Talk 6: Lurie's Geometries and $\mathcal{G}$-Structured ∞-Topoi I

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19 May 2026

Abstract

These are notes for Talk 6 in the seminar on Higher Zariski Geometry organized by Denis-Charles Cisinski and Giovanni Rossanigo in Summer Term 2026.

The goal is to introduce the topos-theoretic background needed for Lurie's notion of geometry, and then to explain the definitions of geometries, $\mathcal{G}$-structures, local maps, and $\mathrm{LTop}(\mathcal{G})$ as used in higher Zariski geometry. The running example is the classical Zariski geometry.

Contents

Overview

Let me begin by connecting this talk to the previous two.

In Talk 4, we saw that to every 2-ring K one can associate a locale $\mathrm{Spec}(K)$, using the study of thick tensor ideals, radical ideals, localizations, and coherent frames. We then showed that the resulting locale is equivalent to the poset of open subsets of an actual spectral topological space, the Balmer spectrum $\mathrm{Spc}(K)$.

In Talk 5, we saw that this topological space can carry more structure. For $K = \mathrm{Perf}(X)$, the Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}(X))$ comes equipped with a sheaf of local rings, turning it into a locally ringed space. Moreover, if X is a topologically Noetherian qcqs scheme, then the locally ringed space associated to $\mathrm{Perf}(X)$ is isomorphic to X itself.

The goal of the next couple of talks is to upgrade this picture. Instead of attaching only a sheaf of local rings, we want to attach a sheaf of local 2-rings. In other words, we want to pass from locally ringed spaces to locally 2-ringed spaces, or more generally to locally 2-ringed ∞-topoi.

To define this cleanly, we use Lurie's notion of a *geometry* $\mathcal{G}$ and a *$\mathcal{G}$-structured ∞-topos*. The point is that $\mathcal{G}$-structures abstract the idea of a sheaf of local rings:

$$\text{Locally ringed spaces} \quad \rightsquigarrow \quad \mathcal{G}\text{-structured } \infty\text{-topoi.}$$

One running example will be the geometry $\mathcal{G}_{\mathrm{Zar}}$ which recovers classical Zarisi geometry. In this case, a $\mathcal{G}$-structure on an ∞-topos is a point-free way of saying "a sheaf of local rings", and a local map of $\mathcal{G}$-structures is the point-free version of a local homomorphism of local rings.

The advantage is that the same language works far beyond ordinary schemes. By changing $\mathcal{G}$, one obtains ordinary Zariski geometry, derived Zariski geometry, spectral geometry, étale geometry, and, for the purpose of this seminar, higher Zariski geometry for 2-rings.

For the general setup, we make the following replacements/generalizations:

$$\begin{array}{ccl} \text{topological space} & \rightsquigarrow & \infty\text{-topos,} \\ \text{sheaf of rings} & \rightsquigarrow & \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})\text{-valued sheaf,} \\ \text{local rings and local maps} & \rightsquigarrow & \mathcal{G}\text{-structures and local morphisms.} \end{array}$$

In Part I of this talk, we will go through some recollections on the theory of ∞-topoi. In Part II, we will define geometries and $\mathcal{G}$-structures.

1 Part I: ∞-topoi

1.1 Definition

Let me begin with the version of the definition that is closest to the paper.

Definition 1.1. An ∞-*topos* is an accessible left exact localization of a presheaf ∞-category. Thus an ∞-topos is an ∞-category $\mathcal{X}$ for which there exists a small ∞-category C and an adjunction

$$L\colon \mathrm{PSh}(C) \rightleftarrows \mathcal{X} : i$$

where $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$, the right adjoint i is fully faithful, and the localization functor L preserves finite limits (equivalently, it preserves the terminal object and pullbacks.)

The fully faithfulness of i means that we may identify $\mathcal{X}$ with a full subcategory of $\mathrm{PSh}(C)$. We may thus think of $\mathcal{X}$ as presheaves on C satisfying some kind of locality/descent condition. The left exactness of L means that the locality interplays nicely with the 'geometric' finite limits.

