Lurie's geometries and $\mathcal{G}$-structured ∞-topoi II

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Contents

Notations and Categorical convention. Through this note, we always means ∞-categories for categories and ∞-topoi for topoi.

• We will denote by An the category of animae(spaces).

• We will denote by LTop the category of topoi, whose objects are topoi and whose morphisms are functors between topoi that admit left exact colimit-preserving left adjoints.

• $\mathrm{Fun}^{\mathrm{R}}(X, Y) \subseteq \mathrm{Fun}(\mathrm{X}, \mathrm{Y})$ denotes the full subcategories of functors that admits right adjoints.

• $\mathrm{Map}_{\mathcal{C}}$ denotes the mapping anima in the category $\mathcal{C}$.

1 Recap: $\mathcal{G}$-structures

In this section We will review some definitions of last talk that will be used.

Definition 1.0.1. Let $\mathcal{G}$ be a mall catgeory, $\mathcal{G}^{\mathrm{ad}}$ a wide subcategory of $\mathcal{G}$, and τ a Grothendieck topology on $\mathcal{G}$. We say that $(\mathcal{G}^{ad}, \tau)$ is an admissibility structure on $\mathcal{G}$ or that $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a geometry if the conditions below are satisfied:

- $\mathcal{G}$ has finite limits and is idempotent-complete;
- τ is generated by morphisms in $\mathcal{G}^{\mathrm{ad}}$;
- $\mathcal{G}^{\mathrm{ad}}$ is closed under base changes in $\mathcal{G}$;
- If f and g are composable morphisms in $\mathcal{G}$ satisfying $g, g \circ f \in \mathcal{G}^{\mathrm{ad}}$, then $f \in \mathcal{G}^{\mathrm{ad}}$;
- If f is a retract of g in $\mathcal{G}^{[1]}$ satisfying $g \in \mathcal{G}^{\mathrm{ad}}$, then $f \in \mathcal{G}^{\mathrm{ad}}$.

We refer to morphisms in $\mathcal{G}^{\mathrm{ad}}$ admissible morphisms and covers in τ admissible covers.

Example 1.0.1. Consider an arbitrary idempotent-complete category $\mathcal{G}$ that admits finite limits. There is an admissibility structure where the only admissible morphisms are equivalences and the topology is discrete. We call this the discrete geometry and denotes $\mathcal{G}_{\mathrm{disc}}$ for it.

Definition 1.0.2. A morphism between geometries is a functor between the underlying categories which preserves finite limits, admissible morphisms and admissible covers.

Example 1.0.2. Consider a geometry $\mathcal{G}$. Then the identity functor determines a morphism $\mathcal{G}_{\mathrm{disc}} \to \mathcal{G}$.

Recall that for an topos $\mathcal{X}$ and a category $\mathcal{C}$ with small limits, the category of sheaves on $\mathcal{X}$ with coefficients in $\mathcal{C}$ is defined as $\mathrm{Shv}(\mathcal{X};\mathcal{C}) := \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$. Suppose that $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ for a small category $\mathcal{G}$, by the adjoint functor theorem we have $\mathrm{Shv}(\mathcal{X};\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ and we have the following identification

$$\begin{aligned}\mathrm{Shv}(\mathcal{X};\mathcal{C}) &\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \\ &\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An})) \\ &\simeq \mathrm{Fun}^{\mathrm{R,lex}}(\mathcal{X}^{\mathrm{op}} \times \mathcal{G}, \mathrm{An}) \\ &\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{An})) \\ &\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}).\end{aligned}$$

This equivalence sends a sheaf $\mathcal{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ to the left exact functor $\mathrm{Map}_{\mathcal{C}}(-, \mathcal{F}) : \mathcal{G} \to \mathcal{X} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{An}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{An})$, $g \mapsto (X \mapsto \mathrm{Map}_{\mathcal{C}}(y(g), \mathcal{F}(X)))$, where y is the yoneda embedding.

Definition 1.0.3. Let $\mathcal{X}$ be a topos and $\mathcal{G}$ a geometry.

