

A RECONSTRUCTION THEOREM

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ABSTRACT. This is the notes of my talk at the seminar of higher Zariski geometry at the university of Regensburg. The goal of this talk is to show that for a quasi compact quasi separated scheme X , the Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}(X))$ is carrying a sheaf of rings $\mathcal{O}_{\mathrm{Perf}(X)}$ and there is an isomorphism of schemes $X \simeq (\mathrm{Spc}(\mathrm{Perf}(X)), \mathcal{O}_{\mathrm{Perf}(X)})$. The proof is done by first identifying the underlying topological spaces and then comparing the structure sheaves. We try to present all the definitions and results using the language of *2-rings* as introduced in the [3].

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Notation 0.1.

- (a) \mathcal{S} denotes the ∞ -category of anima.
- (b) Given a small ∞ -category \mathcal{C} , the notation $\mathcal{P}(\mathcal{C})$ refers to the category of presheaves on \mathcal{C} .
- (c) $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ denotes the ∞ -category of idempotent-complete stable ∞ -categories and exact functors.
- (d) $2\mathrm{CAlg}$ denotes the ∞ -category $\mathrm{CAlg}^{\otimes}(\mathrm{Cat}_{\infty}^{\mathrm{perf}})$ called *2-rings* whose morphisms consist of symmetric monoidal exact functors.
- (e) Let $\mathcal{K} \in 2\mathrm{CAlg}$. We denote by $\mathrm{Idl}(\mathcal{K})$ the poset of *thick tensor ideals* (*tt*-ideals or simply ideals) of \mathcal{K} , by $\mathrm{Rad}(\mathcal{K})$ its full subcategory of radical *tt*-ideals and by $\mathrm{Prm}(\mathcal{K})$ its full subcategory of prime ideals (all ordered by inclusions).
- (f) $\mathrm{Spc} \mathcal{K}$ denotes the *Balmer spectrum* of \mathcal{K} , that is the spectral space associated to the Hochster dual frame $\mathrm{Rad}(\mathcal{K})^{\vee}$. where $(-)^{\vee}$ is the dual functor $L \mapsto L^{\mathrm{op}}$. More explicitly, $\mathrm{Spc} \mathcal{K} := \{\mathcal{P} \in \mathrm{Prm}(\mathcal{K})\}$
- (g) Let X be a scheme. We denote by $\mathrm{Perf}(X)$ the full subcategory of *perfect complexes* of $\mathrm{QCoh}(X)$, the symmetric monoidal stable category of quasi-coherent sheaves on X . The symmetric monoidal structure is given by the derived tensor product.

- (h) Given an ∞ -category \mathcal{C} , we denote by \mathcal{C}^\natural its idempotent completion.
- (i) Let $\mathcal{K} \in 2\text{CAlg}$ and $\mathcal{J} \in \text{Spc } \mathcal{K}$ an ideal. We denote by \mathcal{K}/\mathcal{J} the *Karoubi quotient* of \mathcal{K} by \mathcal{J} .
- (j) Let \mathcal{K} be a 2-ring. $\text{End}_{\mathcal{K}}(1)$ denotes the ring of those endomorphisms of the identity functors which commute with the suspension functor.

1. IDENTIFICATION OF THE TOPOLOGICAL SPACES

1.1. The Zariski topology.

Let \mathcal{K} be a 2-ring.

Definition 1.1. For any family of objects $S \subset \mathcal{K}$, we denote by $\mathbf{Z}(S)$ the following subset of $\text{Spc } \mathcal{K}$

$$\mathbf{Z}(S) := \{\mathcal{P} \in \text{Spc } \mathcal{K} \mid S \cap \mathcal{P} = \emptyset\}$$

Proposition 1.2. *The collection $\{\mathbf{Z}(S) \subset \text{Spc } \mathcal{K} \mid S \subset \mathcal{K}\}$ defines the closed subsets of a topology on $\text{Spc } \mathcal{K}$.*

Proof. First, note that $\mathbf{Z}(\emptyset) = \text{Spc } \mathcal{K}$ and $\mathbf{Z}(\mathcal{K}) = \emptyset$.

