Points of Spec

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Contents

1 Introduction

This talk is intended to present how the points of the Balmer spectra look and behave. The reference is [ABC$^+$25].

Notation 1.1. *1. All the categories and topoi we talk about are in the infinity categorical sense if we don't make additional assumptions.*

2. Rtop, LTop *are categories of topoi, logoi, respectively.* RTop_n *is the category of n-topoi.*
3. $\mathcal{S}$ *is the category of spaces.*
4. *Let X be a topological space. We denote the space-valued sheaf category with respect to the family of open sets on X by* $\mathrm{Shv}(X)$. *The set-valued one will be denoted by* $\mathrm{Shv}(X, \mathrm{Set})$
5. *Let C be a category admitting a Grothendieck topology τ. Then* $\mathrm{Shv}_\tau(C)$ *is the category of τ-sheaves on C.*
6. *Let $\mathcal{X}$ be a topos. Let $\mathcal{C}$ be a category admitting small limits. Then* $\mathrm{Shv}(\mathcal{X}, \mathcal{C}) := \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$.
7. $2\,\mathrm{CAlg}$ *is the category of commutative 2-rings. Let $\mathcal{A}$ be a commutative 2-ring. Then* $\mathrm{Spc}\,\mathcal{A}$ *is the Balmer spectrum of $\mathcal{A}$.*

8. Frm *is the category of frames.*

9. *Let L be a frame. We consider the canonical Grothendieck topology on L: $\{a_i \to a\}_i$ generates a covering sieve if and only if $\bigvee_i a_i = a$. Then we denote the category of sheaves on L by* $\mathrm{Shv}(L)$.

Here is the definition of points in a topos:

Definition 1.2. *Let $\mathcal{X}$ be a topos. A point of $\mathcal{X}$ is a morphism in* RTop*: $\mathcal{S} \to \mathcal{X}$, where $\mathcal{S}$ is the category of spaces.*

Remark 1.3.

1. *We also write the point as a morphism $\mathcal{X} \to \mathcal{S}$ in* LTop *depending on the context.*

2. *$\tau_{\leq 0}\mathcal{X} \to \tau_{\leq 0}\mathcal{S}$ is a point in the classic sense.*

Example 1.4. *Here is an example of the classic version of the points. Let X be a topological space. Let x be a point of X. Then the stalk functor at x is*

$$\begin{aligned}(-)_x^{cl} : \mathrm{Shv}(X, \mathrm{Set}) &\to \mathbf{Set} \\ \mathcal{F} &\mapsto \mathcal{F}_x.\end{aligned}$$

Example 1.5. *Let X be a topological space. Let x be a point of X. Then the stalk functor*

$$\begin{aligned}(-)_x : \mathrm{Shv}(X) &\to \mathcal{S} \\ \mathcal{F} &\mapsto \mathcal{F}_x\end{aligned}$$

is a point. And $(-)_x^{cl} \simeq \tau_{\leq 0}((-)_x)$.

Remark 1.6. *The points we talk about in this remark are in the classic sense. Let X be a topological space. Then the following statements are equivalent:*

1. *X is a sober space, which means that every closed irreducible subset of X has a unique generic point;*

2. *the set of the isomorphic classes of points in* $\mathrm{Shv}(X, \mathrm{Set})$ *is equivalent to X as sets. To be more specific, every point in* $\mathrm{Shv}(X, \mathrm{Set})$ *has the form of a stalk functor $(-)_x^{cl}$ for some $x \in X$.*

2 The points of the Balmer spectra

Definition 2.1. *Let $\mathcal{X}$ be a topos. Let $n \geq 0$ $\mathcal{X}$ is n-localic if and only if*

$$\mathrm{Map}_{\mathrm{Rtop}}(\mathcal{Y}, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Rtop}_n}(\tau_{\leq n-1}\mathcal{Y}, \tau_{\leq n-1}\mathcal{X})$$

for every $\mathcal{Y} \in$ RTop.

