently exists even when its indexing category does not belong to $\mathcal{K}$, and Theorem 3.4.5 identifies $\overline{F}$ with $j_! F$. By Proposition 3.4.2, this gives an adjunction between left Kan extension and restriction on the indicated full subcategories of functor categories.

The unit $F \to j^* j_! F$ is an equivalence by the computation on the objects $j(c)$. For a functor $G \in \mathrm{Fun}^{\mathcal{K}}(\mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C}), \mathcal{D})$, its restriction respects $\mathcal{R}$, and the counit $j_! j^* G \to G$ is an equivalence on these generators. Both functors preserve $\mathcal{K}$-colimits, so the full subcategory on which the counit is an equivalence is closed under them. By generation, the counit is an equivalence everywhere, proving the asserted equivalence of functor categories.

Finally, if all prescribed cocones are colimit cocones in $\mathcal{C}$, each representable $\text{ɣ}_{\mathcal{C}}(c)$ is S-local by the colimit mapping property in $\mathcal{C}$. The reflector fixes these presheaves, so full faithfulness of j follows from Yoneda. Taking $\mathcal{R} = \varnothing$ gives the stated free-completion universal property. □

Remark 4.1.5. Let $\mathcal{K}$ be a small collection of small categories, let $\mathcal{C} \in \mathrm{Cat}$ be a small category, and let $\mathcal{R}$ be a small family of cocones in $\mathcal{C}$ indexed by members of $\mathcal{K}$. The completion $\mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C})$ is small, as the closure of a small family of objects in a locally small category under a small collection of small colimits. In particular, $\mathrm{PSh}_{\mathcal{K}}(\mathcal{C})$ is an object of $\mathrm{Cat}^{\mathcal{K}\text{-colim}}$.

The free completion defines an adjunction when $\mathcal{K}$ is small:

Corollary 4.1.6. Let $\mathcal{K}$ be a small collection of small categories. The free completion functor $\mathrm{PSh}_{\mathcal{K}} : \mathrm{Cat} \to \mathrm{Cat}^{\mathcal{K}\text{-colim}}$ is left adjoint to the forgetful $\mathrm{Cat}^{\mathcal{K}\text{-colim}} \to \mathrm{Cat}$.

Proof. Apply the case $\mathcal{R} = \varnothing$ of Theorem 4.1.4, take cores, and use Remark 4.1.5. □

Remark 4.1.7. Let $\mathcal{K}$ be a small collection of small categories. The inclusion $\mathrm{Cat}^{\mathcal{K}\text{-colim}} \to \mathrm{Cat}$ generally restricts morphisms and is not full, so the adjunction of Corollary 4.1.6 is not generally a reflective localization.

There are some easy examples we can compute:

Example 4.1.8. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Then:

(1) For $\mathcal{K} = \varnothing$, the completion is the essential image of Yoneda and hence is equivalent to $\mathcal{C}$.
(2) If $\mathcal{K}$ consists of all small categories, Proposition 3.5.1 gives $\mathrm{PSh}_{\mathcal{K}}(\mathcal{C}) = \mathrm{PSh}(\mathcal{C})$.

With $\mathcal{K}$ equal to all small categories and $\mathcal{R} = \varnothing$, Theorem 4.1.4 recovers the free-cocompletion universal property.

4.2. **Ind-completions.** Recall that a infinite cardinal κ is *regular* if a union of fewer than κ sets of cardinality less than κ again has cardinality less than κ.

Definition 4.2.1. Let κ be an infinite regular cardinal.

(1) A set is *κ-small* if its cardinality is less than κ.
(2) For a category $\mathcal{A} \in \mathrm{Cat}$, admitting *$\kappa$-small limits* means admitting pullbacks and products indexed by κ-small sets, including the empty set. A functor between categories admitting these limits *preserves κ-small limits* if it preserves these products and pullbacks.
(3) Dually, admitting or preserving *κ-small colimits* means admitting or preserving pushouts and coproducts indexed by κ-small sets.

Finally, a small category $I \in \mathrm{Cat}$ is *κ-filtered* if the colimit functor

$$\mathrm{colim}_I : \mathrm{Fun}(I, \mathrm{An}) \to \mathrm{An}$$

preserves κ-small limits. We say *filtered* when $\kappa = \omega$.

The limits in $\mathrm{Fun}(I, \mathrm{An})$ are computed pointwise by Proposition 3.3.13, so the defining condition says that I-indexed colimits commute with κ-small limits in animae. Preservation of the terminal object gives $\mathrm{colim}_I * \simeq *$, and in particular I is nonempty.

Example 4.2.2. Let κ be an infinite regular cardinal. A small poset $I \in \mathrm{Cat}$, regarded as an ordinary category, is κ-filtered if and only if every subset of cardinality less than κ has an upper bound; see [Ker, Proposition 9.1.5.10] and [Lur09, Section 5.3.1]. For $\kappa = \omega$, this is the usual condition that I is a nonempty directed poset.

Ind-completion is completion under filtered colimits in the sense of Construction 4.1.3.

Definition 4.2.3. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Let $\mathcal{K}$ be the collection of all small κ-filtered categories in Cat. Its *κ-Ind-completion* is

$$\mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{PSh}_{\mathcal{K}}(\mathcal{C}).$$

We put $\mathrm{Ind}(\mathcal{C}) = \mathrm{Ind}_\omega(\mathcal{C})$. The Yoneda embedding factors through this subcategory.

Then:

Theorem 4.2.4. Let κ be an infinite regular cardinal, let $\mathcal{C} \in \mathrm{Cat}$ be small, and let $\mathcal{D} \in \widehat{\mathrm{Cat}}$ be a category admitting small κ-filtered colimits. Restriction along Yoneda induces an equivalence

$$\text{よ}^*_{\mathcal{C}} : \mathrm{Fun}^{\kappa\text{-filt}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where the left side consists of functors preserving small κ-filtered colimits.

