TENSOR TRIANGULATED GEOMETRY AND THE BALMER SPECTRUM

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Abstract. These are the notes prepared by Yiming Wang for the talk *Tensor triangulated geometry and the Balmer spectrum* for the seminar on *Higher Zariski Geometry* organized by Denis-Charles Cisinski and Giovanni Rossanigo during the summer semester 2026 at the University of Regensburg. The author would like to thank Giovanni Rossanigo for leading him to write down every single detail.

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Conventions. In these notes, our notation will mostly follow [Aok+25], with the following exceptions: we refer to *categories* as what are usually called $(\infty, 1)$-*categories*, which we will understand them model-independently. We will not specify *repleteness* of a subcategory as an additional property of a subcategory. Moreover, we will refer to *localization* as the *Dwyer-Kan localization*, and distinguish it from the *(left) Bousfield localization*. We will refer to *ordinary categories* as $(1, 1)$-*categories* in the literature. We will also state the following fact for the sake of completeness of the notes:

Fact 0.1 (ordinary categories as categories)**.** Every ordinary category may be regarded canonically as a category. Under this identification, all ordinary categorical constructions agree with their categorical counterparts, for example, limits, colimits, adjunctions, slice categories, and similar constructions are computed exactly as in the classical sense.

In model-dependent terms, for example, in the model of quasi-categories, this means that the nerve functor preserves the relevant constructions.

1. Thick tensor ideals

The theory of 2-rings is analogous to that of ordinary rings: in particular, it admits a rich theory of ideals, which we now make precise.

Definition 1.1 (Thick tensor ideals)**.** Let $\mathcal{K}$ be a 2-ring. A full subcategory $\mathcal{I} \subseteq \mathcal{K}$ is called a *thick tensor ideal* or *tt-ideal*, if the following conditions hold:

(a) $\mathcal{I} = \mathrm{Thick}(\mathcal{I})$, where $\mathrm{Thick}(\mathcal{I})$ is the smallest full subcategory that contains $\mathcal{I}$ and is closed under cofibers, shifts, and retracts;
(b) for any $x \in \mathcal{K}$ and $x' \in \mathcal{I}$, we have $x \otimes x' \in \mathcal{I}$.

We denote $\mathrm{Idl}(\mathcal{K})$ the poset of tt-ideals, ordered by inclusion.

Date: May 4, 2026.

Recall that in classical ring theory, an ideal of a (commutative) ring R is an *additive* subgroup $I \subseteq R$ such that the *absorption law* holds, that is $rx \in I$ holds for every $r \in R$ and $x \in X$. Furthermore, recall that a full subcategory of a stable category is stable if and only if it is closed under cofibers, shifts and retracts and that a full subcategory of a idempotent complete category is idempotent complete if and only if it is closed under retract. The condition (a) of Definition 1.1 precisely says that $\mathcal{I}$ is a stable idempotent complete subcategory of $\mathcal{K}$, and the condition (b) is analogous to the absorption law in classical ring theory.

Notation 1.2. Let $S \subseteq \mathcal{K}$ be a *subset*, equivalently, a monomorphism $S \subseteq \mathcal{K}^{\simeq}$. We write $\langle S\rangle$ to denote the tt-ideal generated by S, that is, the smallest tt-ideal that contains S. When $S \simeq \{x\}$, we denote $\langle x\rangle$ for $\langle S\rangle$.

The notion radical ideal and principal ideal is analogous to the case of ordinary rings:

Definition 1.3 (Radical/principal ideals)**.** We say a tt-ideal $\mathcal{I} \subseteq \mathcal{K}$ is:

(a) *principal*, if $\mathcal{I} \simeq \langle x\rangle$ for some $x \in \mathcal{K}$;
(b) *radical*, if $x^{\otimes 2}$ implies $x \in \mathcal{I}$ for any $x \in \mathcal{K}$. By induction, $x^{\otimes n} \in \mathcal{I}$ for some $n \geq 1$ implies $x \in \mathcal{I}$.