Then, observe that $\bigcap_{j \in J} \mathbf{Z}(S_j) = \mathbf{Z}\left(\bigcup_{j \in J} S_j\right)$. We prove the result when J is a set of two elements, then, it is an induction argument. Suppose that $\mathcal{P} \in \mathbf{Z}(S_1) \cap \mathbf{Z}(S_2)$. Then, $S_1 \cap \mathcal{P} = \emptyset$ and $S_2 \cap \mathcal{P} = \emptyset$ so that $(S_1 \cap \mathcal{P}) \cup (S_2 \cap \mathcal{P}) = \emptyset$ which implies by distributivity of \cap that $\mathcal{P} \cap (S_1 \cup S_2) = \emptyset$ so $\mathcal{P} \in \mathbf{Z}(S_1 \cup S_2)$. Conversely, suppose that $\mathcal{P} \in \mathbf{Z}(S_1 \cup S_2)$. Then, $\mathcal{P} \cap (S_1 \cup S_2) = \emptyset$ so in particular, $\mathcal{P} \cap S_1 = \emptyset$ and $\mathcal{P} \cap S_2 = \emptyset$, which shows the equality.

Now, we show that $\mathbf{Z}(S_1) \cup \mathbf{Z}(S_2) = \mathbf{Z}(S_1 \oplus S_2)$ where $S_1 \oplus S_2 = \{a_1 \oplus a_2 \mid a_i \in S_i\}$. Suppose that $\mathcal{P} \in \mathbf{Z}(S_1 \oplus S_2)$. If $a \in \mathcal{P} \cap S_1$ and $b \in \mathcal{P} \cap S_2$, then $a \oplus b \in \mathcal{P} \cap (S_1 \oplus S_2)$, so by contradiction $\mathcal{P} \in \mathbf{Z}(S_1) \cup \mathbf{Z}(S_2)$. By a similar argument, we prove the converse. \square

Definition 1.3. The topology defined by the closed subsets $\{\mathbf{Z}(S) \subset \text{Spc } \mathcal{K} \mid S \subset \mathcal{K}\}$ is called *the Zariski topology* on $\text{Spc } \mathcal{K}$.

Definition 1.4. • Let $a \in \mathcal{K}$. The *support* of a is the following closed subset (for the Zariski topology on $\text{Spc } \mathcal{K}$) of $\text{Spc } \mathcal{K}$

$$\text{supp}(a) := \mathbf{Z}(\{a\}) = \{\mathcal{P} \in \mathcal{K} \mid a \notin \mathcal{P}\}$$

- We denote the open complement of $\mathbf{Z}(S)$ by

$$\mathbf{U}(S) := \text{Spc } \mathcal{K} \setminus \mathbf{Z}(S) = \{\mathcal{P} \in \text{Spc } \mathcal{K} \mid S \cap \mathcal{P} \neq \emptyset\}$$

Lemma 1.5. *Let $a, b \in \mathcal{K}$. Then, the assignment $a \mapsto \mathbf{U}(a)$ satisfies the following properties:*

- (a) $\mathbf{U}(0) = \text{Spc } \mathcal{K}$ and $\mathbf{U}(1) = \emptyset$
- (b) $\mathbf{U}(a \oplus b) = \mathbf{U}(a) \cap \mathbf{U}(b)$
- (c) $\mathbf{U}(\Sigma a) = \mathbf{U}(a)$
- (d) $\mathbf{U}(b) \cap \mathbf{U}(c) \subset \mathbf{U}(a)$ for any exact sequence $a \rightarrow b \rightarrow c$
- (e) $\mathbf{U}(a \otimes b) = \mathbf{U}(a) \cup \mathbf{U}(b)$

Proof. We only prove (b). Everything is routine and proved using the axioms of prime ideal.

Let $\mathcal{P} \in \mathbf{U}(a) \cap \mathbf{U}(b)$. Then, a and b are in \mathcal{P} by definition. As \mathcal{P} is thick, it is closed under products so $a \oplus b \in \mathcal{P}$. Hence, $\mathbf{U}(a) \cap \mathbf{U}(b) \subset \mathbf{U}(a \oplus b)$.