Remark 1.2. One can show that an ∞-category $\mathcal{X}$ is an ∞-topos if and only if it is presentable and satisfies descent for all colimits, meaning that the slice functor

$$\mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty, \qquad X \longmapsto \mathcal{X}_{/X}$$

sends colimits to limits. Spelling this out: if $X \simeq \mathrm{colim}_i X_i$ in a topos $\mathcal{X}$, descent says that the natural functor

$$\mathcal{X}_{/X} \longrightarrow \lim_i \mathcal{X}_{/X_i}$$

is an equivalence. In words: an object over X is the same as objects over all the X_i, together with coherent identifications over all overlaps.

This is the higher-categorical version of a very familiar idea: a bundle, a sheaf, or a geometric object on a space can be reconstructed from its restrictions to an open cover, together with gluing data.

1.2 Sheaf topoi

Now let (C, τ) be a small Grothendieck site, which roughly means that C is a small ∞-category equipped with a notion of 'coverings'. For example, the poset of open subsets of a topological space is a Grothendieck site, with the coverings being the usual open coverings. Then there is an associated *sheaf* ∞-*topos*

$$\mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

An object of $\mathrm{Shv}_\tau(C)$ is a presheaf satisfying descent for the covering sieves of τ. For simplicity, let me spell out the descent condition only in the case of a covering family $\{U_i \to X\}_{i\in I}$ for which relevant fiber products that appear all exist in C. (If they do not exist, one first needs to left Kan extend F to all of $\mathrm{PSh}(C)$.) Then a presheaf F is a sheaf if for each such covering family the diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \overset{\rightarrow}{\rightrightarrows} \cdots$$

is a limit diagram. This is just the usual descent, written with animae instead of sets.

Proposition 1.3. *The sheaf category* $\mathrm{Shv}_\tau(C)$ *is an* ∞*-topos.*

Proof. The inclusion

$$\mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C)$$

admits a left adjoint, the sheafification functor L_τ. This exists for formal reasons, as we are localizing at a small collection of maps. We need to show that L preserves terminal objects (which is clear, as the terminal presheaf is a sheaf) and pullbacks. In other words, given maps of presheaves $X \to Y \leftarrow Z$, we must show that the canonical map $X \times_Y Z \to LX \times_{LY} LZ$ is inverted by L. We may factor this map as a composite

$$X \times_Y Z \to X \times_{RLY} Z \to RLX \times_{RLY} Z \to RLX \times_{RLY} RLZ,$$

so it suffices to show each of these three maps is inverted by L. Now note that the second and third maps are base changes of $X \to RLX$ and $Z \to RLZ$, while the first map is a base change of the diagonal of $Y \to RLY$. Since L is a localization, the unit $X \to RLX$ is always inverted by L, and thus it remains to show that maps inverted by L are closed under base change and diagonals.

To see this, let K be the class of morphisms inverted by L. By the general theory of localizations, this class is obtained by starting with the class τ of covering sieve inclusions, and then closing it up under colimits and 2-out-of-3 to get the *strong saturation* of τ. As a first approximation, consider the *saturation* τ^s, where we only close τ up under colimits and composition. The axioms of a Grothendieck topology demand that the covering sieves are closed under base change. One can deduce that the class τ^s is closed under base change. Moreover, since the maps $U \hookrightarrow y(X)$ are monomorphisms, their diagonals are isomorphisms, hence in particular again lie in τ^s. One may deduce that τ^s is closed under diagonals. Any class of morphisms closed under both base change and diagonals is also closed under 2-out-of-3: given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that g and gf are in the class, we may factor f as

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_2} Y,$$

where the first map is a base change of Δ_g, while the second is a base change of gf. It follows that the saturation τ^s already satisfies 2-out-of-3, and thus it agrees with the strong saturation. All in all, we see that the maps inverted by L is given by τ^s, and that it is closed under base change and diagonals, as desired. □

Example 1.4. For a topological space X, take $C = \mathrm{Open}(X)$ with the usual open-cover topology. Then

$$\mathrm{Shv}(X) := \mathrm{Shv}(\mathrm{Open}(X))$$

is the ∞-topos of sheaves of animae on X. This is the basic example of an ∞-topos to keep in mind.

The reason for working with ∞-topoi is that in practice many geometric objects have a natural site but not necessarily an honest underlying topological space, and the sheaf topos is the invariant object.