We write $\mathrm{Prin}(\mathcal{K}) \subseteq \mathrm{Idl}(\mathcal{K})$ to denote the poset of principal ideals and $\mathrm{Rad}(\mathcal{K}) \subseteq \mathrm{Idl}(\mathcal{K})$ for the posets of radical ideals, ordered by inclusions.

As usual, one can product radical ideals by taking the radical of an ideal:

Notation 1.4 (The radical of a subset)**.** Let $S \subseteq \mathcal{K}$ be a subset. We denote $\sqrt{S}$ to be the smallest radical tt-ideal containing S. A radical ideal $\mathcal{I}$ contains S if and only if $\mathcal{I}$ contains $\sqrt{S}$. Therefore, the functor $\sqrt{-}$ assembles into a left adjoint to the inclusion $\mathrm{Rad}(\mathcal{K}) \hookrightarrow \mathrm{Idl}(\mathcal{K})$.

Example 1.5 (Examples of tt-ideals)*.* Let $\mathcal{K} \in 2\,\mathrm{CAlg}$. We have the following examples of tt-ideals.

(a) The smallest tt-ideal of $\mathcal{K}$ is $\langle 0\rangle$.
(b) The largest tt-ideal is $\langle \mathbf{1}\rangle = \mathcal{K}$.
(c) Let $f\colon \mathcal{K} \to \mathcal{L}$ be a map of 2-rings. The *kernel* of f, denoted by $\ker(f)$, is the full subcategory $\mathcal{K} \times_{\mathcal{L}} \{0\} \subseteq \mathcal{K}$. By functorality of pullback, this assembles into a functor $\ker(-)\colon 2\,\mathrm{CAlg}_{\mathcal{K}/} \to \mathrm{Idl}(\mathcal{K})$.

Definition 1.6 (Colon ideal)**.** Let $\mathcal{K}$ be a 2-ring, $\mathcal{I} \subseteq \mathcal{K}$ be a tt-ideal and $x \in \mathcal{K}$. The *colon* tt-*ideal* $(\mathcal{I} : x)$ of $\mathcal{I}$ by x is the full subcategory spanned by the objects $y \in \mathcal{K}$ such that $x \otimes y \in \mathcal{I}$. Since $\otimes$ is exact in both variables and preserves retracts, the colon tt-ideal $(\mathcal{I} : x)$ is an tt-ideal.

Lemma 1.7 (Radical of a product)**.** *Let $\mathcal{K} \in 2\,\mathrm{CAlg}$ and $\sqrt{x} \cap \sqrt{y}$ be the full subcategory of $\mathcal{K}$ spanned by the objects that are both in $\sqrt{x}$ and $\sqrt{y}$, where $x, y \in \mathcal{K}$. Then we have $\sqrt{x \otimes y} \simeq \sqrt{x} \cap \sqrt{y}$.*

Proof. We will first show that $\sqrt{x \otimes y} \subseteq \sqrt{x} \cap \sqrt{y}$. By Definition 1.1, since $\sqrt{x}$ is a tt-ideal, $x \in \sqrt{x}$ implies that $x \otimes y \in \sqrt{x}$. Similarly, we have $x \otimes y \in \sqrt{y}$, hence $x \otimes y \in \sqrt{x} \cap \sqrt{y}$. By definition, the intersection of two radical ideal is again radical, hence $\sqrt{x \otimes y} \subseteq \sqrt{x} \cap \sqrt{y}$.

Conversely, we need to show that $\sqrt{x} \cap \sqrt{y} \subseteq \sqrt{x \otimes y}$. Let $z \in \sqrt{x} \cap \sqrt{y}$. We can find some $n \in \mathbf{N}$ such that $z^{\otimes n} \in \langle x\rangle \cap \langle y\rangle$. We will now show that $a \in \langle x\rangle$ and $b \in \langle y\rangle$ implies $a \otimes b \in \langle x \otimes y\rangle$. Since $(\langle x \otimes y\rangle : y)$ is a tt-ideal and $x \in (\langle x \otimes y\rangle : y)$, we have $\langle x\rangle \subseteq (\langle x \otimes y\rangle : y)$. In particular, we have $a \otimes y \in \langle x \otimes y\rangle$. This implies that $\langle y\rangle \subseteq (\langle x \otimes y\rangle : a)$, hence $a \otimes b \in \langle x \otimes y\rangle$. Apply the above result to $a = b = z^{\otimes n}$, we find that

$$z^{\otimes 2n} = z^{\otimes n} \otimes z^{\otimes n} \in \langle x \otimes y\rangle.$$