Proof. Apply the case $\mathcal{R} = \varnothing$ of Theorem 4.1.4 to the collection of all small κ-filtered categories. □

Our definition of κ-Ind-completion differs from the usual one (that is, the full subcategory of the presheaf category spanned by κ-filtered colimits or representable). Notice that, if $\mathcal{K}$ be the collection of all small κ-filtered categories in Cat, then

$$\{\kappa\text{-filtered colimtis of representables}\} \subseteq \mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{PSh}_{\mathcal{K}}(\mathcal{C})$$

contains already the κ-filtered colimtis of representables, since it contains the representables and is closed under small κ-filtered colimits. To prove the other inclusion we use the:

Proposition 4.2.5. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Every object $P \in \mathrm{Ind}_\kappa(\mathcal{C})$ admits a presentation

$$P \simeq \mathrm{colim}_{i \in I} \text{よ}_{\mathcal{C}}(c_i)$$

for a small κ-filtered category $I \in \mathrm{Cat}$ and a diagram $c_\bullet \in \mathrm{Fun}(I, \mathcal{C})$.

Proof. We give an idea on how to prove this result. The natural approach uses the category of elements. By Proposition 3.5.1, we have

$$P \simeq \mathrm{colim}_{(c,\alpha) \in \mathcal{C}_{/P}} \text{よ}(c).$$

Thus it suffices to show that $\mathcal{C}_{/P} := \mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/P}$ is κ-filtered whenever $P \in \mathrm{Ind}_\kappa(\mathcal{C})$. If $P = \text{よ}(c)$ is representable, then this follows since $\mathcal{C}_{/P} \simeq \mathcal{C}_{/c}$ has a terminal object. For the general case, it suffices to show that presheaves whose categories of elements are κ-filtered are closed under κ-filtered colimits, see [Lur09, Proposition 5.3.5.3]. □

Remark 4.2.6. For an infinite regular cardinal κ, a small category $\mathcal{C} \in \mathrm{Cat}$, small κ-filtered categories $I, J \in \mathrm{Cat}$ and diagrams of objects $a_i, b_j \in \mathcal{C}$, the mapping formula is

$$\mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}\left(\mathrm{colim}_{i \in I} \text{よ}(a_i), \mathrm{colim}_{j \in J} \text{よ}(b_j)\right) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{colim}_{j \in J} \mathrm{Hom}_{\mathcal{C}}(a_i, b_j).$$

By Proposition 4.2.5, every pair of objects in $\mathrm{Ind}_\kappa(\mathcal{C})$ admits such presentations, so this formula computes all mapping animae in the Ind-completion. The colimit mapping property of Proposition 3.3.5 gives the limit in the first variable. In the second variable, Theorem 3.1.3 identifies maps from a representable with evaluation, which preserves presheaf colimits by Proposition 3.3.13.

Definition 4.2.7. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \widehat{\mathrm{Cat}}$ be a large category admitting small κ-filtered colimits. An object $x \in \mathcal{C}$ is *κ-compact* if $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \widehat{\mathrm{An}}$ preserves these colimits. We write $\mathcal{C}^\kappa \subseteq \mathcal{C}$ for the full subcategory of such objects, and say *compact* when $\kappa = \omega$.

The definition shows the following closure property:

Lemma 4.2.8. Let κ be an infinite regular cardinal and let $\mathcal{D} \in \widehat{\mathrm{Cat}}$ be a category with small mapping animae admitting small κ-filtered colimits and κ-small colimits. The full subcategory of κ-compact objects is closed under κ-small colimits in $\mathcal{D}$. In particular, compact objects are closed under finite colimits when $\kappa = \omega$.

Proof. Let $J \in \mathrm{Cat}$ be the category indexing a span or a discrete category with fewer than κ objects, and let $x_\bullet \in \mathrm{Fun}(J, \mathcal{D})$ be a diagram of κ-compact objects. Put $x = \mathrm{colim}_{j \in J} x_j \in \mathcal{D}$. For a small κ-filtered category $I \in \mathrm{Cat}$ and a diagram $F \in \mathrm{Fun}(I, \mathcal{D})$, we have

$$\begin{aligned}
\mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathcal{D}}(x, F(i)) &\simeq \mathrm{colim}_{i \in I} \lim_{j \in J^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(x_j, F(i)) \\
&\simeq \lim_{j \in J^{\mathrm{op}}} \mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathcal{D}}(x_j, F(i)) \\
&\simeq \lim_{j \in J^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}\left(x_j, \mathrm{colim}_{i \in I} F(i)\right) \\
&\simeq \mathrm{Hom}_{\mathcal{D}}\left(x, \mathrm{colim}_{i \in I} F(i)\right).
\end{aligned}$$

The first and last equivalences are the colimit mapping property, and the third uses compactness of each $x_j \in \mathcal{D}$. For the second, Definition 4.2.1 says precisely that $\mathrm{colim}_I : \mathrm{Fun}(I, \mathrm{An}) \to \mathrm{An}$ preserves κ-small limits. □

Remark 4.2.9. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Representables are κ-compact in $\mathrm{Ind}_\kappa(\mathcal{C})$: by Theorem 3.1.3, maps from $\text{ℶ}_{\mathcal{C}}(c)$ for an object $c \in \mathcal{C}$ are computed by evaluation at c, and the relevant colimits are computed pointwise by Construction 4.1.3 and Proposition 3.3.13; compare [Lur09, Proposition 5.3.5.5].

More generally:

Remark 4.2.10. Let κ be an infinite regular cardinal, $\mathcal{C} \in \mathrm{Cat}$ be a small category, $\mathcal{D} \in \widehat{\mathrm{Cat}}$ be a large category admitting small κ-filtered colimits and let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful with κ-compact image. Its extension $\overline{F} : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ is fully faithful: the colimit mapping property and compactness give the same mapping formula in $\mathcal{D}$ as in Remark 4.2.6.

4.3. **Presentable categories.**

Definition 4.3.1. Let κ be an infinite regular cardinal. We say that a large category $\mathcal{C} \in \widehat{\mathrm{Cat}}$ is:

(1) κ-*accessible* if it admits small κ-filtered colimits and contains a small full subcategory $\mathcal{A} \in \mathrm{Cat}$ consisting of κ-compact objects such that every object of $\mathcal{C}$ is a small κ-filtered colimit of objects of $\mathcal{A}$.
(2) It is *accessible* if this holds for some κ.
(3) It is *presentable* if it is accessible and admits all small colimits.