Proposition 2.2. *Let $n \geq 0$*

1. *Let C be a n-category admitting finite limits and a Grothendieck topology τ. Then $\mathrm{Shv}_\tau(C)$ is n-localic.*
2. *Let $\mathcal{X}$ be a n-localic topos. Then there exists a n-category C admitting finite limits and a Grothendieck topology τ such that $\mathcal{X} \simeq \mathrm{Shv}_\tau(C)$.*

Proof. [Lur09, Lemma 6.4.5.6, Proposition 6.4.5.7]. □

Example 2.3. *Let X be a topological space. Then $\mathrm{Shv}(X)$ is a 1-localic topos.*

$$\mathrm{Map}_{\mathrm{RTop}}(\mathcal{S}, \mathrm{Shv}(X)) \simeq \mathrm{Map}_{\mathrm{RTop}_1}(\mathrm{Set}, \mathrm{Shv}(X, \mathrm{Set})).$$

In this case, the classic version of points is the same as that in ∞-topoi.

Remark 2.4. *If X is a sober space. Then the set of the isomorphic classes of points in $\mathrm{Shv}(X)$ is equivalent to X as sets. To be more specific, every point in $\mathrm{Shv}(X)$ has the form of a stalk functor $(-)_x$ for some $x \in X$.*

Let's get back to the main theme.

Notation 2.5. *$\mathcal{G}_{\mathrm{Zar}}$ is the Zariski geometry on $2\,\mathrm{CAlg}$.*

Last time, we proved the following statement:

Proposition 2.6. *Let $\mathcal{A}$ be a commutative 2-ring. Then $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A} \simeq \mathrm{Shv}(\mathrm{Spc}\,\mathcal{A})$.*

Remark 2.7. *The absolute $\mathcal{G}_{\mathrm{Zar}}$-structured spectrum of $\mathcal{A}$ is $(\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A}, \mathcal{O}_{\mathcal{A}})$, where $\mathcal{O}_{\mathcal{A}}$ is*

$$\mathcal{G}_{\mathrm{Zar}} \xrightarrow{\bar{\mathcal{O}}_{\mathcal{A}}} \mathcal{P}(\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A},$$

where $\bar{\mathcal{O}}_{\mathcal{A}}$ is represented by the forgetful functor $(\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}})^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{G}_{\mathrm{Zar}}^{\mathrm{op}}) \simeq 2\,\mathrm{CAlg}$, the second functor is the sheafification functor.

Corollary 2.8. *Let $\mathcal{A}$ be a commutative 2-ring. The following descriptions are equivalent:*

1. *A point in $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A}$: $\mathcal{S} \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A}$.*
2. *A point in $\mathrm{Spc}\,\mathcal{A}$: a prime ideal $\mathcal{P} \subseteq \mathcal{A}$.*

Proof. $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A} \simeq \mathrm{Shv}(\mathrm{Spc}\,\mathcal{A})$ is 1-localic. Then

$$\mathrm{Map}_{\mathrm{RTop}}(\mathcal{S}, \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A}) \simeq \mathrm{Map}_{\mathrm{RTop}_1}(\mathrm{Set}, \mathrm{Shv}(\mathrm{Spc}\,\mathcal{A}, \mathrm{Set})) \simeq \mathrm{Spc}\,\mathcal{A}$$

as sets, since $\mathrm{Spc}\,\mathcal{A}$ is a sober space. □

3 Locally 2-ringed topoi

Notation 3.1. *1. Let $\mathcal{G}$ be a geometry. Let $\mathcal{X}$ be a topos. Then $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X})$ is the category of $\mathcal{G}$-structures on $\mathcal{X}$. And $\mathrm{LTop}(\mathcal{G})$ is the category of $\mathcal{G}$-structured logoi.*

2. We denote the Zariski geometry on $2\,\mathrm{CAlg}$ by $\mathcal{G}_{\mathrm{Zar}}$, the discreet geometry by $\mathcal{G}_{\mathrm{disc}}$.