Since the tt-ideal $\sqrt{\langle x \otimes y\rangle}$ is radical, we obtain $z \in \sqrt{\langle x \otimes y\rangle}$. □

In ordinary ring theory, for a ring R we can construct a topological space $\mathrm{Spec}(R)$ by constructing a topology by hand. However, for 2-rings, the usual strategy fails, and we need to find clever ways to encode the topology.

Topological spaces called *sober spaces* can be reconstructed from its poset of open subsets. Classically, for a ring R, the topological space $\mathrm{Spec}(R)$ has the even better property of being a *spectral space*, that is, a quasi-compact sober T_0 space with a basis $(\mathrm{D}(f))_{f\in R}$ consisting of compact open subsets that is closed under finite intersection. In fact, spectral spaces are precisely the topological spaces that are homeomorphic to $\mathrm{Spec}(R)$ for some ring R.

The poset of open subsets of a spectral space is in fact a *coherent frame* and every coherent frame give rises to a spectral space. We will show that $\mathrm{Rad}(\mathcal{K})$ is a coherent frame and this will be the most important ingredient to construct the *Balmer spectrum* of $\mathcal{K}$. Before that, we will first introduce some lattice-theoretical notions that allow us to proceed.

Remark 1.8 (Limits and colimits of posets). Let P be a poset.

(a) Let $(x_i)_{i\in I}$ be a family in P. We denote $\bigvee_{i\in I} x_i$ its *supremum* (also called *least upper bound* or *joins*), and by $\bigwedge_{i\in I} x_i$ its *infimum* (also called *greatest lower bounds* or *meets*), if they exists.

(b) It follows from the universal property that if we view P as a category, I-indexed colimits and I-indexed limits are given by the suprema and infima indexed by the underlying set I respectively.

(c) If exists, the supremum of the empty family (that is, the colimit of the empty diagram) is the smallest element (the initial object). Dually, the infimum of the empty family is the largest element, if exists.

(d) A poset admits all suprema if and only if it admits all infima. Indeed, the infimum of a family is given by the supremum of all elements below the family and the supremum of a family is given by the infimum of all elements above the family.

(e) A category admits all colimits if and only if it admits filtered colimits and all coproducts. In particular, a poset P admits all colimits if and only if it admits filtered colimits and all coproducts. Specifically, for any family $(x_i)_{i\in I}$ in P,
$$\bigvee_{i\in I} x_i = \bigvee_{J\in \mathrm{Fin}(I)} \Big(\bigvee_{j\in J} x_j\Big),$$
where $\mathrm{Fin}(I)$ is the filtered poset of finite subsets of I.

(f) Let P be a poset with arbitrary colimits. An object $x \in P$ is *compact* if and only if for any family $(v_i)_{i\in I}$, we have $x \le \bigvee_{i\in I} v_i$ implies that $x \le v_{i_1} \vee \ldots \vee v_{i_n}$ for some $i_1, \ldots, i_n \in I$.

Definition 1.9 (Lattices and frames)**.** We have the following definitions:

(a) A *bounded lattice* is a poset admitting finite limits and colimits.

(b) A bounded lattice L is *distributive*, if finite limits *distribute* over finite colimits, that is, for any $u, v, w \in L$,
$$u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w).$$
This is equivalent to requiring that finite colimits distribute over finite limits, see [Joh82, Lemma 1.5]. A morphism of distributive lattices $f\colon L \longrightarrow L'$ is a functor preserving finite limits and colimits.

(c) A *complete lattice* is a poset admitting arbitrary colimits.