A functor between accessible categories is *accessible* if it preserves small κ-filtered colimits for some sufficiently large infinite regular cardinal κ.

Definition 4.3.2. Write Pr^{L} for the category of presentable categories and functors preserving all small colimits, regarded as an object of a further enlarged category. Let κ be an infinite regular cardinal. A presentable category $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$ is κ-*compactly generated* if it is κ-accessible. The subcategory $\mathrm{Pr}^{\kappa} \subseteq \mathrm{Pr}^{\mathrm{L}}$ has these categories as objects and functors preserving all small colimits and κ-compact objects as morphisms. The inclusion $\mathrm{Pr}^{\kappa} \to \mathrm{Pr}^{\mathrm{L}}$ is generally not full.

An accessible category is the Ind-completion of a small full subcategory of compact generators.

Lemma 4.3.3. Let κ be an infinite regular cardinal, let $\mathcal{C} \in \widehat{\mathrm{Cat}}$ be a κ-accessible category, and let $\mathcal{A} \in \mathrm{Cat}$ be a small full subcategory of κ-compact objects generating $\mathcal{C}$ under small κ-filtered colimits. The inclusion extends to an equivalence $\mathrm{Ind}_\kappa(\mathcal{A}) \simeq \mathcal{C}$.

Proof. The extended inclusion is fully faithful by Remark 4.2.10, and the generation assumption makes it essentially surjective. □

We also have the following.

Lemma 4.3.4. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \widehat{\mathrm{Cat}}$ be a κ-accessible category. Then $\mathcal{C}$ is *locally small*: for every pair of objects $x, y \in \mathcal{C}$, its mapping anima $\mathrm{Hom}_{\mathcal{C}}(x, y) \in \widehat{\mathrm{An}}$ belongs to An.

Proof. Choose a small full subcategory $\mathcal{A} \in \mathrm{Cat}$ of κ-compact objects generating $\mathcal{C}$ under small κ-filtered colimits. By Lemma 4.3.3, we have $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{A})$. Since $\mathcal{A}$ is small, the mapping formula of Remark 4.2.6 and closure of An under small limits and colimits imply that $\mathcal{C}$ is locally small. □

Warning 4.3.5. The category Pr^{L} is not locally small. For a presentable category $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}$, the equivalence $\mathrm{An} \simeq \mathrm{PSh}(*)$ and Theorem 3.5.2 give

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}}(\mathrm{An}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{An}, \mathcal{C})^{\simeq} \simeq \mathcal{C}^{\simeq}.$$

Taking $\mathcal{C} = \mathrm{An}$, this mapping anima is not small: its connected components include distinct equivalence classes of discrete animae for every small cardinality. In contrast, Pr^{κ} is locally small for every infinite regular cardinal κ; see [Sch26, Remark 2.8] and Remark 4.5.5.

The following results give basic examples of presentable categories, the existence of their limits, and reflective localizations.

Theorem 4.3.6 ([Lur09, Theorem 5.5.1.1 and Corollary 5.5.2.4])**.** One has the following.

(1) Let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be a small category admitting κ-small colimits. Then $\mathrm{Ind}_{\kappa}(\mathcal{C})$ is presentable.
(2) Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Then $\mathrm{PSh}(\mathcal{C})$ is presentable.
(3) Let $\mathcal{D} \in \widehat{\mathrm{Cat}}$ be a presentable category. Then $\mathcal{D}$ admits all small limits.
(4) Let $\mathcal{A} \in \mathrm{Pr}^{\mathrm{L}}$ be a presentable category and let S be a small set of its morphisms. The full subcategory $S^{-1}\mathcal{A} \subseteq \mathcal{A}$ of S-local objects, in the sense of Definition 3.6.4, is presentable, and its inclusion admits an accessible left adjoint $L : \mathcal{A} \to S^{-1}\mathcal{A}$.

In particular, $\mathrm{An} \simeq \mathrm{PSh}(*)$ is presentable.

Proof. For (1), let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be a small category admitting κ-small colimits. The case $\mathcal{R} = \varnothing$ of Theorem 4.1.4, applied to small κ-filtered categories, gives the free completion $\mathrm{Ind}_{\kappa}(\mathcal{C})$ of Definition 4.2.3, which admits these colimits. The representables form a small full subcategory of κ-compact objects by Remark 4.2.9, and every object is a small κ-filtered colimit of representables by Proposition 4.2.5. Thus $\mathrm{Ind}_{\kappa}(\mathcal{C})$ is κ-accessible by Definition 4.3.1. To show that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ is presentable it remains to show that it has all colimits. For this use [Lur09, Theorem 5.5.1.1, implication (4) $\Rightarrow$ (5)]: it asserts that the inclusion $\mathrm{Ind}_{\kappa}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ admits a left adjoint L. Consequently, for a small category $I \in \mathrm{Cat}$ and a diagram $D : I \to \mathrm{Ind}_{\kappa}(\mathcal{C})$, its colimit is obtained by first computing it into presheaves, and then applying L.

For (2), let $\mathcal{C} \in \mathrm{Cat}$ be a small category. The case $\mathcal{R} = \varnothing$ of Theorem 4.1.4, with $\mathcal{K}$ the collection of all small categories, identifies $\mathrm{PSh}(\mathcal{C})$ as the free completion under small colimits, so it remains to prove accessibility. Since the representables form a small collection of compact objects (since the Yoneda lemma Theorem 3.1.3 identifies maps from a representable with evaluation, which preserves colimits) and since every presheaf is a small colimit of representables by density, Proposition 3.5.1, the claim follows if every presheaf can be built via a κ-filtered colimit of representables for some infinite regular cardinal κ. For this, it suffices to consider the full subcategory of $\mathrm{PSh}(\mathcal{C})$ generated by representables under finite colimits (this is small by Remark 4.1.5 and consists of compact objects by Lemma 4.2.8) since every small colimit can be buld from filtered colimits and finite colimits. In particular, chose κ to be ω.

For (3), let $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$ be a presentable category and choose an infinite regular cardinal κ for which it is κ-accessible. Choose a small full subcategory $\mathcal{A} \in \mathrm{Cat}$ of κ-compact objects generating $\mathcal{D}$ under small κ-filtered colimits.