Suppose now that $\mathcal{P} \in \mathbf{U}(a \oplus b)$. By assumption, $a \oplus b \in \mathcal{P}$, and as \mathcal{P} is a prime

ideal, it is in particular closed under retracts. Thus, $a \in \mathcal{P}$ and $b \in \mathcal{P}$ as they are retracts of $a \oplus b$. So we have $\mathbf{U}(a \oplus b) \subset \mathbf{U}(a) \cap \mathbf{U}(b)$ which proves the equality. \square

Remark 1.6. *Since for any $S \subset \mathcal{K}$, we have $\mathbf{U}(S) = \bigcup_{a \in S} \mathbf{U}(a)$, it follows from the last proposition that $\{\mathbf{U}(a) \mid a \in \mathcal{K}\}$ is a basis of the topology on $\mathrm{Spc} \mathcal{K}$. Equivalently, their complements $\{\mathrm{supp}(a) \mid a \in \mathcal{K}\}$ form a basis of closed subsets.*

Proposition 1.7 (closure). *Let $W \subset \mathrm{Spc} \mathcal{K}$ be a subset of the Balmer spectrum. Then, the closure may be computed as:*

$$\overline{W} := \bigcap_{(a \in \mathcal{K} \mid W \subset \mathrm{supp}(a))} \mathrm{supp}(a)$$

Proof. Given a basis \mathcal{B} of closed subsets, the closure of a subset W is the intersection of all of those $B \in \mathcal{B}$ such that $W \subset B$. The last remark provides such basis, thus concluding the proof. \square

Proposition 1.8 (closure of a point). *For any point $\mathcal{P} \in \mathrm{Spc} \mathcal{K}$, its closure in $\mathrm{Spc} \mathcal{K}$ is*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc} \mathcal{K} \mid \mathcal{Q} \subset \mathcal{P}\}$$

In particular, if $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$, then $\mathcal{P}_1 = \mathcal{P}_2$. In particular, the space $\mathrm{Spc} \mathcal{K}$ is T_0 .

Proof. Let $S_0 := \mathcal{K} \setminus \mathcal{P}$. Clearly, $\mathcal{P} \in \mathbf{Z}(S_0)$ and if $\mathcal{P} \in \mathbf{Z}(S)$, then $S \subset S_0$ and hence $\mathbf{Z}(S_0) \subset \mathbf{Z}(S)$. So, $\mathbf{Z}(S_0)$ is the smallest closed subset which contains the point \mathcal{P} , ie $\overline{\{\mathcal{P}\}} = \mathbf{Z}(S_0) = \{\mathcal{Q} \in \mathrm{Spc} \mathcal{K} \mid \mathcal{Q} \subset \mathcal{P}\}$. The second assertion is immediate. \square

1.2. Functoriality of the Balmer spectrum.

Definition 1.9 (support data). *A support data on \mathcal{K} is a pair (X, σ) where X is a topological space and σ an assignment that takes object $a \in \mathcal{K}$ to a closed subset $\sigma(a) \subseteq X$ with the following properties:*

- (i) $\sigma(0) = \emptyset$ and $\sigma(1) = X$
- (ii) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$
- (iii) $\sigma(\Sigma a) = \sigma(a)$ where $\Sigma : \mathcal{K} \rightarrow \mathcal{K}$ is the suspension functor
- (vi) $\sigma(a) \subset \sigma(b) \cup \sigma(c)$, when given a exact sequence $a \rightarrow b \rightarrow c$
- (v) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$

Definition 1.10. *A morphism $f : (X, \sigma) \rightarrow (Y, \tau)$ of support data on \mathcal{K} is a continuous map $f : X \rightarrow Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all $a \in \mathcal{K}$.*

Remark 1.11. *A morphism between support data is an isomorphism if and only if f is a homeomorphism.*

The goal is now to prove that the pair $(\mathrm{Spc} \mathcal{K}, \mathrm{supp})$ is final as a support data. We need two lemmas.