(d) A *frame* F is a complete lattice in which finite limits distributes over arbitrary colimits. In other words, for any $u \in F$ and any family $(v_i)_{i\in I}$ in F,
$$u \wedge \Big(\bigvee_{i\in I} v_i\Big) = \bigvee_{i\in I} (u \wedge v_i).$$
A morphism $f\colon F \longrightarrow F'$ is an order-preserving function preserving finite limits and all colimits.

(e) A frame is *coherent* if the compact objects are closed under finite limits and every object is a colimit of compact objects. In other words, F is *compactly generated* in the sense that $F \simeq \mathrm{Ind}(F^{\omega})$, where F^{ω} admits finite limits and finite colimits.

A morphism of coherent frames $f\colon F \longrightarrow F'$ is a morphism of frames which furthermore preserves compact objects.

We write DLat, Frm, $\mathrm{Frm}^{\mathrm{coh}}$ to refer to the categories of distributive lattices, frames, and coherent frames, respectively.

We will roughly compare these lattice-theoretical notions with its topological counterparts as follows. We will make precise this analogy later when we discuss *Stone duality*

Remark 1.10 (Topological analogy). Let T be a topological space.

(a) The union of arbitrary open subsets is again open and the finite intersection of open subsets is again open, this is analogous to the condition (a) and (c) above.
(b) Finite intersection of sets distribute over arbitrary unions, this corresponds to (b) and (d) above.
(c) In the poset of open subsets of T, the compact object is precisely the quasi-compact opens, this is immediate from the definition. The poset is compactly generated precisely means that the topological space admits a basis consisting of quasi-compact opens. This is analogous to the condition (e) above.

Before we prove that $\mathrm{Rad}(\mathcal{K})$ is a coherent frame, where $\mathcal{K}$ is a 2-ring, we will prove the following lemma to simplify our proof in showing that $\mathrm{Rad}(\mathcal{K})$ is distributive.

Lemma 1.11 (Colon ideal lemma)**.** *Let $\mathcal{K}$ be a 2-ring, $\mathcal{I} \subseteq \mathcal{K}$ be a radical* tt*-ideal and $x \in \mathcal{K}$. The colon* tt*-ideal $(\mathcal{I} : x)$ is a radical* tt*-ideal.*

Proof. Let $y^n \in \langle \mathcal{I} : x\rangle$, we need to show that $y \in \langle \mathcal{I} : x\rangle$. In other words, we need to show that $x \otimes y^n \in \mathcal{I}$ implies $x \otimes y \in \mathcal{I}$. Since $\mathcal{I}$ is a tt-ideal, it is closed under tensor product, hence

$$x^{n-1} \otimes (x \otimes y^n) = (x \otimes y)^n \in \mathcal{I}.$$

Since $\mathcal{I}$ is a radical tt-ideal, we furthermore have $x \otimes y \in \mathcal{I}$, which is the same as $y \in \mathcal{J}$. □

Proposition 1.12. *Let $\mathcal{K} \in 2\,\mathrm{CAlg}$, we have:*

(a) $\mathrm{Idl}(\mathcal{K})$ *is a complete lattice with compact objects* $\mathrm{Idl}(\mathcal{K})^\omega = \mathrm{Prin}(\mathcal{K})$*;*
(b) $\mathrm{Rad}(\mathcal{K})$ *is a coherent frame with compact objects the radicals of principal* tt*-ideals.*

Proof. (a) It is easy to check that the intersection of tt-ideals is a tt-ideals, hence, one can show that the poset $\mathrm{Idl}(\mathcal{K})$ admits finite limits given by the finite intersections. Moreover, by unwinding the construction, we see that the coproduct of $\mathcal{I}, \mathcal{J} \in \mathrm{Idl}(\mathcal{K})$ is given by $\langle \mathcal{I} \cup \mathcal{J}\rangle$. Since the operations defining a tt-ideal are finitary, any union of tt-ideals along a nested sequence of inclusions is a tt-ideal, hence $\mathrm{Idl}(\mathcal{K})$ admits filtered colimits. By Remark 1.8, the poset $\mathrm{Idl}(\mathcal{K})$ admits all colimits, hence, it is a complete lattice.