The restricted Yoneda functor

$$R : \mathcal{D} \to \mathrm{PSh}(\mathcal{A}), \qquad R(d)(a) = \mathrm{Hom}_{\mathcal{D}}(a, d)$$

is fully faithful by the Ind-completion description of accessible categories and Remark 4.2.6. By Theorem 3.5.2, the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$ extends to a colimit-preserving functor $L : \mathrm{PSh}(\mathcal{A}) \to \mathcal{D}$. For a presheaf $P \in \mathrm{PSh}(\mathcal{A})$, choose a small category $I \in \mathrm{Cat}$ and a diagram $a \in \mathrm{Fun}(I, \mathcal{A})$ with $P \simeq \mathrm{colim}_{i \in I} \yoneda_{\mathcal{A}}(a_i)$. For every object $d \in \mathcal{D}$, the colimit mapping property and Yoneda give natural equivalences

$$\mathrm{Hom}_{\mathcal{D}}(LP, d) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(a_i, d) \simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{A})}(P, Rd).$$

Thus $L \dashv R$, and $\mathcal{D}$ is a reflective full subcategory of $\mathrm{PSh}(\mathcal{A})$. Local objects are characterized by the condition that mapping into them inverts every unit morphism $P \to RLP$. This condition is preserved

under small limits, since mapping into a limit is computed as a limit of mapping animae. Hence the essential image of R is closed under small limits in $\mathrm{PSh}(\mathcal{A})$, and $\mathcal{D}$ admits all small limits.

Finally assertion (4) is the presentable localization theorem, [Lur09, Proposition 5.5.4.15], which we use without proof. □

One of the magic of presentable categories is that colimit preserving functors are the same of left adjoints, giving a converse to Proposition 3.3.12.

Theorem 4.3.7 ([Lur09, Corollary 5.5.2.9])**.** Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ be presentable categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if it preserves small colimits. It admits a left adjoint if and only if it is accessible and preserves small limits.

4.4. **Idempotent completion.** Idempotent completion is completion under a particular class of colimits.

Notation 4.4.1. Let $\mathrm{Idem}^+ \in \mathrm{Cat}$ be the following ordinary category

$$y \underset{}{\overset{i}{\rightleftarrows}} x \overset{p}{\rightarrow} y.$$

Write $\mathrm{Idem} \in \mathrm{Cat}$ for its full subcategory on x, whose endomorphisms are 1 and $e = ip$, with $e^2 = e$. For a category $\mathcal{C} \in \mathrm{Cat}$, an *idempotent* is a functor $E : \mathrm{Idem} \to \mathcal{C}$. A *splitting* of E is an extension $\widetilde{E} : \mathrm{Idem}^+ \to \mathcal{C}$ together with an identification $\widetilde{E}|_{\mathrm{Idem}} \simeq E$. We say that E *admits a splitting* when such an extension exists.

The object $y \in \mathrm{Idem}^+$ is terminal, so a splitting supplies a cocone on its idempotent diagram with vertex $\widetilde{E}(y)$. The relation between this cocone and colimits is the following result.

Lemma 4.4.2 ([Lur09, Corollaries 4.4.5.14–4.4.5.15])**.** Let $\mathcal{C} \in \mathrm{Cat}$ be a category and let $E : \mathrm{Idem} \to \mathcal{C}$ be an idempotent. The cocone supplied by a splitting of E is a colimit cocone, and every colimit cocone on E arises from a splitting. In particular, E admits a splitting if and only if it admits a colimit, and every functor preserves these colimits.

Proof. Write $j : \mathrm{Idem} \to \mathrm{Idem}^+$ for the inclusion. We first check that j is cofinal, so that restriction along j does not change colimits. By the cofinality criterion, it suffices that $\mathrm{Idem} \times_{\mathrm{Idem}^+} (\mathrm{Idem}^+)_{a/}$ have contractible groupoid completion for each object $a \in \mathrm{Idem}^+$; see [Ker, Theorem 7.2.3.1]. For $a = x$, this category consists of the arrows $1, e : x \to x$, and $(x, 1)$ is initial: the unique morphism from $(x, 1)$ to (x, f) is f, for $f \in \{1, e\}$. For $a = y$, its only object is $i : y \to x$, and both 1 and e are endomorphisms of this object, since $ei = i$. It is therefore isomorphic to Idem. The endomorphism e defines a natural transformation from the identity functor of Idem to the constant functor at x, since $ef = e$ for $f \in \{1, e\}$. After groupoid completion, this gives a homotopy between the identity and a constant map, proving contractibility.

If $\widetilde{E} : \mathrm{Idem}^+ \to \mathcal{C}$ is a splitting, terminality of $y \in \mathrm{Idem}^+$ and cofinality of j give

$$\mathrm{colim}_{\mathrm{Idem}} E \simeq \mathrm{colim}_{\mathrm{Idem}^+} \widetilde{E} \simeq \widetilde{E}(y).$$

The resulting cocone is the one induced by the arrows to y, hence the cocone supplied by the splitting.

Conversely, suppose that E admits a colimit cocone with vertex $Y \in \mathcal{C}$. We construct a splitting by the pointwise left Kan extension of E along j, using Theorem 3.4.5. At x, the indexing category $\mathrm{Idem} \times_{\mathrm{Idem}^+} (\mathrm{Idem}^+)_{/x}$ has terminal object $(x, 1)$, so the required colimit is $E(x)$. At y, the indexing category has the single object $p : x \to y$, with endomorphisms $1, e$ since $pe = p$; it is isomorphic to Idem, and its diagram in $\mathcal{C}$ is E. Thus both required colimits exist, and $j_! E$ restricts to E and takes y to Y, with the prescribed colimit cocone. This is a splitting: its images of i and p satisfy the relations $pi = 1_y$ and $ip = e$ together with their coherences. Finally, applying any functor to a splitting again gives a splitting, so the first part shows that every functor preserves these colimits. □

In a splitting, the maps $\widetilde{E}(i)$ and $\widetilde{E}(p)$ exhibit the colimit object $\widetilde{E}(y)$ as a retract of $E(x)$.

Definition 4.4.3. A category $\mathcal{C} \in \mathrm{Cat}$ is *idempotent-complete* if it admits Idem-indexed colimits. Equivalently, every idempotent in $\mathcal{C}$ admits a splitting, by Lemma 4.4.2.