Definition 3.2. *1. $\mathrm{LTop}_{2\,\mathrm{CAlg}} := \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}})$ is the category of 2-ringed logoi. $\mathrm{RTop}_{2\,\mathrm{CAlg}} := \mathrm{LTop}^{\mathrm{op}}_{2\,\mathrm{CAlg}}$ is the category of 2-ringed topoi.*

2. $\mathrm{LTop}^{\mathrm{loc}}_{2\,\mathrm{CAlg}} := \mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$ is the category of locally 2-ringed logoi. $\mathrm{RTop}^{\mathrm{loc}}_{2\,\mathrm{CAlg}} := (\mathrm{LTop}^{\mathrm{loc}}_{2\,\mathrm{CAlg}})^{\mathrm{op}}$ is the category of locally 2-ringed topoi.

Remark 3.3. $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{S}) \hookrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Zar}}, \mathcal{S}) \simeq 2\,\mathrm{CAlg}$.

3.1 Commutative local 2-rings

Now we will introduce the notion of 'localness' that could describe the elements of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{S})$ as commutative 2-rings.

Definition 3.4. *Let $\mathcal{A}$ be a commutative 2-ring. If $\{0\} \subseteq \mathcal{A}$ is a prime ideal, then we say that $\mathcal{A}$ is local. $2\,\mathrm{CAlg}^{\mathrm{loc}} \subseteq 2\,\mathrm{CAlg}$ is the subcategory of local commutative 2-rings, in which the morphisms are conservative maps.*

Proposition 3.5. $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{S}) \simeq 2\,\mathrm{CAlg}^{\mathrm{loc}}$.

Proof. First, we need to check that $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{S}) \subseteq 2\,\mathrm{CAlg}^{\mathrm{loc}}$.

Let $\mathcal{A}$ be a $\mathcal{G}_{\mathrm{Zar}}$-structure on $\mathcal{S}$. Let $\{\mathcal{B} \to \mathcal{B}/\langle b_i \rangle\}_i$ be a finite admissible covering in $\mathcal{G}_{\mathrm{Zar}}$. Here $\otimes_i b_i \in \sqrt{0}$. Then the map

$$\bigsqcup_i \pi_0 \,\mathrm{Map}(\mathcal{B}/\langle b_i \rangle, \mathcal{A}) \to \pi_0 \,\mathrm{Map}(\mathcal{B}, \mathcal{A})$$

is surjective. We assume $(\otimes_{1 \le i \le k} a_i)^{\otimes l} \simeq 0$ in $\mathcal{A}$. Then

$$\{\mathrm{Sp}^{\omega}\{x_1, ..., x_k\}/\langle (\otimes_j x_j)^{\otimes l} \rangle \to (\mathrm{Sp}^{\omega}\{x_1, ..., x_k\}/\langle (\otimes_j x_j)^{\otimes l} \rangle)/\langle x_i \rangle\}_{1 \le i \le k}$$

is a finite admissible covering. Let $f : \mathrm{Sp}^{\omega}\{x_1, ..., x_k\}/\langle (\otimes_j x_j)^{\otimes l} \rangle \to \mathcal{A}$ be the morphism mapping x_i to a_i. Then there exists i_0 such that f could be factored into

$$\mathrm{Sp}^{\omega}\{x_1, ..., x_k\}/\langle (\otimes_j x_j)^{\otimes l} \rangle \to (\mathrm{Sp}^{\omega}\{x_1, ..., x_k\}/\langle (\otimes_j x_j)^{\otimes l} \rangle)/\langle x_{i_0} \rangle \to \mathcal{A}.$$