In order to show that $\mathrm{Idl}(\mathcal{K}) = \mathrm{Prin}(\mathcal{K})$, we will first show that $\mathrm{Prin}(\mathcal{K}) \subseteq \mathrm{Idl}(\mathcal{K})^\omega$. In other words, for every tt-ideal $\langle a\rangle$, we need to show that

$$\mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\langle a\rangle, \bigcup_{i\in I} \mathcal{I}_i) \simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\langle a\rangle, \mathcal{I}_i).$$

Since $\mathrm{Idl}(\mathcal{K})$ is a poset, the mapping animae are sets, hence we can show this by hand. By cofinality, it suffices to show that the principal tt-ideal $\langle a\rangle$ is contained in the filtered colimit $\bigcup_{i\in I} \mathcal{I}_i$ if and only if it is contained in $\mathcal{I}_i$ for some $i \in I$. This can be checked after applying $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ and use π_0 commutes with all colimits, since it is the right adjoint of the inclusion $\iota\colon\ \mathrm{Set} \hookrightarrow \mathrm{An}$.

Conversely, we will show that $\mathrm{Idl}(\mathcal{K})^\omega \subseteq \mathrm{Prin}(\mathcal{K})$. Let $\mathcal{I} \in \mathrm{Idl}(\mathcal{K})^\omega$. Since, by definition, every $a \in \mathcal{I}$ is contained in the tt-ideal $\langle a\rangle$, we have $\mathcal{I} = \bigvee_{a\in\mathcal{I}}\langle a\rangle$. Furthermore, since $\mathcal{I}$ is closed under retracts, for $a, b \in \mathcal{I}$, we have $\langle a\rangle, \langle b\rangle \subseteq \langle a \oplus b\rangle$. Since $a \oplus b$ is the cofiber of the zero map $\Omega a \to b$ and tt-ideals are closed under shifts and cofibers, we have $a \oplus b \in \mathcal{I}$. Therefore, the tt-ideal $\mathcal{I}$ is a filtered colimit of principal ideals. Assume that $\mathcal{I}$ is a compact object of $\mathrm{Idl}(\mathcal{K})$, there is a canonical isomorphism

$$\mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I}, \bigvee_{a\in\mathcal{I}}\langle a\rangle) \simeq \operatorname*{colim}_{a\in\mathcal{I}} \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I}, \langle a\rangle).$$

The right hand side is a set, hence the inclusion $\mathcal{I} \subseteq \bigvee_{a\in\mathcal{I}}\langle a\rangle$ corresponds to an inclusion $\mathcal{J} \subseteq \langle a\rangle$ for some $a \in \mathcal{I}$. In other words, the tt-ideal $\mathcal{I} = \langle a\rangle$ is radical.

(b) By Notation 1.4, the functor $\sqrt{(-)}\colon\ \mathrm{Idl}(\mathcal{K}) \to \mathrm{Rad}(\mathcal{K})$ is a Bousfield localization. Since $\mathrm{Rad}(\mathcal{K}) \hookrightarrow \mathrm{Idl}(\mathcal{K})$ is fully faithful, colimits in $\mathrm{Rad}(\mathcal{K})$ can be computed by embedding the diagram into $\mathrm{Idl}(\mathcal{K})$ via ι and applying the colimit-preserving functor $\sqrt{(-)}$. Since $\mathrm{Idl}(\mathcal{K})$ is a complete lattice, the reflective subcategory $\mathrm{Rad}(\mathcal{K}) \subseteq \mathrm{Idl}(\mathcal{K})$ is again a complete lattice.