Construction 4.4.4. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. Its *idempotent completion* is $\mathcal{C}^{\natural} = \mathrm{PSh}_{\{\mathrm{Idem}\}}(\mathcal{C})$, with the canonical functor $よ_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\natural}$ induced by Yoneda as in Construction 4.1.3. Notice that this category is small by Remark 4.1.5.

Then, as before:

Proposition 4.4.5. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. The inclusion $よ_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\natural}$ is fully faithful and the category $\mathcal{C}^{\natural}$ is idempotent-complete. Furthermore, precomposition with the Yoneda embedding induces an equivalence

$$(よ_{\mathcal{C}})^* : \mathrm{Fun}(\mathcal{C}^{\natural}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for every idempotent-complete $\mathcal{D} \in \mathrm{Cat}$.

Proof. Full faithfulness follows from the case $\mathcal{R} = \varnothing$ of Theorem 4.1.4, and idempotent completeness follows from Construction 4.1.3 and Definition 4.4.3. Applying the case $\mathcal{R} = \varnothing$ of Theorem 4.1.4 with $\mathcal{K} = \{\mathrm{Idem}\}$ gives an equivalence

$$(よ_{\mathcal{C}})^* : \mathrm{Fun}^{\{\mathrm{Idem}\}}(\mathcal{C}^{\natural}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The right-hand side already contains all functors. By Lemma 4.4.2, every functor $\mathcal{C}^{\natural} \to \mathcal{D}$ preserves Idem-indexed colimits, so the left-hand side is the whole functor category $\mathrm{Fun}(\mathcal{C}^{\natural}, \mathcal{D})$. □

The completion also has a description in terms of retracts.

Proposition 4.4.6. Let $\mathcal{C} \in \mathrm{Cat}$ be a small category. The full subcategory $\mathcal{C}^{\natural} \subseteq \mathrm{PSh}(\mathcal{C})$ consists precisely of retracts of representables.

Proof. A retract $P \in \mathrm{PSh}(\mathcal{C})$ of a representable $よ_{\mathcal{C}}(c)$, for an object $c \in \mathcal{C}$, determines an idempotent on $よ_{\mathcal{C}}(c)$ split by P; see [Ker, Section 8.5]. By Lemma 4.4.2, P is its colimit and therefore belongs to $\mathcal{C}^{\natural}$. Conversely, retracts of representables form a full subcategory containing the representables and closed under Idem-indexed colimits: such a colimit is a retract of its diagram's object, and a retract of a retract of a representable is again a retract of that representable, by composing the retraction maps. Minimality in Construction 4.1.3 gives the reverse inclusion. □

Compactness identifies precisely which retracts occur in an Ind-completion. This is [Lur09, Lemma 5.4.2.4].

Proposition 4.4.7. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \mathrm{Cat}$ be small. The full subcategory of κ-compact objects of $\mathrm{Ind}_{\kappa}(\mathcal{C})$ is the retract closure of the representables, so there is a canonical equivalence

$$\big(\mathrm{Ind}_{\kappa}(\mathcal{C})\big)^{\kappa} \simeq \mathcal{C}^{\natural}.$$

Proof. The category Idem $\in \mathrm{Cat}$ is κ-filtered for every infinite regular κ: its absorbing endomorphism e supplies a cocone on every diagram; see [Lur09, Example 5.3.1.9]. A splitting object is the colimit of its idempotent diagram by Lemma 4.4.2. Since $\mathrm{Ind}_{\kappa}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is closed under small κ-filtered colimits, it is therefore closed under retracts. Representables are κ-compact, and a retract of a κ-compact object is κ-compact: its comparison map for each κ-filtered colimit is a retract of an equivalence.

Conversely, by Proposition 4.2.5, write a κ-compact object $x \in \mathrm{Ind}_{\kappa}(\mathcal{C})$ as $x \simeq \mathrm{colim}_{i \in I} よ_{\mathcal{C}}(c_i)$, for a small κ-filtered category $I \in \mathrm{Cat}$ and a diagram of objects $c_i \in \mathcal{C}$. Compactness gives an equivalence

$$\mathrm{colim}_{i \in I} \mathrm{Hom}(x, よ_{\mathcal{C}}(c_i)) \simeq \mathrm{Hom}(x, x).$$

On connected components, id_x factors through some $よ_{\mathcal{C}}(c_i)$, exhibiting x as a retract of a representable. □

Corollary 4.4.8. Let κ be an infinite regular cardinal and let $\mathcal{C} \in \widehat{\mathrm{Cat}}$ be a κ-accessible category. Its full subcategory $\mathcal{C}^\kappa \in \mathrm{Cat}$ of κ-compact objects is small, and the inclusion induces an equivalence $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \simeq \mathcal{C}$.

Proof. Choose a small full subcategory $\mathcal{A} \in \mathrm{Cat}$ of κ-compact generators. By Lemma 4.3.3, the inclusion induces an equivalence $\mathrm{Ind}_\kappa(\mathcal{A}) \simeq \mathcal{C}$. By Proposition 4.4.7, this identifies $\mathcal{C}^\kappa$ with $\mathcal{A}^\natural$, which is small by Construction 4.4.4. Since $\mathcal{C}^\kappa$ contains $\mathcal{A}$, it also generates $\mathcal{C}$ under small κ-filtered colimits, so Lemma 4.3.3 applies again. □

4.5. **An equivalence.** Our next goal is to show that κ-presentable categories are generated by a small amout of data.

Remark 4.5.1. Every category admitting telescopes is idempotent-complete. Indeed, let $e : x \to x$ be an idempotent. Consider the sequential diagram $x \xrightarrow{e} x \xrightarrow{e} x \xrightarrow{e} \cdots$ and let

$$y := \mathrm{colim}(x \xrightarrow{e} x \xrightarrow{e} x \to \cdots).$$

The colimit maps induce a morphism $p : x \to y$. The idempotent diagram supplies a compatible cocone with components $e : x \to x$, inducing a morphism $i : y \to x$. Then one has

$$ip \simeq e, \qquad pi \simeq \mathrm{id}_y .$$

Thus e is split. Hence every idempotent admits a splitting.