1.3 The categories LTop and RTop

There are two useful directions for morphisms of topoi. Since this talk follows the notation of the paper, I will use LTop as the primary convention.

Definition 1.5. The ∞-category LTop has:

- objects: ∞-topoi;
- morphisms $\mathcal{X} \to \mathcal{Y}$: left exact left adjoint functors $f^* \colon \mathcal{X} \longrightarrow \mathcal{Y}$.

The ∞-category RTop is the opposite category:

$$\mathrm{RTop} := \mathrm{LTop}^{\mathrm{op}}.$$

Equivalently, a morphism $\mathcal{Y} \to \mathcal{X}$ in RTop is a right adjoint functor

$$f_* \colon \mathcal{Y} \longrightarrow \mathcal{X}$$

whose left adjoint f^* is left exact.

If $f \colon X \to Y$ is a continuous map of topological spaces, then inverse image gives a left exact left adjoint

$$f^* \colon \ \mathrm{Shv}(Y) \longrightarrow \mathrm{Shv}(X),$$

i.e. this assignment defines a contravariant functor

$$\mathrm{Shv} \colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{LTop}.$$

By passing to right adjoints, we obtain a map in the RTop-direction:

$$f_* \colon \ \mathrm{Shv}(X) \longrightarrow \mathrm{Shv}(Y).$$

Since this points in the same direction as the map on topological spaces, the map f_* is often referred to as the "geometric" direction.

1.4 Points

The terminal object of RTop is the ∞-topos An of animae. This motivates the following definition.

Definition 1.6. A *point* of an ∞-topos $\mathcal{X}$ is a morphism

$$x_* \colon \mathrm{An} \longrightarrow \mathcal{X}$$

in RTop. Equivalently, it is a left exact left adjoint

$$x^*\colon \mathcal{X} \longrightarrow \mathrm{An}$$

in LTop. We write

$$\mathrm{Pt}(\mathcal{X}) := \mathrm{Map}_{\mathrm{RTop}}(\mathrm{An}, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{LTop}}(\mathcal{X}, \mathrm{An}).$$

Example 1.7. For $\mathcal{X} = \mathrm{Shv}(X)$, where X is a topological space, a point $x \in X$ gives the stalk functor

$$x^*\colon\ \mathrm{Shv}(X) \longrightarrow \mathrm{An}, \qquad F \longmapsto F_x \simeq \mathrm{colim}_{x\in U} F(U),$$

giving a point of $\mathrm{Shv}(X)$. (Warning: Not all points of $\mathrm{Shv}(X)$ are of this form if X is not sober.)

Definition 1.8. We say an ∞-topos $\mathcal{X}$ *has enough points* if the stalk functors $x^*\colon \mathcal{X} \to \mathrm{An}$ are jointly conservative.

Not every ∞-topos has enough points. But when points are available, they usually simplify the mathematics: many locality conditions can be tested after applying all stalk functors x^*.

2 Part II: Geometries and Structured Topoi

Classical Zariski geometry is built from three ingredients:

- affine schemes $\mathrm{Spec}\, R$;
- basic open immersions $D(f) \hookrightarrow \mathrm{Spec}\, R$;
- finite Zariski covers by basic opens.

Lurie's definition of a geometry $\mathcal{G}$ attempts to abstract these ingredients:

$$\begin{aligned} \text{affine test objects} &\rightsquigarrow \text{objects of } \mathcal{G}, \\ \text{basic open inclusions} &\rightsquigarrow \text{admissible morphisms}, \\ \text{families of basic opens covering an affine} &\rightsquigarrow \text{admissible covers.} \end{aligned}$$

Definition 2.1. Let $\mathcal{G}$ be a small ∞-category, let $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ be a wide subcategory, and let τ be a Grothendieck topology on $\mathcal{G}$. We say that $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a *geometry* if:

(1) $\mathcal{G}$ admits finite limits and is idempotent complete;

(2) The topology τ is generated by morphisms in $\mathcal{G}^{\mathrm{ad}}$;

(3) $\mathcal{G}^{\mathrm{ad}}$ is closed under base change in $\mathcal{G}$;

(4) $\mathcal{G}^{\mathrm{ad}}$ is left cancellable: if $f\colon X \to Y$ and $g\colon Y \to Z$ are composable and both g and $g \circ f$ are admissible, then f is admissible;

(5) admissible morphisms are closed under retracts in the arrow category $\mathrm{Fun}([1], \mathcal{G})$.