We will now show that $\mathrm{Rad}(\mathcal{K})^\omega = \sqrt{\mathrm{Prin}(\mathcal{K})}$. For $\sqrt{\mathrm{Prin}(\mathcal{K})} \subseteq \mathrm{Rad}(\mathcal{K})^\omega$, we first note that the inclusion $\iota\colon \mathrm{Idl}(\mathcal{K}) \hookrightarrow \mathrm{Rad}(\mathcal{K})$ preserves filtered colimits. It suffices to show that a filtered colimits of radical ideals is again radical. Choose an object $x^{\otimes 2} \in \bigcup_{i\in I} \mathcal{I}_i$, where $(\mathcal{I}_i)_{i\in I}$ is a filtered system of radical ideals. In particular, $x^{\otimes 2} \in \mathcal{I}_i$ for some $i \in I$. Therefore, we have $x \in \mathcal{I}_i$ and $x \in \bigcup_{i\in I} \mathcal{I}_i$, which proves that $\bigcup_{i\in I} \mathcal{I}_i$ is radical. Now we can move on further to show that $\sqrt{\langle a\rangle}$ is compact in $\mathrm{Rad}(\mathcal{K})$, where $a \in \mathcal{K}$. In fact, by combining the result above, we obtain the following sequence of equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Rad}(\mathcal{K})}(\sqrt{\langle a\rangle}, \operatorname*{colim}_{i\in I} \mathcal{I}_i) &\simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\langle a\rangle, \iota(\operatorname*{colim}_{i\in I} \mathcal{I}_i)) \\
&\simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\langle a\rangle, \operatorname*{colim}_{i\in I} \iota(\mathcal{I}_i)) \\
&\simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Idl}}(\langle a\rangle, \iota(\mathcal{I}_i)) \\
&\simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Rad}(\mathcal{K})}(\sqrt{\langle a\rangle}, \mathcal{I}_i) \\
&\simeq \mathrm{Map}_{\mathrm{Rad}(\mathcal{K})}(\sqrt{\langle a\rangle}, \operatorname*{colim}_{i\in I} \mathcal{I}_i).
\end{aligned}$$

Conversely, for $\mathrm{Rad}(\mathcal{K})^\omega \subseteq \sqrt{\mathrm{Prin}(\mathcal{K})}$, consider $\mathcal{I} \in \mathrm{Rad}(\mathcal{K})^\omega$. By (a) above, the tt-ideal $\mathcal{I}$ can be written as a filtered colimit $\mathcal{I} = \bigvee_{a\in\mathcal{I}} \langle a\rangle$ and the functor $\sqrt{(-)}$ preserves filtered colimits, hence

$$\mathcal{I} \simeq \sqrt{\mathcal{I}} \simeq \bigvee_{a\in\mathcal{I}} \sqrt{\langle a\rangle}.$$

Since $\mathcal{I}$ is a compact object of $\mathrm{Rad}(\mathcal{K})$, the inclusion $\mathcal{I} \subseteq \bigvee_{a\in\mathcal{I}} \sqrt{\langle a\rangle}$ corresponds to an inclusion $\mathcal{I} \subseteq \sqrt{\langle a\rangle}$ for some $a \in \mathcal{I}$. Therefore, the tt-ideal $\mathcal{I} = \sqrt{\langle a\rangle}$ is the radical of a principal tt-ideal.

We will show that the poset $\mathrm{Rad}(\mathcal{K})$ is a coherent frame. It suffices to show that the compact objects of $\mathrm{Rad}(\mathcal{K})$ is closed under finite limits, generates $\mathrm{Rad}(\mathcal{K})$ under colimits and finite limits in $\mathrm{Rad}(\mathcal{K})$ distributes over arbitrary colimits. That the compact objects of $\mathrm{Rad}(\mathcal{K})$ generates $\mathrm{Rad}(\mathcal{K})$ under colimits follows from that every tt-ideal $\mathcal{I}$ can be written as a filtered colimit $\mathcal{I} = \bigvee_{a\in\mathcal{I}} \langle a\rangle$ and that the left adjoint $\sqrt{(-)}$ preserves all colimit. That compact objects of $\mathrm{Rad}(\mathcal{K})$ sare closed under finite limits follows from finite limits in $\mathrm{Rad}(\mathcal{K})$ being the finite intersections and Lemma 1.7.