Then:

Notation 4.5.2. Let κ be an infinite regular cardinal. Write $\mathrm{Cat}^{\mathrm{rex}(\kappa)} \in \widehat{\mathrm{Cat}}$ for the category of small idempotent-complete categories admitting κ-small colimits and functors preserving these colimits. For $\kappa = \omega$, this is the full subcategory of $\mathrm{Cat}^{\mathrm{rex}}$ spanned by idempotent-complete categories; the notation $\mathrm{Cat}^{\mathrm{rex}}$ retains its meaning from Notation 3.3.10. For $\kappa > \omega$, idempotent completeness follows from the existence of κ-small colimits by Remark 4.5.1, since sequential diagrams are κ-small.

We use the following criterion from [Lur09, Proposition 5.5.1.9] without proof.

Proposition 4.5.3. Let κ be an infinite regular cardinal, let $\mathcal{D} \in \mathrm{Pr}^\kappa$ be a κ-compactly generated presentable category, and let $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ be presentable. A functor $F : \mathcal{D} \to \mathcal{E}$ preserves all small colimits if and only if it preserves small κ-filtered colimits and its restriction to $\mathcal{D}^\kappa$ preserves κ-small colimits.

The criterion expresses the passage from colimits of κ-compact objects to arbitrary colimits.

Theorem 4.5.4. Let κ be an infinite regular cardinal. Ind-completion induces an equivalence

$$\mathrm{Ind}_\kappa : \mathrm{Cat}^{\mathrm{rex}(\kappa)} \to \mathrm{Pr}^\kappa$$

whose inverse sends $\mathcal{D} \in \mathrm{Pr}^\kappa$ to $\mathcal{D}^\kappa$.

Proof. For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{rex}(\kappa)}$, the category $\mathrm{Ind}_\kappa(\mathcal{C})$ is presentable by Theorem 4.3.6. Its κ-compact objects form $\mathcal{C}^\natural \simeq \mathcal{C}$ by Proposition 4.4.7, and Proposition 4.2.5 gives κ-accessibility. For $\mathcal{D} \in \mathrm{Pr}^\kappa$, the full subcategory $\mathcal{D}^\kappa$ is small and closed under κ-small colimits and retracts, hence belongs to $\mathrm{Cat}^{\mathrm{rex}(\kappa)}$. The inclusion extends to $\mathrm{Ind}_\kappa(\mathcal{D}^\kappa) \to \mathcal{D}$ by Theorem 4.2.4. It is essentially surjective by generation and fully faithful by Remark 4.2.10.

For $\mathcal{C}, \mathcal{B} \in \mathrm{Cat}^{\mathrm{rex}(\kappa)}$, a functor $f : \mathcal{C} \to \mathcal{B}$ preserving κ-small colimits extends uniquely to a κ-filtered-colimit-preserving functor between their Ind-completions. Yoneda into an Ind-completion preserves existing κ-small colimits by [Lur09, Proposition 5.3.5.14], so Proposition 4.5.3 makes this extension preserve all small colimits. It preserves κ-compact objects by the identifications with $\mathcal{C}$ and $\mathcal{B}$. Conversely, a functor preserving all small colimits and κ-compact objects restricts to one preserving κ-small colimits. Theorem 4.2.4 gives the correspondence on natural transformations as well, proving the equivalence. □

Remark 4.5.5. Let κ be an infinite regular cardinal. The category Pr^{κ} is locally small by Theorem 4.5.4. Indeed, for small categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{rex}(\kappa)}$, the mapping anima $\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{rex}(\kappa)}}(\mathcal{C}, \mathcal{D})$ is the core of the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors preserving κ-small colimits, and is therefore small.

Corollary 4.5.6. Let κ be an infinite regular cardinal, let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{rex}(\kappa)}$ be small, and let $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ be presentable. Restriction along Yoneda induces an equivalence

$$\text{よ}^{*}_{\mathcal{C}} : \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}^{\kappa\text{-colim}}(\mathcal{C}, \mathcal{E}),$$

where the superscripts mean preservation of all small colimits and of κ-small colimits, respectively.

Proof. Apply Theorem 4.2.4 and then Proposition 4.5.3, using $(\mathrm{Ind}_{\kappa}(\mathcal{C}))^{\kappa} \simeq \mathcal{C}$. □

4.6. **Tensor products, colimit completion and localization.** We follow Lurie's construction of tensor products and change of colimit requirements in [Lur17, Section 4.8.1, including Variant 4.8.1.11]. The collection $\mathcal{K}$ of small categories need not itself be small. The category $\widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ allows large categories, while the required colimits are still indexed by members of $\mathcal{K}$ in the original Cat.

Construction 4.6.1. Let $\mathcal{K}$ be a collection of small categories and let $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ be categories admitting these colimits. Let $\mathcal{R}$ be the family of cocones $(c_i, d)_{i \in I} \to (c, d)$ and $(c, d_i)_{i \in I} \to (c, d)$ in $\mathcal{C} \times \mathcal{D}$, for small categories $I \in \mathcal{K}$, colimit cocones $(c_i)_{i \in I} \to c$ in $\mathcal{C}$ and $(d_i)_{i \in I} \to d$ in $\mathcal{D}$, and fixed objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ in the other variable. Apply the relative completion of Construction 4.1.3 in a category large enough that $\mathcal{C} \times \mathcal{D}$, $\mathcal{K}$, and $\mathcal{R}$ are small relative to it. Define

$$\mathcal{C} \otimes_{\mathcal{K}} \mathcal{D} = \mathrm{PSh}_{\mathcal{K},\mathcal{R}}(\mathcal{C} \times \mathcal{D}),$$

with its canonical functor $t : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathcal{K}} \mathcal{D}$. Here the presheaf category and its localization are formed in the enlarged category, but the completion adjoins only the colimits indexed by members of the original collection $\mathcal{K}$. Explicitly, localize at the comparison maps from the colimits of representables associated with $\mathcal{R}$, then close the localized representables under $\mathcal{K}$-colimits. The defining relations make t preserve $\mathcal{K}$-colimits separately in each variable. The resulting category is an object of $\widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, with the enlargement understood as in Notation 2.6.2.