(a) A **$\mathcal{G}$-structure** on $\mathcal{X}$ is a left exact functor $\mathcal{O} : \mathcal{G} \to \mathcal{X}$ such that for any admissible cover $\{U_i \to X\}_{i \in I}$ the induced map $\bigsqcup_{i \in I} \mathcal{O}(U_i) \to \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$. We denote $\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \subseteq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$ for the full subcategory spanned by the $\mathcal{G}$-structures.

(b) Let $\mathcal{O}$ and $\mathcal{O}'$ be two $\mathcal{G}$-structures on $\mathcal{X}$. A morphism $\alpha : \mathcal{O} \to \mathcal{O}'$ is said to be **local** If for any admissible map $U \to X$, the natural diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}'(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{O}'(X) \end{array}$$

is a pullback square. We write $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X})$ for the wide subcategory of $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ spanned by local morphisms.

Definition 1.0.4. Fix a geometry $\mathcal{G}$. We define the category $\mathrm{LTop}(\mathcal{G})$ of $\mathcal{G}$-structured topoi as

(a) An object $(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G})$ if and only if $\mathcal{X}$ is a topos and the functor $\mathcal{O} : \mathcal{G} \to \mathcal{X}$ is a $\mathcal{G}$-structure.

(b) A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \mathrm{LTop}(\mathcal{G})$[1] if and only if $f^* : \mathcal{X} \to \mathcal{Y}$ is in LTop and $f^\sharp : f^*\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$ is a local morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{Y})$, that is for every admissible morphism $U \to X$ in $\mathcal{G}$, the diagram below is a pulback square in $\mathcal{Y}$

$$\begin{array}{ccc} f^*\mathcal{O}_{\mathcal{X}}(U) & \longrightarrow & \mathcal{O}_{\mathcal{Y}}(U) \\ \downarrow & & \downarrow \\ f^*\mathcal{O}_{\mathcal{X}}(X) & \longrightarrow & \mathcal{O}_{\mathcal{Y}}(X). \end{array}$$

Formally, $\mathrm{LTop}(\mathcal{G})$ is defined as the unstraightening of the functor

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(-) : \mathrm{LTop} \to \mathrm{Cat}.$$

From these definitions we have $\mathrm{LTop}(\mathcal{G}) \times_{\mathrm{LTop}} \{\mathcal{X}\} \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X})$ as the condition (b) is automatically true.

2 Affine spectra

2.1 Relative spectrum functor as left adjoint

Before stating the definition of the relative spectrum functor, let us see the construction of the restriction funtor between the categories of $\mathcal{G}$-structured topoi that is induced by a morphism of geometries.

Construction 2.1.1. Consider a morphism of geometries $\alpha : \mathcal{G} \to \mathcal{G}'$. Precomposition with α gives a natural transformation $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}'}(-) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(-)$ since α preserves admissible morphisms and admissible covers. Taking unstraightening we get a functor $\mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G})$.

Explicitly, as we see that the definition of $\mathcal{G}$-structured topoi Definition 1.0.4 involves admissible morphisms, admissible cover and finite limits, and α as a morphism between geometries preserves finite limits, admissible morphisms and admissible covers, $\mathcal{O} \circ \alpha$ also satisfies the definition of $\mathcal{G}$-structured topoi. We may restrict the precomposing functor to obtain the following map $\mathrm{res}_\alpha : \mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G})$. We call refer to this map as the **restriction along** α, and it is pointwise given by $(\mathcal{X}, \mathcal{O}) \mapsto (\mathcal{X}, \mathcal{O} \circ \alpha)$.

The key result for relative spectrum functors is the following theorem [Lur09a, Theorem 2.1.1]:

Theorem 2.1.1. *For a morphism $\mathcal{G} \to \mathcal{G}'$ of geometries, the restrion functor we defined above $\mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G})$ admits a left adjoint, denoted by $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}$. We refer to this functor as the **relative spectrum functor**.*

2.2 $\mathcal{G}$-structured global sections functor

From now on we will invetigate to the case of discrete geometries and fix a geometry $\mathcal{G} = (\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$.