Lemma 1.12. *Let X be a set and $f_1, f_2 : X \rightarrow \mathrm{Spc} \mathcal{K}$ be two maps such that $f_1^{-1}(\mathrm{supp}(a)) = f_2^{-1}(\mathrm{supp}(a))$. Then, $f_1 = f_2$.*

Proof. Let $x \in X$. Observe first that for any object $a \in \mathcal{K}$, we have by assumption $f_1(x) \in \mathrm{supp}(a)$ if and only if $f_2(x) \in \mathrm{supp}(a)$. This implies that the two following closed subsets of $\mathrm{Spc} \mathcal{K}$ coincide:

$$\bigcap_{f_1(x) \in \mathrm{supp}(a)} \mathrm{supp}(a) = \bigcap_{f_2(x) \in \mathrm{supp}(a)} \mathrm{supp}(a)$$

The left-hand side is nothing but $\overline{\{f_1(x)\}}$ and the right-hand side is $\overline{\{f_2(x)\}}$ by Proposition 1.7. Hence, $\overline{\{f_1(x)\}} = \overline{\{f_2(x)\}}$ in $\mathrm{Spc} \mathcal{K}$ and by Proposition 1.8, $f_1(x) = f_2(x)$. \square

Lemma 1.13. *Let (X, σ) be a support data on \mathcal{K} and $Y \subset X$ be any subset. Then, the full subcategory of \mathcal{K} with objects $\{a \in \mathcal{K} \mid \sigma(a) \subset Y\}$ is a tt-ideal of \mathcal{K} .*

Proof. It follows directly from Definition 1.9. \square

Theorem 1.14. *The spectrum $(\mathrm{Spc} \mathcal{K}, \mathrm{supp})$ is the final support data on \mathcal{K} . In other words, $(\mathrm{Spc} \mathcal{K}, \mathrm{supp})$ is a support data and for any other support data (X, σ) on \mathcal{K} , there exists a unique continuous map $f : X \rightarrow \mathrm{Spc} \mathcal{K}$ such that $\sigma(a) = f^{-1}(\mathrm{supp}(a))$ for any object $a \in \mathcal{K}$. Explicitly, the map f is defined for all $x \in X$ by*

$$f(x) = \{a \in \mathcal{K} \mid x \notin \sigma(a)\}$$

Proof. Thanks to Lemma 1.5, $(\mathrm{Spc} \mathcal{K}, \mathrm{supp})$ is a support data on \mathcal{K} . For the universal property, let (X, σ) be a support data on \mathcal{K} . Uniqueness of the morphism $f : X \rightarrow \mathrm{Spc} \mathcal{K}$ follows from Lemma 1.12, so let us check that the announced map $f(x) := \{a \in \mathcal{K} \mid x \notin \mathrm{supp}(a)\}$ is as we wanted. Applying Lemma 1.13 to $Y = X \setminus \{x\}$, we get that $f(x) \in \mathrm{Idl}(\mathcal{K})$. To see that $f(x) \in \mathrm{Prm}(\mathcal{K})$, take $a \otimes b \in f(x)$: this means $x \notin \sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ and hence $x \notin \sigma(a)$ or $x \notin \sigma(b)$, that is $a \in f(x)$ or $b \in f(x)$. By definition, we have $f(x) \in \mathrm{supp}(a)$ if and only if $a \notin f(x)$ if and only if $x \in \sigma(a)$, hence $f^{-1}(\mathrm{supp}(a)) = \sigma(a)$. This also gives continuity by the definition of the topology on $\mathrm{Spc} \mathcal{K}$, see Remark 1.6. \square

Remark 1.15. *Now, we want to change the 2-ring \mathcal{K} . The next theorem shows that the Balmer spectrum is functorial at the level of the homotopy categories. Indeed, the homotopy category of a stable infinity category is a tensor-triangulated category and a symmetric monoidal exact functor between ∞ -categories gives rise to a tensor-triangulated functor at the level of the homotopy categories.*

Theorem 1.16. *The Balmer spectrum is functorial at the level of 2-rings. That is, given a symmetric monoidal exact functor $F : \mathcal{K} \rightarrow \mathcal{L}$, the map*

$$\begin{aligned} \mathrm{Spc} F : \mathrm{Spc} \mathcal{K} &\longrightarrow \mathrm{Spc} \mathcal{L} \\ \mathcal{Q} &\longmapsto F^{-1}\mathcal{Q} \end{aligned}$$

is well-defined, continuous, and for all objects $a \in \mathcal{K}$, we have:

$$(\mathrm{Spc} F)^{-1}(\mathrm{supp}_{\mathcal{K}}(a)) = \mathrm{supp}_{\mathcal{L}}(F(a))$$

in $\mathrm{Spc} \mathcal{L}$. This defines a contravariant functor $\mathrm{Spc}(-)$ from the category of 2-rings to the category of topological spaces.