Theorem 4.6.2. Let $\mathcal{K}$ be a collection of small categories and let $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ be categories admitting $\mathcal{K}$-colimits. For every category $\mathcal{E} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ admitting these colimits, restriction along the functor t of Construction 4.6.1 is an equivalence

$$t^{*} : \mathrm{Fun}^{\mathcal{K}}(\mathcal{C} \otimes_{\mathcal{K}} \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathcal{K},\mathcal{K}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}),$$

where the right side consists of functors preserving $\mathcal{K}$-colimits separately in the two variables. These tensor products give a closed symmetric monoidal structure on $\widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, with unit $\mathrm{PSh}_{\mathcal{K}}(*)$ and internal Hom $\mathrm{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{D})$. If $\mathcal{K}$ is small, this structure restricts to $\mathrm{Cat}^{\mathcal{K}\text{-colim}}$.

Proof. Apply Theorem 4.1.4 in the enlarged category to the cocones of Construction 4.6.1. A functor respects these cocones precisely when it preserves $\mathcal{K}$-colimits separately, giving the asserted equivalence of functor categories. Only the specified colimits enter this universal property, so enlarging the category does not impose preservation of additional colimits.

For a finite family of categories $\mathcal{C}_1, \dots, \mathcal{C}_n \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, the same construction on $\prod_{r=1}^{n} \mathcal{C}_r$ uses the colimit cocones in each variable with the other variables fixed. Its universal property represents functors preserving $\mathcal{K}$-colimits separately in all variables. For the empty family, the product is $*$ and there are no relations, giving $\mathrm{PSh}_{\mathcal{K}}(*)$. This explains the tensor operations and their unit. Lurie's cocartesian-fibration construction supplies their full symmetric monoidal compatibility: the relative completions define the universal operations, and transitivity of relative completion makes

these operations stable under composition [Lur17, Proposition 4.8.1.3, Corollary 4.8.1.4, and Variant 4.8.1.11]; see also [Lur09, Proposition 5.3.6.11].

Pointwise $\mathcal{K}$-colimits of $\mathcal{K}$-colimit-preserving functors again preserve $\mathcal{K}$-colimits, by interchange of colimits. Currying therefore gives the internal Hom $\mathrm{Fun}^{\mathcal{K}}(\mathcal{C},\mathcal{D})$ [Lur17, Remark 4.8.1.6]. If $\mathcal{K}$ and the input categories are small, the family $\mathcal{R}$ is small, and Remark 4.1.5 makes the tensor product and unit small. The internal Hom is then a full subcategory of a small functor category, so it is small as well. □

Remark 4.6.3. Let $\mathcal{K} \subseteq \mathcal{K}'$ be collections of small categories. The inclusion $\widehat{\mathrm{Cat}}^{\mathcal{K}'\text{-colim}} \to \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$ admits a left adjoint, obtained by adjoining $\mathcal{K}'$-colimits with the existing $\mathcal{K}$-colimit cocones as prescribed relations. This follows from Theorem 4.1.4 in a category containing the input category and the families of indexing categories and relations as small objects. The left adjoint is symmetric monoidal by [Lur17, Remark 4.8.1.8 and Variant 4.8.1.11]. If both collections are small, it restricts to a symmetric monoidal left adjoint $\mathrm{Cat}^{\mathcal{K}\text{-colim}} \to \mathrm{Cat}^{\mathcal{K}'\text{-colim}}$. The inclusions are generally not full. For $\mathcal{K} = \varnothing$, the tensor product is cartesian and the left adjoint is $\mathrm{PSh}_{\mathcal{K}'}$. Consequently, for categories $\mathcal{C},\mathcal{D} \in \widehat{\mathrm{Cat}}$, one has

$$\mathrm{PSh}_{\mathcal{K}'}(\mathcal{C}\times\mathcal{D}) \simeq \mathrm{PSh}_{\mathcal{K}'}(\mathcal{C}) \otimes_{\mathcal{K}'} \mathrm{PSh}_{\mathcal{K}'}(\mathcal{D}),$$

with the completions formed in the enlarged category.

Example 4.6.4. Every functor preserves Idem-indexed colimits by Lemma 4.4.2, so the idempotent-complete categories form a full reflective subcategory of Cat with reflector $(-)^{\natural}$. Taking $\mathcal{K} = \varnothing$ and $\mathcal{K}' = \{\mathrm{Idem}\}$ in Remark 4.6.3 shows that this reflector is symmetric monoidal for cartesian products. For small categories $\mathcal{C},\mathcal{D} \in \mathrm{Cat}$, it follows that $(\mathcal{C}\times\mathcal{D})^{\natural} \simeq \mathcal{C}^{\natural}\times\mathcal{D}^{\natural}$.

Applying Remark 4.6.3 to finite colimits and adjoining Idem gives the symmetric monoidal reflective localization

$$(-)^{\natural} : \mathrm{Cat}^{\mathrm{rex}} \to \mathrm{Cat}^{\mathrm{rex}(\omega)}.$$

Idempotent completion preserves the existence of finite colimits and its canonical inclusion preserves them, by [Lur09, Propositions 5.3.5.14 and 5.5.7.10]. Its restriction universal property for right exact functors is the localization assertion of [Lur09, Proposition 5.5.7.10]. Writing $\otimes_{\mathrm{fin}}$ for the tensor product on $\mathrm{Cat}^{\mathrm{rex}}$ from Theorem 4.6.2, the tensor product on $\mathrm{Cat}^{\mathrm{rex}(\omega)}$ is therefore

$$\mathcal{C} \otimes_{\omega} \mathcal{D} \simeq (\mathcal{C} \otimes_{\mathrm{fin}} \mathcal{D})^{\natural} \qquad (\mathcal{C},\mathcal{D} \in \mathrm{Cat}^{\mathrm{rex}(\omega)}).$$

Remark 4.6.5. Let $\mathcal{K}$ be the collection of all small categories $I \in \mathrm{Cat}$. Then Theorem 4.6.2 gives a symmetric monoidal structure on $\widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, the category of large categories admitting small colimits and functors preserving them. The required colimits are indexed by categories $I \in \mathrm{Cat}$, even though the categories admitting them are objects of $\widehat{\mathrm{Cat}}$. Its tensor product represents functors preserving these colimits separately, and its unit is $\mathrm{PSh}(*) \simeq \mathrm{An}$. The notation $\mathrm{Fun}^{\mathrm{L,L}}$ denotes the full subcategory of functors preserving small colimits separately in each variable. Restriction to Pr^{L} requires the additional assertion that the tensor product of two presentable categories is presentable.