For a topos $\mathcal{X}$ we can define the global sections functor $\Gamma : \mathcal{X} \to \mathrm{An}$ as the functor corepresented by the terminal object $\mathbf{1}$ of $\mathcal{X}$. Now consider the functor

$$\mathrm{LTop}(\mathcal{G}) \times \mathcal{G} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \times \mathcal{G} \xrightarrow{\mathrm{ev}} \mathcal{X} \xrightarrow{\Gamma} \mathrm{An},$$

which is pointwise given by $((\mathcal{X}, \mathcal{O}), X) \mapsto \mathrm{Hom}_{\mathcal{X}}(\mathbf{1}, \mathcal{O}(X)))$. The currying of this functor produces a functor $\mathrm{LTop}(\mathcal{G}) \to \mathrm{Fun}(\mathcal{G}, \mathrm{An})$ which factors through $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An}) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ since $\mathcal{O}$ and $\mathrm{Hom}_{\mathcal{X}}(\mathbf{1}, -)$ are all left exact functor.

$$\begin{array}{ccc} \mathrm{LTop}(\mathcal{G}) & \longrightarrow & \mathrm{Fun}(\mathcal{G}, \mathrm{An}) \\ {\scriptstyle \Gamma_{\mathcal{G}}}\downarrow & & \uparrow \\ \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An}) & \xrightarrow{\simeq} & \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \end{array}$$

We refer to $\Gamma_{\mathcal{G}}$ as the **$\mathcal{G}$-structured global sections functor**. Now we want to see the restriction of $\Gamma_{\mathcal{G}}$ to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{LTop}(\mathcal{G}) \times_{\mathrm{LTop}} \{\mathcal{X}\}$:

Lemma 2.2.1. *Given a topos $\mathcal{X}$, the composite*

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{LTop}(\mathcal{G}) \times_{\mathrm{LTop}} \{\mathcal{X}\} \hookrightarrow \mathrm{LTop}(\mathcal{G}) \xrightarrow{\Gamma_{\mathcal{G}}} \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$$

is identified with the evaluation functor

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \subseteq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \xrightarrow{\sim} \mathrm{Shv}(\mathcal{X}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \xrightarrow{\mathrm{ev}_{\mathbf{1}}} \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$$

where $\mathrm{ev}_{\mathbf{1}}$ sends a sheaf $\mathcal{F}$ to its value on the terminal object $\mathbf{1} \in \mathcal{X}$.

Proof. It suffices to show that the map $\Gamma \circ (-) : \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An})$ identifies with the map $\mathrm{ev}_{\mathbf{1}}$. By the definition of Γ we have the following commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Gamma} & \mathrm{An} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{An}) & \xrightarrow{(-)\circ\Gamma^*} & \mathrm{Fun}^{\mathrm{R}}(\mathrm{An}^{\mathrm{op}}, \mathrm{An}) \end{array}$$

where Γ^* denotes the left adjoint of Γ and the vertical equivalences are induced from the Yoneda embedding.

Now we consider the embedding $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, -) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{Fun}^{\mathrm{R}}((-)^{\mathrm{op}}, \mathrm{An})) \hookrightarrow \mathrm{Fun}(\mathcal{G} \times (-)^{\mathrm{op}}, \mathrm{An})$. The map $\Gamma \circ (-)$ is identified with the restriction of

$$(-) \circ (\mathrm{id} \times \Gamma^*) : \mathrm{Fun}(\mathcal{G} \times \mathcal{X}^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Fun}(\mathcal{G} \times \mathrm{An}^{\mathrm{op}}, \mathrm{An})$$

and when rectricting to $\mathrm{Shv}(-;\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \hookrightarrow \mathrm{Fun}(\mathcal{G}\times(-)^{\mathrm{op}},\mathrm{An})$ it is identified with the functor

$$\mathrm{Shv}(\mathcal{X};\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \to \mathrm{Shv}(\mathrm{An};\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$$

given by sending a sheaf $\mathcal{F}$ to its value on $\Gamma^*(*) \simeq \mathbf{1} \in \mathcal{X}$. □

A important recipe of the functor $\Gamma_{\mathcal{G}}$ is the following adjunction:

Lemma 2.2.2. *The functor $\Gamma_{\mathcal{G}}$ is right adjoint to the inclusion*

$$\mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{disc}}}(\mathrm{An}) \simeq \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}}) \times_{\mathrm{LTop}} \{\mathrm{An}\} \subseteq \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}}).$$