It means that $a_{i_0} \simeq 0$ in $\mathcal{A}$. Then $\mathcal{A}$ is prime.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{G}_{\text{Zar}}$-structures on $\mathcal{S}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism between $\mathcal{G}_{\text{Zar}}$-structures. Then we have the following Cartesian diagram:

$$\begin{tikzcd}
\operatorname{Map}(\mathcal{C}/\langle c\rangle, \mathcal{A}) \arrow[r] \arrow[d] & \operatorname{Map}(\mathcal{C}/\langle c\rangle, \mathcal{B}) \arrow[d] \\
\operatorname{Map}(\mathcal{C}, \mathcal{A}) \arrow[r] & \operatorname{Map}(\mathcal{C}, \mathcal{B})
\end{tikzcd}$$

for $\mathcal{C} \in 2\operatorname{CAlg}^{\omega}$ and $c \in \mathcal{C}$. Let's assume $\mathcal{C} \simeq \operatorname{Sp}^{\omega}\{x\}$ and $c \simeq x$. Then the diagram above shows us that if $f(a) \simeq 0$ in $\mathcal{B}$ then $a \simeq 0$ in $\mathcal{A}$. So f is conservative.

Now, we need to prove $2\operatorname{CAlg}^{\text{loc}} \subseteq \operatorname{Str}^{\text{loc}}_{\mathcal{G}_{\text{Zar}}}(\mathcal{S})$.

Let $\mathcal{A}$ be a commutative local 2-ring. Let $\mathcal{B}$ be a compact object in $2\operatorname{CAlg}$ and $\{\mathcal{B} \to \mathcal{B}/\langle b_i\rangle\}_{1\leq i\leq k}$ be a finite admissible covering, where $\otimes_i b_i \in \sqrt{0}$. So we assume $(\otimes_i b_i)^{\otimes l} \simeq 0$. Then we have the morphism $f : \operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle \to \mathcal{B}$ mapping x_i to b_i. So we have the following commutative diagram:

$$\begin{tikzcd}
\bigsqcup_i \pi_0 \operatorname{Map}(\mathcal{B}/\langle b_i\rangle, \mathcal{A}) \arrow[r] \arrow[d] & \pi_0 \operatorname{Map}(\mathcal{B}, \mathcal{A}) \arrow[d] \\
\bigsqcup_i \pi_0 \operatorname{Map}((\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle)/\langle x_i\rangle, \mathcal{A}) \arrow[r] & \pi_0 \operatorname{Map}(\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle, \mathcal{A})
\end{tikzcd}$$

The horizontal map at the bottom is surjective, since $\{0\}$ is prime in $\mathcal{A}$. Let g be an arbitrary morphism from $\mathcal{B}$ to $\mathcal{A}$. Then there exists i_0 such that $f \circ g$ could be factored into

$$\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle \to (\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle)/\langle x_{i_0}\rangle \to \mathcal{A}.$$

And we have the following Cartesian diagram:

$$\begin{tikzcd}
\operatorname{Map}(\mathcal{B}/\langle b_{i_0}\rangle, \mathcal{A}) \arrow[r] \arrow[d] & \operatorname{Map}(\mathcal{B}, \mathcal{A}) \arrow[d] \\
\operatorname{Map}((\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle)/\langle x_{i_0}\rangle, \mathcal{A}) \arrow[r] & \operatorname{Map}(\operatorname{Sp}^{\omega}\{x_1, ..., x_k\}/\langle(\otimes_j x_j)^{\otimes l}\rangle, \mathcal{A}).
\end{tikzcd}$$

So there exists $g_0 : \mathcal{B}/\langle b_{i_0}\rangle \to \mathcal{A}$ such that g could be factorized into

$$\mathcal{B} \to \mathcal{B}/\langle b_{i_0}\rangle \xrightarrow{g_0} \mathcal{A}.$$