Remark 1.17. *It is important to mention that the above result only depends on the tensor-triangulated structure of the homotopy categories $h\mathcal{K}$ and $h\mathcal{L}$. Namely, the construction of $\mathrm{Spc} F$ goes as follows. We still denote by \mathcal{K} and \mathcal{L} the 2-rings. Recall that one has the adjunction*

$$\mathrm{Cat}_{\infty} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{N} \end{array} \mathrm{Cat}_1$$

where N denotes the nerve functor which is fully faithful and h the realization functor. Let $u : \mathrm{id}_{\mathrm{Cat}_{\infty}} \rightarrow Nh$ be the unit of this adjunction. We construct the map at the level of the homotopy category by first, composing with u : $\mathrm{Hom}_{2\mathrm{CAlg}}(\mathcal{K}, \mathcal{L}) \rightarrow$

$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Nh}\mathcal{K}, \mathrm{Nh}\mathcal{L})$ and then using the fact that the nerve functor is fully faithful $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Nh}\mathcal{K}, \mathrm{Nh}\mathcal{L}) \rightarrow \mathrm{Hom}_{\mathrm{Cat}_1}(h\mathcal{K}, h\mathcal{L})$. Thus, by composition Spc sends a map $hf \in \mathrm{Hom}_{\mathrm{Cat}_1}(h\mathcal{K}, h\mathcal{L})$ to a map $\mathrm{Spc}(hf) \in \mathrm{Hom}_{\mathrm{Top}}(\mathrm{Spc}(h\mathcal{K}), \mathrm{Spc}(h\mathcal{L}))$. But as $\mathrm{Spc}(\mathcal{K}) := \mathrm{Spc}(h\mathcal{K})$ and $\mathrm{Spc}(f) := \mathrm{Spc}(hf)$, we proved that the functoriality of the Balmer spectrum at the level of 2-rings is induced by its functoriality at the level of the homotopy category.

1.3. Description of the Balmer spectrum for perfect complexes.

We discuss another type of support, namely the *classifying support*, that helps us to state a result giving a description of the Balmer spectrum.

Definition 1.18. Let X be a topological space. A subset $Y \subset X$ is *specialization closed* if it is a union of closed subsets or equivalently, $y \in Y \implies \overline{\{y\}} \subset Y$.

Definition 1.19. We say that a support data (X, σ) of a 2-ring \mathcal{K} is a *classifying support data* if:

- (a) The topological space X is noetherian and any non-empty irreducible closed subset $Z \subset X$ has a unique generic point.
- (b) There is a bijection $\theta : \{Y \subset X \mid Y \text{ specialization closed}\} \rightarrow \{\mathcal{J} \subset \mathcal{K} \text{ radical ideal}\}$ defined by $Y \mapsto \{a \in \mathcal{K} \mid \sigma(a) \subset Y\}$ with inverse $\mathcal{J} \mapsto \sigma(\mathcal{J}) := \cup_{a \in \mathcal{J}} \sigma(a)$.

Before proving one of the main result of this section, we need the following lemma:

Lemma 1.20. Let (X, σ) be a classifying support data on \mathcal{K} . Then, any closed subset $Z \subset X$ is of the form $Z = \sigma(a)$ for some $a \in \mathcal{K}$.

Proof. Since $\sigma(a_1) \cup \dots \cup \sigma(a_n) = \sigma(a_1 \oplus \dots \oplus a_n)$ and since the space X is noetherian, it suffices to prove the claim for an irreducible $Z = \overline{\{x\}}$ for some $x \in X$. Now, $\overline{\{x\}} = Z = \theta^{-1}(\theta(Z)) = \cup_{a \in \theta(Z)} \sigma(a)$ forces the existence of some object $a \in \mathcal{K}$ such that $x \in \sigma(a) \subset Z$. Hence, $\overline{\{x\}} \subset \sigma(a) \subset Z = \overline{\{x\}}$ which proves the lemma. \square

Theorem 1.21. Suppose that (X, σ) is a classifying support data on \mathcal{K} . Then, the canonical map $f : X \rightarrow \mathrm{Spc}\mathcal{K}$ is a homeomorphism.