Proof. We note that for every $(\mathcal{X},\mathcal{O}) \in \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}})$, the map $(\mathrm{An},\mathcal{O}') \to (\mathcal{X},\mathcal{O}) \in \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}})^{[1]}$ is identified with the map $\Gamma^*\circ\mathcal{O}' \to \mathcal{O} \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathcal{X})^{[1]}$. Moreover using the adjunction $\Gamma^* \dashv \Gamma$ this map is equivalent to $\mathcal{O}' \to \Gamma\circ\mathcal{O} = \Gamma_{\mathcal{G}}(\mathcal{X},\mathcal{O}) \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathrm{An}) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. From theses relations and [Lur09b, Proposition 5.2.7.8] we know that the inclusion $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \hookrightarrow \mathrm{LTop}(\mathcal{G}_{\mathrm{disc}})$ is a fully faithful left adjoint, with right adjoint $\Gamma_{\mathcal{G}}$. □

Combining Theorem 2.1.1 and Lemma 2.2.2 we obtain:

Lemma 2.2.3. *There is an adjunction of the form*

$$\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}_{\mathrm{disc}}} \dashv \Gamma_{\mathcal{G}} : \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \rightleftarrows \mathrm{LTop}(\mathcal{G}).$$

2.3 Absolute spectrum

The next goal is to compute $\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}_{\mathrm{disc}}}$ explicitly.

Definition 2.3.1. A morphism $f : U \to X$ in $\mathrm{Pro}(\mathcal{G})$ is called pro-admissible if there exists a pushout square in $\mathrm{Pro}(\mathcal{G})$ of the form

$$\begin{array}{ccc} U & \longrightarrow & j(U') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j(f')} \\ X & \longrightarrow & j(X') \end{array}$$

where $f' : U' \to X'$ is an admissible morphism in $\mathcal{G}$ and j the yoneda embedding.

We denote $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}$ for the wide subcategory of pro-admissible morphism in $\mathrm{Pro}(\mathcal{G})$. From [Lur09a, Lemma 2.2.4] we know that $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}$ contains the equivalences, that is stable under pullbacks and compositions, and satisfies left cancellation property.

For $X \in \mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}$, let $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}} \subseteq \mathrm{Pro}(\mathcal{G})_{/\mathrm{X}}$ be the full subcategory spanned by the pro-admissible morphisms. Furthermore, we can endow $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}$ with the coarsest Grothendick topology: for any $U \to X$, every admissible cover $\{V_i' \to U'\}_{i\in I}$ in $\mathcal{G}$, and every morphism $U \to j(U')$ in $\mathrm{Pro}(\mathcal{G})$, the collection $\{j(V_i')\times_{j(U')} U \to U\}_{i\in I}$ generates a covering sieve on $U \in \mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}$.

Now we can define the absolute $\mathcal{G}$-spectrum:

Definition 2.3.2. Consider $\mathcal{G}$ a geometry and X an object of $\mathrm{Pro}(\mathcal{G})$, the **absolute $\mathcal{G}$-spectrum** of X is defined as the following pair

$$\mathrm{Spec}^{\mathcal{G}}(\mathrm{X}) := (\mathrm{Shv}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{\mathrm{X}}), \mathcal{O}_{\mathrm{X}}),$$

$\mathcal{O}_X$ is defined as the composition

$$\mathcal{O}_X : \mathcal{G} \xrightarrow{\widetilde{\mathcal{O}}_X} \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}) \xrightarrow{L} \mathrm{Shv}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}) = \mathrm{Spec}^{\mathcal{G}}(\mathrm{X})$$

where $\widetilde{\mathcal{O}}_X$ is the currying of

$$\mathcal{G}\times(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathcal{G}\times(\mathrm{Pro}(\mathcal{G}))^{\mathrm{op}} = \mathcal{G}\times\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathrm{An}) \xrightarrow{\mathrm{ev}} \mathrm{An}$$

and L is the left adjoint of the canonical inclusion $\mathrm{Shv}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}) \subseteq \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})$, that is the sheafification of the topology described above.