Example 4.6.6. Let $\mathcal{A},\mathcal{B} \in \mathrm{Cat}$ be small categories. In the symmetric monoidal category of Remark 4.6.5, one has

$$\mathrm{PSh}(\mathcal{A}) \otimes \mathrm{PSh}(\mathcal{B}) \simeq \mathrm{PSh}(\mathcal{A}\times\mathcal{B}).$$

For every category $\mathcal{E} \in \widehat{\mathrm{Cat}}^{\mathcal{K}\text{-colim}}$, where $\mathcal{K}$ is the collection of all originally small categories, the free-cocompletion universal property in each variable identifies colimit-preserving functors from either side to $\mathcal{E}$ with $\mathrm{Fun}(\mathcal{A}\times\mathcal{B},\mathcal{E})$. Here we use the case $\mathcal{R} = \varnothing$ of Theorem 4.1.4; the extensions preserve colimits separately by interchange of colimits.

Theorem 4.6.7 ([Lur17, Proposition 4.8.1.15])**.** The full subcategory Pr^{L} of the category in Remark 4.6.5 contains the unit and is closed under tensor products, so it inherits a symmetric monoidal structure. For presentable categories $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is presentable, and its canonical functor $t : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ induces an equivalence

$$t^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

for every large category $\mathcal{E} \in \widehat{\mathrm{Cat}}$ admitting small colimits. This symmetric monoidal structure is closed, with unit An and internal Hom $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$.

Proof. The unit An is presentable by Theorem 4.3.6. By Example 4.6.6, the tensor product of two presheaf categories on small categories is again a presheaf category, hence presentable. Every presentable category is a localization of such a presheaf category at a small family of morphisms; we use [Lur09, Theorem 5.5.1.1 and Proposition 5.5.4.15] for this presentation. It therefore suffices to check that presentability of a tensor product is preserved when either factor is localized at a small family of morphisms.

Let $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$ be presentable categories whose tensor product $\mathcal{C} \otimes \mathcal{D}$ is presentable, and let S be a small family of morphisms in $\mathcal{C}$. Choose a small family M of objects $d \in \mathcal{D}$ generating $\mathcal{D}$ under small colimits, and let T be the small family of morphisms $t(s, \mathrm{id}_d)$ with $s \in S$ and $d \in M$. The localization $T^{-1}(\mathcal{C} \otimes \mathcal{D})$ is presentable by Theorem 4.3.6. We claim that

$$(S^{-1}\mathcal{C}) \otimes \mathcal{D} \simeq T^{-1}(\mathcal{C} \otimes \mathcal{D}).$$

To check this, let $\mathcal{E} \in \widehat{\mathrm{Cat}}$ be a large category admitting small colimits. By [Lur09, Proposition 5.5.4.20] and the tensor universal property, colimit-preserving functors from the right-hand side to $\mathcal{E}$ correspond to separately colimit-preserving functors $G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that $G(s, d)$ is an equivalence for every $s \in S$ and $d \in M$. For a fixed morphism $s : a \to b$ in S, with objects $a, b \in \mathcal{C}$, both functors $G(a, -)$ and $G(b, -)$ preserve small colimits, so the objects on which $G(s, -)$ is an equivalence are closed under these colimits. Since M generates $\mathcal{D}$, the condition holds for every object $d \in \mathcal{D}$. Applying the localization universal property in the first variable therefore identifies these functors with $\mathrm{Fun}^{\mathrm{L,L}}(S^{-1}\mathcal{C} \times \mathcal{D}, \mathcal{E})$. This identification includes natural transformations; separate preservation follows by currying into the category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})$, whose small colimits are computed pointwise. The tensor universal property proves the claimed equivalence. Applying this argument in each variable to presheaf presentations proves closure of Pr^{L} under tensor products.

The symmetric monoidal structure restricts to Pr^{L}, and the asserted universal property is inherited with it. Finally, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is presentable by [Lur09, Proposition 5.5.3.8], and currying identifies it with the internal Hom; see also [Lur17, Remark 4.8.1.18]. □

Proposition 4.6.8. Let κ be an infinite regular cardinal and let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{rex}(\kappa)}$ be small categories admitting κ-small colimits, assumed idempotent-complete when $\kappa = \omega$. For $\kappa > \omega$, let $\otimes_\kappa$ be the tensor product for κ-small colimits from Theorem 4.6.2; for $\kappa = \omega$, take the idempotent-completed tensor of Example 4.6.4. Then there is a canonical equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C} \otimes_\kappa \mathcal{D}) \simeq \mathrm{Ind}_\kappa(\mathcal{C}) \otimes \mathrm{Ind}_\kappa(\mathcal{D}).$$

The equivalence of Theorem 4.5.4 is symmetric monoidal, with units An^κ and An, respectively.

Proof. Let $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ be a presentable category. By Corollary 4.5.6 and the tensor universal properties of Theorem 4.6.2 and Theorem 4.6.7, colimit-preserving functors from either side to $\mathcal{E}$ correspond to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving κ-small colimits separately. For the right-hand side, apply Corollary 4.5.6 in each variable; pointwise extension and interchange of colimits preserve the condition in the other variable. For $\kappa = \omega$, use also Example 4.6.4, since $\mathcal{E}$ is idempotent-complete. These correspondences include natural transformations and are natural in $\mathcal{E}$, giving the claimed equivalence.

Ind-completion is the completion adjoining all small colimits while preserving the existing κ-small colimits; see [Lur09, Example 5.3.6.8]. Thus Remark 4.6.3, applied to the collection of all originally small categories, supplies the symmetric monoidal comparisons with their higher compatibility data. The tensor of two functors preserving κ-compact objects is the Ind-extension of their tensor on compact objects, so again preserves κ-compact objects by Theorem 4.5.4. Finally, An is κ-compactly generated by [Lur09, Proposition 5.3.5.12], and its compact objects give the unit An^{κ} on the small-category side. □

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