Then $\bigsqcup_i \pi_0 \operatorname{Map}(\mathcal{B}/\langle b_i\rangle, \mathcal{A}) \to \pi_0 \operatorname{Map}(\mathcal{B}, \mathcal{A})$ is surjective. So $\mathcal{A}$ is a $\mathcal{G}_{\text{Zar}}$-structure on $\mathcal{S}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative local 2-rings. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism between commutative local 2-rings. It suffices to prove the following diagram is Cartesian:

$$\begin{tikzcd}
\operatorname{Map}(\mathcal{C}/\langle c\rangle, \mathcal{A}) \arrow[r] \arrow[d] & \operatorname{Map}(\mathcal{C}/\langle c\rangle, \mathcal{B}) \arrow[d] \\
\operatorname{Map}(\mathcal{C}, \mathcal{A}) \arrow[r] & \operatorname{Map}(\mathcal{C}, \mathcal{B})
\end{tikzcd}$$

for every $\mathcal{C} \in 2\,\mathrm{CAlg}^{\omega}$ and $c \in \mathcal{C}$. We choose $g : \mathcal{C} \to \mathcal{A}$ in $\mathrm{Map}(\mathcal{C}, \mathcal{A})$. Then

$$\{g\} \times_{\mathrm{Map}(\mathcal{C},\mathcal{A})} \mathrm{Map}(\mathcal{C}/\langle c\rangle, \mathcal{A}) \simeq \begin{cases} *, & g(c) \simeq 0 \\ \emptyset, & \text{otherwise} \end{cases}$$

$$\{f \circ g\} \times_{\mathrm{Map}(\mathcal{C},\mathcal{B})} \mathrm{Map}(\mathcal{C}/\langle c\rangle, \mathcal{B}) \simeq \begin{cases} *, & f \circ g(c) \simeq 0 \\ \emptyset, & \text{otherwise} \end{cases}.$$

So $\{g\} \times_{\mathrm{Map}(\mathcal{C},\mathcal{A})} \mathrm{Map}(\mathcal{C}/\langle c\rangle, \mathcal{A}) \simeq \{f \circ g\} \times_{\mathrm{Map}(\mathcal{C},\mathcal{B})} \mathrm{Map}(\mathcal{C}/\langle c\rangle, \mathcal{B})$, since f is conservative. Then the statement is proved. □

Construction 3.6. Let $\mathcal{G}$ be a geometry. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a morphism in LTop. Then we have the following adjunctions:

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{Y}),$$

which are induced by postcomposing f^*, f_*. This pair of adjunctions is equivalent to

$$\mathrm{Shv}(\mathcal{X}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Shv}(\mathcal{Y}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})).$$

Restricting f^* on $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X})$, we get $f^* : \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{Y})$.

3.2 Sheaves on frames

Proposition 3.7. *We have the following adjunctions:*

$$\mathbf{Frm} \underset{\tau_{\leq -1}}{\overset{\mathrm{Shv}(\text{-})}{\rightleftarrows}} \mathbf{LTop}.$$

And the left adjoint is fully faithful. The essential image of the left adjoint is the category of 0-localic topoi.

Proof. Apply [Lur09, Theorem 6.4.2.1] and proposition 2.2. □

Remark 3.8. *Let L be a frame. Then*

$$\mathrm{Map}_{\mathrm{LTop}}(\mathrm{Shv}(L), \mathcal{S}) \simeq \mathrm{Map}_{\mathrm{LTop}_n}(L, \tau_{\leq -1}\mathcal{S}) \simeq \mathrm{Map}_{\mathrm{Frm}}(L, [1]).$$

So a point in $\mathrm{Shv}(L)$ *is equivalent to a morphism of frames* $L \to [1]$.

Let $x^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A} \to \mathcal{S}$ be a point. We want to compute $x^*\mathcal{O}_{\mathcal{A}}$. So we need $\bar{\mathcal{O}}_{\mathcal{A}}$ and the following lemma:

Lemma 3.9. *Let L be a frame. $x : L \to [1]$ is a point. Let $\mathcal{G}$ be a small category admitting finite limits. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Fun}(L^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \arrow[r, "x_!"] \arrow[d] & \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \arrow[d] \\
\mathrm{Shv}(L; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \arrow[r, "x^*"'] & \mathrm{Shv}([1]; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}),
\end{tikzcd}$$

where $x_!$ is the left Kan extension functor along $x^{\mathrm{op}} : L^{\mathrm{op}} \to [1]^{\mathrm{op}}$ and the two vertical functors are sheafification functors.