[Lur09a, Proposition 2.2.11] told us that this definition gives a $\mathcal{G}$-structure on $\mathrm{Spec}^{\mathcal{G}}(\mathrm{X})$:

Proposition 2.3.1. *$\mathcal{O}_X$ equips $\mathrm{Spec}^{\mathcal{G}}(\mathrm{X})$ with a $\mathcal{G}$-structure.*

Proof. $\widetilde{\mathcal{O}}_X$ and L are all left exact functors, so $\mathcal{O}_X$ is a left exact functor. The rest is to show that if $\{V_\alpha \to Y\}$ is a admmisible cover, then the induced map $\coprod_\alpha \mathcal{O}_X(V_\alpha) \to \mathcal{O}_X(Y)$ is an effective epimorphism. By [Lur09b, Lemma 6.2.4.5] it suffices to prove that: for $U \in \mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}$ and $\eta \in \pi_0\mathcal{O}_X(Y)(U)$, we can find an admissible cover $\{U_\alpha \to U\}$ and a collection of $\eta_\alpha \in \mathcal{O}_X(V_\alpha)(U_\alpha)$ having the same images of η. We may assume that η arises from a map $U \to j(Y)$ in $\mathrm{Pro}(\mathcal{G})$. Then $U_\alpha := j(V_\alpha) \times_{j(Y)} U$ form an admissible cover of U by the definition of coarsest Grothendick topology, and each $\widetilde{\eta_\alpha} := \eta|_{U_\alpha} \in \pi_0\mathcal{O}_X(Y)(U_\alpha)$ lefts to $\eta_\alpha \in \pi_0\mathcal{O}_X(V_\alpha)(U_\alpha)$ since $U_\alpha \to U \to j(Y)$ factors through $j(V_\alpha)$. □

Lemma 2.3.1. *Let $\mathcal{X}$ be the underlying topos of $\mathrm{Spec}^{\mathcal{G}}(\mathrm{X})$ for some $X \in \mathrm{Pro}(\mathcal{G})$. The object $\mathcal{O}_X \in \mathrm{Shv}(\mathcal{X}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))$ corresponds to the sheafification of the forgetful functor $(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathrm{Pro}(\mathcal{G})^{\mathrm{op}} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$.*

Proof. There is a commutaive square of the form

$$\begin{array}{ccccc}
\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathcal{X}) & \longrightarrow & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})) & & \\
\downarrow \simeq & & \downarrow \simeq & \searrow \simeq & \\
\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) & \xrightarrow{(-)\circ L^{\mathrm{op}}} & \mathrm{Fun}((\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) & \xleftarrow[\simeq]{} & \mathrm{Fun}^{\mathrm{R}}((\mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}))^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))
\end{array}$$

where the top arrow comes from postcomposition with the canonical inclusion $\iota : \mathrm{Shv}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}) \to \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})$. Passing to left adjoints, we see that the functor $L \circ (-) : \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathcal{X})$ is identified with the sheafification functor $\mathrm{Fun}((\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \to \mathrm{Shv}(\mathcal{X}^{\mathrm{op}}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))$, as it is the left adjoint of the canonical forgetful functor.

The rest is to consider $\widetilde{\mathcal{O}}_X \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}))$. Applying the following equivalence

$$\mathrm{Fun}(\mathcal{G}, \mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{G} \times (\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}}, \mathrm{An}) \xrightarrow{\simeq} \mathrm{Fun}((\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}}, \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))$$

to $\widetilde{\mathcal{O}}_X$ which is defined by

$$\mathcal{G} \times (\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathcal{G} \times (\mathrm{Pro}(\mathcal{G}))^{\mathrm{op}} = \mathcal{G} \times \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An}) \xrightarrow{\mathrm{ev}} \mathrm{An}$$

we see that $\widetilde{\mathcal{O}}_X$ corresponds to the forgetful functor $(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathrm{Pro}(\mathcal{G})^{\mathrm{op}} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. □

Combining Lemma 2.2.1 and Lemma 2.3.1 we obtain a natural identification

$$\Gamma_{\mathcal{G}}(\mathcal{P}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}}), \widetilde{\mathcal{O}}_{\mathrm{X}}) \simeq X.$$

In fact, the left term is evaluating the forgetful functor $(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ on the terminal object, which is just id_X. Thus we have a map

$$\alpha : X \to \Gamma_{\mathcal{G}}(\mathrm{Spec}^{\mathcal{G}}(\mathrm{X}), \mathcal{O}_{\mathrm{Spec}^{\mathcal{G}}(\mathrm{X})}) \in \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})^{[1]}$$

inducing by the sheafification map $\widetilde{\mathcal{O}}_X \to \mathcal{O}_X$ on the terminal object in virtue of Lemma 2.2.1.