We will now show that finite limit distributes over arbitrary colimits in $\mathrm{Rad}(\mathcal{K})$. It is obvious that

$$\bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i) \subseteq \mathcal{J} \cap \left(\bigvee_{i\in I} \mathcal{I}_i\right),$$

where $\mathcal{I}_i$ and $\mathcal{J}$ are radical tt-ideals. In order to show the reverse inclusion, assume that $x \in \mathcal{J} \cap (\bigvee_{i\in I} \mathcal{I}_i)$. By that $x \in \mathcal{J}$ and the intersection of tt-ideals is again a tt-ideal, we find that

$$x \otimes y \in \mathcal{J} \cap \mathcal{I}_i \subseteq \bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i),$$

for every $i \in I$ and $y \in \mathcal{I}_i$. This implies that, for every $i \in I$, we have

$$\mathcal{I}_i \subseteq (\bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i) : x),$$

where the right hand side is a radical ideal, by Lemma 1.11. This induces an inclusion of the supremum

$$x \in \bigvee_{i\in I} \mathcal{I}_i \subseteq (\bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i) : x).$$

But then this implies that $x \otimes x \in \bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i)$, hence $x \in \bigvee_{i\in I} (\mathcal{J} \cap \mathcal{I}_i) \in \mathrm{Rad}(\mathcal{K})$.

□

2. Localization theory for 2-rings

Before constructing the Balmer spectrum, we first introduce the localization theory of 2-rings. The guiding idea is that tt-ideals are precisely the subcategories that can be quotiented out, in analogy with the role of ideals in ordinary commutative algebra.

Definition 2.1 (Karoubi quotient)**.** Let $\mathcal{K} \in \mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{J} \subseteq \mathcal{K}$ be a stable subcategory closed under retract. Let W denote the collection of all arrows whose cofiber lies in $\mathcal{J}$. The *Karoubi quotient by* $\mathcal{J}$ is the idempotent completion of $\mathcal{K}[W^{-1}]$, the localization at the class of morphisms W. We denote this construction by $\mathrm{L}_{\mathcal{J}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{J}$, or simply by $\mathcal{K}/\mathcal{J}$.

Proposition 2.2. *Let $\mathcal{K} \in \mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{J} \subseteq \mathcal{K}$ a stable subcategory closed under retract. The following holds:*

(a) The Karoubi quotient $\mathcal{K}/\mathcal{J}$ is a stable category, and the quotient $\mathrm{L}_{\mathcal{J}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{J}$ is exact.

(b) The induced functor

$$\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \colon \mathrm{Ind}(\mathcal{K}) \to \mathrm{Ind}(\mathcal{K}/\mathcal{J})$$

is a Bousfield localization with kernel $\mathrm{Ind}(\mathcal{J})$ *(the full subcategory spanned by the objects sent to* 0*).*

The right adjoint is given by the unique colimit preserving functor

$$\mathrm{Ind}(\mathcal{K}/\mathcal{J}) \to \mathrm{Ind}(\mathcal{K})$$

which extends the functor $\mathcal{K}/\mathcal{J} \to \mathrm{Ind}(\mathcal{K})$ given by the Yoneda embedding $x \mapsto \mathrm{Map}_{\mathcal{K}/\mathcal{J}}(\mathrm{L}_{\mathcal{J}}(-), x)$.

(c) Let $\mathcal{K} \in 2\,\mathrm{CAlg}$ and $\mathcal{J} \subseteq \mathcal{K}$ a tt*-ideal. There is a unique way to simultaneously endow the category $\mathcal{K}/\mathcal{J}$ and the functor $\mathcal{K} \to \mathcal{K}/\mathcal{J}$ with a stably symmetric monoidal structure.*

If $\mathcal{L}$ is another 2-ring, then composition with $\mathcal{K} \to \mathcal{K}/\mathcal{J}$ induces an equivalence between $\mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}/\mathcal{J}, \mathcal{L})$ and the full subcategory of $\mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}, \mathcal{L})$ on those functors which send the objects of $\mathcal{J}$ to 0.