Proof. Theorem 1.14 tell us that the map f is continuous and satisfies $f^{-1}(\mathrm{supp}(a)) = \sigma(a)$ for all objects $a \in \mathcal{K}$.

For $x \in X$, define $Y(x) := \{y \in X \mid x \notin \overline{\{y\}}\}$. It is easy to check that $Y(x)$ is specialization closed. Let $a \in \mathcal{K}$. Let us see that $\sigma(a) \subset Y(x) \iff x \notin \sigma(a)$. Since $x \notin Y(x)$, we have $\sigma(a) \subset Y(x) \implies x \notin \sigma(a)$ as $Y(x)$ is specialization closed (second property). Conversely, since $\sigma(a)$ is (specialization) closed, and since $x \notin \sigma(a)$, we have $x \notin \overline{\{y\}}$ for all $y \in \sigma(a)$, which exactly means $\sigma(a) \subset Y(x)$ by definition of the latter. So we have established:

$$(1) \quad \theta(Y(x)) := \{a \in \mathcal{K} \mid \sigma(a) \subset Y(x)\} = \{a \in \mathcal{K} \mid x \notin \sigma(a)\} =: f(x)$$

In particular, if $f(x_1) = f(x_2)$ then $Y(x_1) = Y(x_2)$ which immediately implies that $\overline{\{x_1\}} = \overline{\{x_2\}}$ and $x_1 = x_2$ since X is T_0 . Hence, f is injective.

Let \mathcal{P} be a prime ideal of \mathcal{K} . By assumption, there exists a specialization closed subset $Y \subset X$ such that $\mathcal{P} = \theta(Y)$. The complement $X \setminus Y$ is non-empty as $\mathcal{P} \neq \mathcal{K}$. Let $x, y \in X \setminus Y$. By Lemma 1.20, there exists objects $a, b \in \mathcal{K}$ such that $\overline{\{x\}} = \sigma(a)$ and $\overline{\{y\}} = \sigma(b)$. Since x and y are outside Y , the objects a, b do not belong to $\theta(Y) = \mathcal{P}$. The latter being a prime, we then have $a \otimes b \notin \mathcal{P}$. So $\sigma(a \otimes b) \not\subset Y$, ie, there is a point $z \in X \setminus Y$ such that $z \in \sigma(a \otimes b) = \sigma(a) \cap \sigma(b) = \overline{\{x\}} \cap \overline{\{y\}}$, and

hence, $\overline{\{z\}} \subset \overline{\{x\}}$ and $\overline{\{z\}} \subset \overline{\{x\}}$. In short, we have established that the non-empty family of closed subsets

$$\mathcal{F} := \{\overline{\{x\}} \subset X \mid x \in X \setminus Y\}$$

has the property that any two elements admit a lower bound for inclusion. On the other hand, since X is noetherian, there exists a minimal element in \mathcal{F} which is then the lower bound of \mathcal{F} by the above reasoning. This shows that there exists a point $x \in X \setminus Y$ such that $X \setminus Y \subset \overline{\{y \in X \mid x \in \overline{\{y\}}\}}$. The reverse inclusion also holds because $x \notin Y$, which is specialization closed. In other words,

$$Y = \{y \in X \mid x \notin \overline{\{y\}}\} = Y(x).$$

Therefore, $\mathcal{P} = \theta(Y) = f(x)$ by (1) which proves the surjectivity of f .

The relation $f^{-1}(\text{supp}(a)) = \sigma(a)$ now gives $f(\sigma(a)) = \text{supp}(a)$. This shows that f is a closed map, since we know from Lemma 1.20 that any closed subset of X is of the form $\sigma(a)$. Hence, f is a homeomorphism. \square

Now we are ready to give a description of the perfect complexes.

Notation 1.22. Let X be a scheme. For any perfect complex $a \in \text{Perf}(X)$, we denote by $\text{supph}(a) := \{x \in X \mid a_x \not\cong 0 \text{ in } \text{Perf}(\mathcal{O}_{X,x})\} \subset X$ the *homological support* of a .

Theorem 1.23. *Let X be a topologically noetherian scheme. The pair (X, supph) is a classifying support data on $\text{Perf}(X)$.*

Proof. The condition (a) in Definition 1.19 is immediate. Let θ denotes the map as in (b) in Definition 1.19 and ψ be the map in the other direction.