The follwoing theorem from [Lur09a, Theorem 2.2.12] shows that the left adjoint in Lemma 2.2.3 $\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}_{\mathrm{disc}}}$ can be calculated as $\mathrm{Spec}^{\mathcal{G}}(\mathrm{X})$:

Theorem 2.3.1. *Let $\mathcal{G}$ be a geometry and let $\mathcal{G}_{\mathrm{disc}} \to \mathcal{G}$ be the canonical functor from the discrete geometry $\mathcal{G}_{\mathrm{disc}}$ on $\mathcal{G}$ to $\mathcal{G}$. The for any $X \in \mathrm{Pro}(\mathcal{G})$ the map α above is adjoint to an equivalence $\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}_{\mathrm{disc}}}\mathrm{X} \simeq \mathrm{Spec}^{\mathcal{G}}\mathrm{X}$.*

From this theorem we have the following straightforward corollary:

Corollary 2.3.1. *For any $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \mathrm{LTop}(\mathcal{G})$ and $X \in \mathrm{Pro}(\mathcal{G})^{\mathrm{op}}$, there is an equivalence*

$$\mathrm{Map}_{\mathrm{LTop}(\mathcal{G})}((\mathrm{Spec}^{\mathcal{G}}\mathrm{X}, \mathcal{O}_{\mathrm{Spec}^{\mathcal{G}}\mathrm{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})^{\mathrm{op}}}(\mathrm{X}, \Gamma_{\mathcal{G}}(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

induced by applying α.

3 Examples of geometries

3.1 Classical Zariski Geometry

Now let's return to classical Zariski geometry.

Definition 3.1.1. The classical Zariski geometry consists of the following data:
(a) $\mathcal{G}_{\mathrm{cZar}} = (\mathrm{CAlg}^{\omega})^{\mathrm{op}}$, the opposite of the category of compact commutative ring spectra;
(b) Admissible morphisms correspond to localization maps $R \to R[x^{-1}]$ for $x \in \pi_0 R$;
(c) A finite collection $\{R \to R[x_i^{-1}]\}_{i\in I}$ is declared to generate a covering sieve if the set $\{x_i\}_{i\in I} \subseteq \pi_0 R$ generates the unit ideal.

If we take $\mathcal{G}$ in Lemma 2.2.3 with $\mathcal{G}_{\mathrm{cZar}}$ and use Theorem 2.3.1, we have the following adjunction

$$\mathrm{Spec} := \mathrm{Spec}^{\mathcal{G}_{\mathrm{cZar}}} : \mathrm{CAlg} \rightleftarrows \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}}) : \Gamma_{\mathcal{G}_{\mathrm{cZar}}}.$$

We consider the left adjoint $\mathrm{Spec} : \mathrm{CAlg} \to \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ and in fact this functor can be identified with the familiar presentation. The following proposition is a special case of [Lur11, Theorem 2.40]:

Proposition 3.1.1. *Given $R \in \mathrm{CAlg}$, $\mathrm{Spec(R)} \in \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ may be identified with the pair $(\mathrm{Shv}(\mathrm{Spec}(\pi_0\mathrm{R})), \mathcal{O})$ where*
(a) $\mathrm{Spec}(\pi_0\mathrm{R})$ refers to the ordinary Zariski spectrum of prime ideals;
(b) $\mathcal{O}$ is the unique sheaf of commuttaive ring spectra on $\mathrm{Spec}(\pi_0\mathrm{R})$ satisfying $\mathcal{O} : D(f) \mapsto R[f^{-1}]$ for $D(f)$ a basic open set of $\pi_0 R$ associated to some element f.

From this proposition, we find that the adjunction above recovers the traditional characterication of affine schemes.