Morphisms in $\mathcal{G}^{\mathrm{ad}}$ are called *admissible morphisms*; covers in τ are called *admissible covers*.

The topology being generated by admissible morphisms says that all covers can be refined by covers made from the admissible morphisms.

Example 2.2 (Discrete geometry)**.** If $\mathcal{G}$ is idempotent complete and has finite limits, there is a discrete geometry $\mathcal{G}_{\mathrm{disc}}$ on the same underlying category: the admissible morphisms are only the equivalences, and the topology is the trivial topology. This example is useful because $\mathcal{G}_{\mathrm{disc}}$-structures are just ordinary $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaves.

Definition 2.3. A *morphism of geometries*

$$\alpha\colon \mathcal{G} \longrightarrow \mathcal{G}'$$

is a functor which preserves finite limits, sends admissible morphisms to admissible morphisms, and sends admissible covers to admissible covers.

2.1 The classical Zariski geometry

Now for the running example.

Definition 2.4. The *classical Zariski geometry* $\mathcal{G}_{\mathrm{Zar}}$ is given as follows:

(1) The underlying ∞-category is

$$\mathcal{G}_{\mathrm{Zar}} = (\mathrm{CAlg}^{\omega})^{\mathrm{op}},$$

where CAlg^{ω} denotes the compact, or finitely presented, commutative $\mathbb{E}_\infty$-rings.

(2) Admissible morphisms in $\mathcal{G}_{\mathrm{Zar}}$ correspond, on rings, to principal localizations

$$R \longrightarrow R[f^{-1}],$$

where $f \in \pi_0(R)$. Geometrically, this is the principal open immersion

$$D(f) = \mathrm{Spec}\, R[f^{-1}] \hookrightarrow \mathrm{Spec}\, R.$$

(3) A finite family

$$\{R \to R[f_i^{-1}]\}_{i \in I}$$

generates a covering sieve if the elements f_i generate the unit ideal in $\pi_0 R$.

Lemma 2.5. $\mathcal{G}_{\mathrm{Zar}}$ *defines a geometry.*

Proof. The ∞-category $\mathcal{G}_{\mathrm{Zar}}$ has finite limits, since compact objects are closed under finite colimits. The topology is by assumption generated by admissible maps. Stability under base change is clear: the pushout of $R \to R[f^{-1}]$ along $\varphi\colon R \to S$ is $S \to S[\varphi(f)^{-1}]$. For left cancellability, consider a commutative diagram of the form

$$\begin{tikzcd} & R \arrow[dl, "\psi"'] \arrow[dr, "\varphi\psi"] & \\ R[f^{-1}] \arrow[rr, "\varphi"] & & R[g^{-1}]. \end{tikzcd}$$

Since ψ inverts f, so does $\varphi\psi$, and thus f is already invertible in $R[g^{-1}]$, giving

$$R[g^{-1}] \cong R[g^{-1}][f^{-1}] \cong R[f^{-1}][g^{-1}].$$

So also the map φ is admissible. It remains to check closure under retracts: consider a retract diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & R \arrow[r, "\rho"] \arrow[d, "\ell"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\beta"'] & R[f^{-1}] \arrow[r, "\sigma"'] & B \end{tikzcd}$$

in CAlg^{ω}. Then the restriction map

$$\varphi^*\colon \mathrm{Map}_{\mathrm{CAlg}}(B, C) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}}(A, C)$$

is a retract of the restriction map

$$\ell^*\colon \mathrm{Map}_{\mathrm{CAlg}}(R[f^{-1}], C) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}}(R, C).$$

By the universal property, ℓ^* is a monomorphism whose image consists of those maps $R \to C$ sending f to a unit in $\pi_0 C$. It follows that also φ^* is a monomorphism. Moreover, an easy diagram chase shows that a map $A \to C$ extends to $B \to C$ if and only if it inverts the element $a := \rho(f) \in \pi_0 A$, so that $B \simeq A[a^{-1}]$, as desired. $\square$

2.2 $\mathcal{G}$-structures

Fix a geometry $\mathcal{G}$. We now define $\mathcal{G}$-structures on a topos.