It is easy to see that θ and ψ preserve order on $\text{Perf}(X)$ and on X given by inclusions of subcategories and of subspaces. Hence, we have $\theta(\psi(\mathcal{J})) \subset \mathcal{J}$ and $\psi(\theta(Y)) \subset Y$. It remains to show that θ and ψ are inverse to each other, for which suffices to prove the reverse inclusions $\mathcal{J} \subset \theta(\psi(\mathcal{J}))$ and $Y \subset \psi(\theta(Y))$ for $Y \subset X$ specialization closed and $\mathcal{J} \subset \text{Perf}(X)$ a radical ideal. We refer the reader to [5] Theorem 3.15 for details of the proof. \square

Corollary 1.24. *There is a homeomorphism $f : X \rightarrow \text{Spc}(\text{Perf}(X))$ with*

$$f(x) = \{a \in \text{Perf}(X) \mid a_x \simeq 0 \text{ in } \text{Perf}(\mathcal{O}_{X,x})\} \quad \forall x \in X.$$

Moreover, for any perfect complex $a \in \text{Perf}(X)$, the closed subset $\text{supph}(a)$ in X corresponds via f to the closed subset $\text{supp}(a)$ in $\text{Spc} \text{Perf}(X)$.

Proof. Theorems 1.14, 1.21 and 1.23. \square

Remark 1.25. *The hypothesis of the scheme being topologically noetherian is necessary. Indeed, consider the polynomial ring in infinitely many variables $R = k[x_1, x_2, \dots]$ over a field k . We want to show that in this case, $\text{Perf}(R) \not\cong \bigoplus_{i \in \mathbb{N}} \text{Perf}(k[x_i])$. There is an obvious map $R \rightarrow k$ which sends x_i to 0 for all i . We can consider the support data $\{m\}$, consisting of the singleton closed point which is the maximal ideal. There is no finite complex (X is noetherian so the homotopy category of perfect complexes is a bounded derived category) of projectives whose support is only m . Thus, there is no radical ideals of $\bigoplus_{i \in \mathbb{N}} \text{Perf}(k[x_i])$ corresponding to $\{m\}$.*

2. IDENTIFICATION OF THE STRUCTURE SHEAVES

2.1. Reconstruction of perfect complexes of an open subscheme.

Notation 2.1. Let $Y \subset X$ be a subscheme, denote by $\text{Perf}_Y(X) = \{\mathcal{P} \in \text{Perf}(X) \mid \text{supph}(\mathcal{P}) \subset Y\}$

We start by giving an outline of this section. We first prove a key result, answering the following question. Let $U \subset X$ be an open subscheme. One can ask if we can retrieve the category $\text{Perf}(U)$ using only the datum of the category $\text{Perf}(X)$ and its restriction to the open complement of U , $\text{Perf}_{X-U}(X)$. It is indeed possible, and this result combined with the fact that the category $\text{Perf}(U)$ is *rigid* allow us to compute the unit \mathcal{O}_U as the endomorphism of the unit $\text{End}_{\text{Perf}(U)}(\mathcal{O}_U, \mathcal{O}_U)$. Then, by evaluating on an open basis of X , we get the main result.

Theorem 2.2. *Let X be a qcqs scheme, $i : U \subset X$ an open subscheme and $Z = X - U$ its closed complement. Then, the following sequence of symmetric monoidal ∞ -categories is exact:*

$$\text{Perf}_Z(X) \longleftarrow \text{Perf}(X) \xrightarrow{i^*} \text{Perf}(U)$$

Moreover, the functor i^* induces an equivalence

$$(\text{Perf}(X)/\text{Perf}_Z(X))^{\natural} \longrightarrow \text{Perf}(U)$$

Proof. The restriction functor $i^* : \text{Perf}(X) \rightarrow \text{Perf}(U)$ induces a functor $\text{Perf}(X)/\text{Perf}_Z(X) \rightarrow \text{Perf}(U)$. The functor is fully faithful using Proposition 5.2 and 5.3 in [4]. Also, observe that $\text{Perf}(U)$ is idempotent complete. It follows that the idempotent completion of $\text{Perf}(X)/\text{Perf}_Z(X)$ is a full subcategory of $\text{Perf}(U)$. Then, using Proposition 5.2.(a) in [4], every object of $\text{Perf}(U)$ is a direct summand of the restriction of an object of X which concludes. \square

Remark 2.3. *Let $\mathcal{K} \in 2\text{CAlg}$ and \mathcal{J} be an ideal of \mathcal{K} . The sequence*

$$\mathcal{J} \longleftarrow \mathcal{K} \longrightarrow (\mathcal{K}/\mathcal{J})^{\natural}$$

gives a symmetric monoidal structure on $(\mathcal{K}/\mathcal{J})^{\natural}$. In particular, it gives to this localisation a structure of 2-ring.