3.2 Spectral Dirac Geometry

The definition below is a spectral invariant of the underlying geometry in [HP25]

Definition 3.2.1. The spectral Dirac geometry on $\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$ consists of the following data:
(a) $\mathcal{G}_{\mathrm{Dir}} = (\mathrm{CAlg}^{\omega})^{\mathrm{op}}$, the opposite of the category of compact commutative ring spectra;
(b) Admissible morphisms correspond to localization maps $R \to R[x^{-1}]$ for $x \in \pi_* R$;
(c) A finite collection $\{R \to R[x_i^{-1}]\}_{i\in I}$ is declared to generate a cobering sieve if the set $\{x_i\}_{i\in I} \subseteq \pi_* R$ generates the unit ideal.

Definition 3.2.2. A commutative ring spectrum R is Dirac-local if $\pi_{2*}R$ is a local ring. A map $R \to S$ of Dirac-local ring spectra is a Dirac-local map if the induced map on π_{2*} is a local map of rings.

The following proposition is a modification of [Lur11, Theorem 2.40] combining with [HP25, Remark 2.25, Theorem 2.26, Proposition 2.35]:

Proposition 3.2.1. *Given $R \in \mathrm{CAlg}$, $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}}(\mathrm{R}) \in \mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$ may be identified with the pair $(\mathrm{Shv}(\mathrm{Spec}^{\mathrm{h}}(\pi_{2*}\mathrm{R})), \mathcal{O}_{\mathrm{R}})$ where*
(a) $\mathrm{Spec}^{\mathrm{h}}(\pi_{2*}\mathrm{R})$ is the homogeneous spectrum of the homogeneous prime ideal of the graded ring $\pi_{2*}R$;
(b) $\mathcal{O}_R$ is the unique sheaf of commutative ring spectra on $\mathrm{Spec}(\pi_{2*}\mathrm{R})$ satisfying $\mathcal{O} : D(f) \mapsto R[f^{-1}]$ for $D(f) := \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(\pi_{2*}\mathrm{R}) | f^2 \notin \mathfrak{p}\}$ where $f \in \pi_* R$.

The following lemma is a spectral incarnation of [HP25, Theorem 2.26] with the identical proof:

Lemma 3.2.1. *Given $R \in \mathrm{CAlg}$ and point $x^* : \mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}}(\mathrm{R}) \to \mathrm{An}$ corresponding to a homogeneous prime ideal $\mathfrak{p} \subseteq \pi_{2*}R$, the pullback $x^*\mathcal{O}_R \in \mathrm{Str}_{\mathcal{G}_{\mathrm{Dir}}}(\mathrm{An})$ corresponds to the Dirac-local ring spectrum $R_{\mathfrak{p}}$.*

Finally, let's identify the local $\mathcal{G}_{\mathrm{Dir}}$-structure on An:

Lemma 3.2.2. *The category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Dir}}}(\mathrm{An})$ is naturally equivalent to the subcategory $\mathcal{C} \subseteq \mathrm{CAlg}$ of Dirac-local commutative ring spectra with Dirac-local maps between them.*

Proof. Under the equivalence $\mathrm{CAlg} \xrightarrow{\sim} \mathrm{Ind}(\mathrm{CAlg}^{\omega}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Dir}}, \mathrm{An}), \mathrm{R} \mapsto \mathrm{Map}_{\mathrm{CAlg}}(-, \mathrm{R})$, a commutative ring spectra R cooresponds to a $\mathcal{G}_{\mathrm{Dir}}$-structure means exactly that R is Dirac-local. Moreover a map of Dirac-local ring spectra $R \to S$ corresponds to local morphism if and only if for an arbitrary map $R' \to S$ where R' is compact and an element $f \in \pi_* R'$, the following is a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(R'[f^{-1}], R) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(R'[f^{-1}], S) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(R', R) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(R', S), \end{array}$$

and this pullback condition is equivalent to $R' \to R$ inverts f in $\pi_* S$ only when it inverts f in $\pi_* R$. As R' is arbitrary, we find that $R \to S$ corresponds to local morphism if and only if $R \to S$ is a Dirac-local map. $\square$

References

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