Remark 3.10. *x^* is equivalent to*

$$\mathrm{Shv}(\mathrm{Shv}(L), \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \xrightarrow{x^*} \mathrm{Shv}(\mathrm{Shv}([1]), \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))$$

induced by the logoi morphism $x^* : \mathrm{Shv}(L) \to \mathrm{Shv}([1]) \simeq \mathcal{S}$.

If $\mathcal{G} \simeq *$. Then $x^* \simeq \mathrm{Shv}(x) : \mathrm{Shv}(L) \to \mathrm{Shv}([1])$.

Proof. It suffices to prove the case when $\mathcal{G} \simeq *$, since we only need to use $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, -)$ to get the general result.

We have the following commutative diagram:

$$\begin{tikzcd}
L \arrow[r, "x"] \arrow[d, "j_1"] & {[1]} \arrow[d, "j_2"] \\
\mathrm{Fun}(L^{\mathrm{op}}, \mathcal{S}) \arrow[r, "x_{\#}"] \arrow[d, "a_1"] & \mathrm{Fun}([1]^{\mathrm{op}}, \mathcal{S}) \arrow[d, "a_2"] \\
\mathrm{Shv}(L) \arrow[r, "\mathrm{Shv}(x)"] & \mathrm{Shv}([1]),
\end{tikzcd}$$

where j_1 and j_2 are Yoneda embeddings, a_1 and a_2 are sheafification functors, and $x_{\#}$ is the left Kan extension of $j_2 x$ along j_1. Both of $x_{\#}$ and x^* are left adjoints of precomposing functor $x^{\mathrm{op}} \circ (-) : \mathrm{Fun}([1]^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(L^{\mathrm{op}}, \mathcal{S})$. So $x^* \simeq x_{\#}$. □

3.3 The map of points

Lemma 3.11. *Let $\mathcal{A}$ be a commutative 2-ring. Let $x^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A} \to \mathcal{S}$ be a point represented by the prime ideal $\mathcal{P}$ of $\mathcal{A}$. Then $x^* \mathcal{O}_{\mathcal{A}} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{S})$ is represented by $\mathcal{A}/\mathcal{P}$.*

Proof.

$$\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}} \to \mathcal{P}(\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A},$$

where the first functor is the Yoneda embedding, the second one is the sheafification functor. It induces the map $i : \mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}} \to \tau_{\leq -1} \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}} \mathcal{A}$.

We have the following diagrams:

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}}, 2\,\mathrm{CAlg}) \arrow[r, "f_1"] \arrow[d, "a_1"] & \mathrm{Fun}(\tau_{\leq -1}\,\mathrm{Spec}^{\mathcal{G}}_{\mathrm{Zar}}\,\mathcal{A}, 2\,\mathrm{CAlg}) \arrow[r, "f_2"] \arrow[d, "a_2"] & \mathrm{Fun}([1]^{\mathrm{op}}, 2\,\mathrm{CAlg}) \arrow[d, "a_3"] \\
\mathrm{Shv}(\mathrm{Spec}^{\mathcal{G}}_{\mathrm{Zar}}\,\mathcal{A}, 2\,\mathrm{CAlg}) \arrow[r, "\mathrm{id}"] & \mathrm{Shv}(\mathrm{Spec}^{\mathcal{G}}_{\mathrm{Zar}}\,\mathcal{A}, 2\,\mathrm{CAlg}) \arrow[r, "x^*"] & \mathrm{Shv}([1]^{\mathrm{op}}, 2\,\mathrm{CAlg}),
\end{tikzcd}$$

where a_1, a_2, a_3 are sheafification functors, f_1, f_2 are left Kan extension functors along i, $\tau_{\leq -1}x^*$, respectively.