2.2. Structural sheaf of the Balmer spectrum.

First, we show that the endomorphism of the unit $\mathcal{O}_U \in \text{Perf}(U)$ has a structure of commutative ring. Then, we sheafify the presheaf of rings $U \mapsto \text{End}_{\text{Perf}(U)}(1_{\mathcal{O}_U})$ to get a sheaf $\mathcal{O}_{\mathcal{X}}$, namely, the structural sheaf associated to the topological space $\text{Spc } \mathcal{K}$.

Lemma 2.4. *Let \mathcal{K} be a 2-ring with unit 1. Then, the endomorphism ring $\text{End}_{\mathcal{K}}(1)$ is commutative.*

Proof. Let $f, g \in \text{End}_{\mathcal{K}}(1)$. Observe that the following diagram is commutative in $\mathcal{K} \times \mathcal{K}$

$$\begin{array}{ccc} (1, 1) & \xrightarrow{(f, id)} & (1, 1) \\ (id, g) \downarrow & \searrow (f, g) & \downarrow (id, g) \\ (1, 1) & \xrightarrow{(f, id)} & (1, 1) \end{array}$$

Apply the functor $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and replace 1×1 by 1 using the fact that both isomorphisms $1 \times 1 \rightarrow 1$ coincide. The result follows. \square

Corollary 2.5. *The association $U \mapsto \text{End}_{\text{Perf}(U)}(1_{\mathcal{O}_U})$ is a presheaf of rings on $\text{Perf}(U)$.*

Definition 2.6. Let X be a qcqs noetherian scheme, $i : U \subset X$ an open subscheme and $Z = X - U$ its closed complement. We denote by $\mathcal{O}_{\text{Perf}(X)}$ the sheafification of the following presheaf of rings

$$U \mapsto \text{End}_{(\text{Perf}(X)/\text{Perf}_Z(X))^{\sharp}}(1).$$

We have defined a locally ringed space

$$\text{Spec}(\text{Spc}(\text{Perf}(X)), \mathcal{O}_{\text{Perf}(X)})$$

Remark 2.7. *It is important to note that we cannot sheafify $U \mapsto \text{End}_{\text{Perf}(U)}(1_{\mathcal{O}_U})$ and get a similar result as $\text{Perf}(U)$ itself doesn't have a structure of 2-ring.*

We can now state the main theorem of this paper.

Theorem 2.8. *Let X be a qcqs noetherian scheme. Then, we have a natural equivalence*

$$\text{Spec}(\text{Spc}(\text{Perf}(X)), \mathcal{O}_{\text{Perf}(X)}) \simeq X$$

Proof. It suffices to show that the structure sheaves are isomorphic as by Corollary 1.24, the underlying topological spaces of $\text{Spc}(\text{Perf}(X))$ and X are homeomorphic. For this sake, we use Theorem 2.2 in the following way. Let $(U_i)_{i \in I}$ be an open basis of X . Then, we can compute $\text{End}_{\text{Perf}(U)}(1_{\mathcal{O}_U}) = \mathcal{O}_U \otimes \check{\mathcal{O}}_U \simeq \mathcal{O}_U \otimes \mathcal{O}_U \simeq \mathcal{O}_U$ using the fact that $\text{Perf}(U)$ is rigid so every object is dualisable. Hence, the structure sheaf $\mathcal{O}_{\text{Perf}(X)}$ coincides with the structure sheaf \mathcal{O}_X on U and as sheaves are determined by their values on a basis, we can conclude that the structure sheaves are isomorphic. \square

Remark 2.9. *The preceding discussion might be generalised when X is a spectral scheme. We proved that when X is a 0-truncated qcqs noetherian spectral scheme, then, we have the equivalence above.*

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