Definition 2.6. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be an ∞-category admitting small limits. The ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ is

$$\mathrm{Shv}(\mathcal{X}; \mathcal{C}) := \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}),$$

the full subcategory of functors preserving small limits.

This definition is elegant because the topology has already been absorbed into $\mathcal{X}$. If $\mathcal{X}$ happens to be a sheaf topos $\mathrm{Shv}_\tau(D)$, one can show that C-valued sheaves on $\mathcal{X}$ are nothing but C-valued τ-sheaves $D^{\mathrm{op}} \to C$. (See [Lur18, Corollary 1.3.1.8].)

The case we need is

$$C = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) := \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{An}).$$

Then one has a canonical equivalence

$$\mathrm{Shv}(\mathcal{X}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{An})) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}).$$

In other words, given a sheaf $\mathcal{F}$, we consider the left exact functor

$$\mathcal{G} \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}, \mathrm{An}), \qquad U \mapsto \mathrm{Hom}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(U, \mathcal{F}(-)),$$

and then use that the any limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{An}$ is representable so that this lands in $\mathcal{X}$.

In the classical Zariski case, we have $\mathrm{Ind}(\mathcal{G}_{\mathrm{Zar}}^{\mathrm{op}}) \simeq \mathrm{CAlg}$. So a sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O} \in \mathrm{Shv}(\mathcal{X}; \mathrm{CAlg})$ can be viewed as a left-exact functor $(\mathrm{CAlg}^\omega)^{\mathrm{op}} \to \mathcal{X}$ by sending a finitely presented ring A to the sheaf

$$U \longmapsto \mathrm{Map}_{\mathrm{CAlg}}(A, \mathcal{O}(U)).$$

So this is a functor-of-points perspective on the ring object $\mathcal{O}$.

Now we add locality.

Definition 2.7. Let $\mathcal{G}$ be a geometry and let $\mathcal{X}$ be an ∞-topos. A *$\mathcal{G}$-structure* on $\mathcal{X}$ is a left exact functor

$$\mathcal{O} \colon \mathcal{G} \longrightarrow \mathcal{X}$$

such that for every admissible cover $\{U_i \to X\}_{i \in I}$ in $\mathcal{G}$, the induced map

$$\coprod_{i \in I} \mathcal{O}(U_i) \longrightarrow \mathcal{O}(X)$$

is an effective epimorphism in $\mathcal{X}$. In other word, the following diagram is a colimit diagram:

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{i,j,k} \mathcal{O}(U_i \times_X U_j \times_X U_k) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{i,j} \mathcal{O}(U_i \times_X U_j) \rightrightarrows \coprod_i \mathcal{O}(U_i) \longrightarrow \mathcal{O}(X).$$

We write $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ for the full subcategory of $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$ spanned by the $\mathcal{G}$-structures.

Example 2.8. When $\mathcal{G} = \mathcal{G}_{\mathrm{disc}}$, there are no nontrivial admissible covers to check, so $\mathcal{G}$-structures are just $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaves:

$$\mathrm{Str}_{\mathcal{G}_{\mathrm{disc}}}(\mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \simeq \mathrm{Shv}(\mathcal{X}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})).$$

Example 2.9. Let's now see what happens when $\mathcal{G} = \mathcal{G}_{\mathrm{Zar}}$ and $\mathcal{X} = \mathrm{An}$. We then have

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Zar}}, \mathrm{An}) \simeq \mathrm{Ind}(\mathcal{G}_{\mathrm{Zar}}^{\mathrm{op}}) \simeq \mathrm{CAlg},$$

a $\mathcal{G}_{\mathrm{Zar}}$-structure on An corresponds to a commutative ring spectrum R with some additional property. For an admissible cover

$$\{S \to S[f_i^{-1}]\}_{i \in I}, \qquad (f_i) = 1,$$

this property says that the map

$$\coprod_i \mathrm{Map}_{\mathrm{CAlg}}(S[f_i^{-1}], R) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}}(S, R)$$

is surjective on π_0. So the property that R is a $\mathcal{G}_{\mathrm{Zar}}$-structure is saying that for a map $\varphi\colon S \to R$ and elements $f_i \in \pi_0(S)$ that generate the unit ideal, at least one of the elements $\varphi(f_i) \in \pi_0(R)$ is invertible.