The right diagram is commutative according to the previous lemma. The left one could be proved to be commutative by a similar argument. And $a_1\bar{\mathcal{O}}_{\mathcal{A}} \simeq \mathcal{O}_{\mathcal{A}}$ because of [ABC$^+$25, Lemma 3.27]: $\mathcal{O}_{\mathcal{A}}$ is the sheafification of the forgetful functor $\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}} \to 2\,\mathrm{CAlg}$.

$\mathrm{Pro}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{ad}}_{/\mathcal{A}} \xrightarrow{i} \tau_{\leq -1}\,\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{A} \xrightarrow{\tau_{\leq -1}x^*} [1]$ is

$$\tau_{\leq -1}x^* \circ i(\mathcal{I}) = \begin{cases} \{1\}, & \mathcal{P} \supseteq \sqrt{\mathcal{I}} \\ \{0\}, & \text{otherwises,} \end{cases}$$

here $\mathcal{I} \in \mathrm{Prin}(\mathcal{A})$. We know that $\bar{\mathcal{O}}_{\mathcal{A}}(\mathcal{I}) \simeq \mathcal{A}/\mathcal{I}$. Then

$$x^*\mathcal{O}_{\mathcal{A}} \simeq \operatorname*{colim}_{\sqrt{\mathcal{I}} \in \mathrm{Prin}(\mathcal{A})_{/\mathcal{P}}} \mathcal{A}/\mathcal{I} \simeq \operatorname*{colim}_{\mathcal{I} \in \mathrm{Rad}(\mathcal{A})^{\omega}_{/\mathcal{P}}} \mathcal{A}/\mathcal{I} \simeq \mathcal{A}/\mathcal{P}$$

the first equivalence is according to the above commutative diagram, the middle one follows from cofinality, and the last one follows from the fact that $\mathcal{A}/(-)$ preserves colimits and $\mathrm{Rad}(\mathcal{A})$ is a coherent frame. ☐

Now, we could characterize the map of points induced by the morphism of commutative 2-rings $\mathcal{A} \to \mathcal{B}$.

Proposition 3.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative 2-rings. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism between commutative 2-rings. Let $x^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{B} \to \mathcal{S}$ be a point represented by $\mathcal{P}$. Then $x^*f^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{A} \to \mathcal{S}$ is presented by $f^{-1}\mathcal{P}$, where $f^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{A} \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{B}$ is a morphism in* LTop *and induced by f.*

Proof. f induces the morphism between $\mathcal{G}_{\mathrm{Zar}}$-structured spectra: $(f^*, u_f) : (\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to (\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{B}, \mathcal{O}_{\mathcal{B}})$ in $\mathrm{LTop}^{\mathrm{loc}}_{2\,\mathrm{CAlg}}$, where $f^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{A} \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\,\mathcal{B}$, $u_f : f^*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ in $\mathrm{Fun}(\mathrm{Shv}(\mathrm{Spc}\,\mathcal{B})^{\mathrm{op}}, 2\,\mathrm{CAlg})$.

$x^*u_f : x^*f^*\mathcal{O}_{\mathcal{A}} \to x^*\mathcal{O}_{\mathcal{B}}$ is a morphism in $2\,\mathrm{CAlg}^{\mathrm{loc}}$. We assume x^*f^* is represented by the prime ideal $\mathcal{Q}$ in $\mathcal{A}$. Then x^*u_f is presented by the morphism of commutative local 2-rings: $\mathcal{A}/\mathcal{Q} \to \mathcal{B}/\mathcal{P}$. This is conservative and fits in the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{A}/\mathcal{Q} \arrow[r] & \mathcal{B}/\mathcal{P}.
\end{tikzcd}$$

So $\mathcal{Q} = f^{-1}\mathcal{P}$. ☐

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