We claim that this is equivalent to $\pi_0(R)$ being a local ring (i.e. that $\pi_0(R)$ has a unique maximal ideal, or equivalently that the nonunits form an ideal). Indeed, in a local ring, if $f_1, \ldots, f_n$ generate the unit ideal, not all of them can lie in the maximal ideal, so one is a unit. Conversely, if this condition holds for every finite partition of the unit, then the nonunits are closed under addition; hence the nonunits form the unique maximal ideal.

The previous example shows why the definition of a $\mathcal{G}$-structure is so nice: it means that a $\mathcal{G}_{\mathrm{Zar}}$-structure on a general ∞-topos $\mathcal{X}$ is a 'point-free' sheaf of local rings. If $\mathcal{X} = \mathrm{Shv}(X)$ has enough points, applying a point

$$x^*\colon \mathcal{X} \to \mathrm{An}$$

to the structure sheaf gives a local ring spectrum at every stalk:

$$x^*\mathcal{O} \in \mathrm{Str}_{\mathcal{G}_{\mathrm{Zar}}}(\mathrm{An}) \simeq \{\text{ local ring spectra }\}.$$

This recovers the usual idea of a locally ringed space, but expressed without making points part of the definition.

Warning 2.10. A $\mathcal{G}_{\mathrm{Zar}}$-structure is *not* a sheaf valued in local rings: The individual rings of sections $\mathcal{O}(U)$ are usual not local rings, only the stalks $x^*\mathcal{O}$ are.

Remark 2.11. The category $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ admits a factorization system in which the right class of maps consists of the local morphisms.

Compare: every map of rings $A \to B$ can be uniquely factored as

$$A \to A[S^{-1}] \to B,$$

where S is the collection of elements of A that become invertible in B. Note that the second map is local.

2.3 Local morphisms

Recall that the category of locally ringed spaces is not the full subcategory of ringed spaces whose objects have local stalks: We need to require the morphisms to also induce local morphisms on stalks. The following definition is the point-free version of that condition.

Definition 2.12. Let $\mathcal{O}, \mathcal{O}' \in \mathrm{Str}_{\mathcal{G}}(\mathcal{X})$. A natural transformation

$$\alpha \colon \mathcal{O} \longrightarrow \mathcal{O}'$$

is called *local* if, for every admissible morphism $U \to X$ in $\mathcal{G}$, the square

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}'(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{O}'(X) \end{array}$$

is a pullback square in $\mathcal{X}$.

We write

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \subseteq \mathrm{Str}_{\mathcal{G}}(\mathcal{X})$$

for the wide subcategory with the same objects and only the local morphisms.

Let's again have a look at the Zariski example to understand this definition.

Example 2.13 (Local maps in the Zariski example)**.** Take $\mathcal{X} = \mathrm{An}$ and let a map of $\mathcal{G}_{\mathrm{Zar}}$-structures correspond to a ring map

$$R \longrightarrow R',$$

where R and R' are local $\mathbb{E}_\infty$-rings. Consider the admissible map represented by a localization $S \to S[f^{-1}]$. Then the locality condition on $R \to R'$ demands that the square

$$\begin{array}{ccc} \mathrm{Map}(S[f^{-1}], R) & \longrightarrow & \mathrm{Map}(S[f^{-1}], R') \\ \downarrow & & \downarrow \\ \mathrm{Map}(S, R) & \longrightarrow & \mathrm{Map}(S, R') \end{array}$$

is a pullback square. In other words: a ring map $S \to R$ sends f to a unit in R if and only if the composite $S \to R \to R'$ sends f to a unit in R'. Thus $R \to R'$ reflects invertibility.

For local ordinary rings, reflecting invertibility is equivalent to being a local homomorphism. Equivalently, the preimage of the maximal ideal of R' is the maximal ideal of R. More generally, we see that $R \to R'$ is local in the above sense if and only if $\pi_0(R) \to \pi_0(R')$ is local in the classical sense.

It follows that if $\alpha \colon \mathcal{O} \to \mathcal{O}'$ is a local morphism of $\mathcal{G}_{\mathrm{Zar}}$-structures on an ∞-topos $\mathcal{X}$, then the map on stalks $\alpha_x \colon x^*\mathcal{O} \to x^*\mathcal{O}'$ is a local morphism of $\mathbb{E}_\infty$-rings for each point x of $\mathcal{X}$.

2.4 The category $\mathbf{LTop}(\mathcal{G})$

As the final thing in the talk, we will assemble $\mathcal{G}$-structures topoi into an ∞-category.

Definition 2.14. Let $\mathcal{G}$ be a geometry. We define an ∞-category $\mathrm{LTop}(\mathcal{G})$ of *$\mathcal{G}$-structured ∞-topoi*:

- An object is a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ of an ∞-topos $\mathcal{X}$ with a $\mathcal{G}$-structure
$$\mathcal{O}_\mathcal{X} \colon \mathcal{G} \to \mathcal{X}.$$
- A morphism from $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ to $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ in $\mathrm{LTop}(\mathcal{G})$ is a pair (f^*, φ), where $f^* \colon \mathcal{X} \to \mathcal{Y}$ is in LTop and
$$f^\sharp \colon f^*\mathcal{O}_\mathcal{X} \longrightarrow \mathcal{O}_\mathcal{Y}$$
is a local morphism of $\mathcal{G}$-structures on $\mathcal{Y}$, i.e. for every admissible map $U \to X$ in $\mathcal{G}$, the square
$$\begin{array}{ccc} f^*\mathcal{O}_\mathcal{X}(U) & \xrightarrow{f^\sharp_U} & \mathcal{O}_\mathcal{Y}(U) \\ \downarrow & & \downarrow \\ f^*\mathcal{O}_\mathcal{X}(X) & \xrightarrow{f^\sharp_X} & \mathcal{O}_\mathcal{Y}(X) \end{array}$$
must be cartesian in $\mathcal{Y}$.

More formally, $\mathrm{LTop}(\mathcal{G})$ is defined as the unstraightening of the functor
$$\mathrm{Str}^{\mathrm{loc}}_\mathcal{G}(-) \colon \mathrm{LTop} \to \mathrm{Cat}_\infty.$$
Here the functoriality in a morphism $f^* \colon \mathcal{X} \to \mathcal{Y}$ in LTop is given as follows:

- Postcomposition with f^* induces $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{Y})$;
- It restricts to $\mathcal{G}$-structures, since f^* preserves effective epimorphisms (any left exact colimit preserving functor does);
- It also preserves local morphisms as these are expressed in terms of pullbacks.

Remark 2.15. In the main reference, they for some reason describe $\mathrm{LTop}(\mathcal{G})$ in a more confusing way as a subcategory of
$$\mathrm{LTop} \times_{\mathrm{Fun}(\mathcal{G}, \mathrm{LTop})} \mathrm{Fun}(\mathcal{G}, \overline{\mathrm{LTop}}),$$
where $\overline{\mathrm{LTop}} \to \mathrm{LTop}$ is the unstraightening of the forgetful functor $\mathrm{LTop} \to \mathrm{Cat}_\infty, \mathcal{X} \mapsto \mathcal{X}$.

Remark 2.16. This definition looks slightly opposite to the usual geometric direction because LTop is the left-adjoint convention. Passing to

$$\mathrm{RTop}(\mathcal{G}) := \mathrm{LTop}(\mathcal{G})^{\mathrm{op}}$$

gives the more familiar form: a morphism in the geometric direction has an underlying geometric morphism of topoi

$$f_* : \mathcal{Y} \to \mathcal{X}$$

and a map of structure sheaves

$$f^\sharp : \mathcal{O}_{\mathcal{X}} \longrightarrow f_*\mathcal{O}_{\mathcal{Y}}$$

which is local.

In the classical Zariski case, $\mathrm{RTop}(\mathcal{G}_{\mathrm{Zar}})$ is the ∞-categorical version of locally ringed topoi, and its 0-localic ordinary part recovers locally ringed spaces.