Proof. (a) In this case, we claim that finite limits and colimits in $\mathcal{K}[W^{-1}]$ are computed in $\mathcal{K}$[1] (see [NS18, Theorem I.3.3]). Since stability is a condition expressed in terms of finite limits and finite colimits, this implies that if $\mathcal{K}$ is stable, the localization $\mathcal{K}[W^{-1}]$ is again stable and the functor $\mathcal{K} \to \mathcal{K}[W^{-1}]$ is exact.

Since the Yoneda embedding

$$よ_{\mathcal{K}[W^{-1}]} \colon \mathcal{K}[W^{-1}] \to \mathrm{Ind}(\mathcal{K}[W^{-1}])$$

preserves finite colimits and all limits, in particular it preserves the zero object and hence $\mathrm{Ind}(\mathcal{K}[W^{-1}])$ is pointed and cofibers exists. Being a colimit, the suspension functor $\Sigma \colon \mathcal{K}[W^{-1}] \to \mathcal{K}[W^{-1}]$ commutes with filtered colimits, hence, it induces a functor

$$\Sigma \colon \mathrm{Ind}(\mathcal{K}[W^{-1}]) \to \mathrm{Ind}(\mathcal{K}[W^{-1}]),$$

which is precisely the suspension functor on $\mathrm{Ind}(\mathcal{K}[W^{-1}])$. In order to show that $\mathrm{Ind}(\mathcal{K}[W^{-1}])$ is stable, it sufficies to show that the above functor is an equivalence. We find that the loop space functor $\Omega \colon \mathcal{K}[W^{-1}] \to \mathcal{K}[W^{-1}]$ commutes with filtered colimits, as finite limits (which is the same as finite colimits) commutes with filtered colimits. Therefore, it induces a functor

$$\Omega \colon \mathrm{Ind}(\mathcal{K}[W^{-1}]) \to \mathrm{Ind}(\mathcal{K}[W^{-1}]),$$

which is precisely the loop space functor.Since $\mathcal{K}[W^{-1}]$ is stable, they form inverses to each other.

We will now show that the full subcategory

$$\mathcal{K}/\mathcal{J} = \mathrm{Ind}(\mathcal{K}[W^{-1}])^{\omega} \subseteq \mathrm{Ind}(\mathcal{K}[W^{-1}])$$

is closed under finite colimits and shifts. Closure under finite colimits follows from that compact objects are closed under finite colimits in general. Therefore, it suffices to show that compact objects in

[1] In the literature, there is no easier way to prove this.

$(\mathrm{Ind}(\mathcal{K}[W^{-1}]))^{\omega}$ is closed under taking loop spaces. Let $x \in (\mathrm{Ind}(\mathcal{K}[W^{-1}]))^{\omega}$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{K}[W^{-1}])}(\Omega(x), \operatorname*{colim}_{i\in I} y_j) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}[W^{-1}])}(x, \Sigma(\operatorname*{colim}_{i\in I} y_j)) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}[W^{-1}])}(x, \operatorname*{colim}_{i\in I} \Sigma(y_j)) \\
&\simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}[W^{-1}])}(x, \Sigma(y_j)) \\
&\simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}[W^{-1}])}(\Omega(x), y_j),
\end{aligned}$$

which follows from stability of $\mathrm{Ind}(\mathcal{K}[W^{-1}])$. A full-subcategory of a stable category that is closed under shifts and finite colimits is again stable. This implies that $\mathrm{Ind}(\mathcal{K}[W^{-1}])$ is stable, and the functor

$$\mathrm{L}_{\mathcal{J}} \colon \mathcal{K} \to \mathcal{K}[W^{-1}] \xrightarrow{\text{よ}_{\mathcal{K}[W^{-1}]}} \mathcal{K}/\mathcal{J}$$

is exact, since the Yoneda embedding $\text{よ}_{\mathcal{K}[W^{-1}]}$ is left exact, hence exact.

(b) By (a), the functor $\mathrm{L}_{\mathcal{J}}$ is exact. By the universal property, there exists a unique filtered colimit preserving functor $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\mathrm{L}_{\mathcal{J}}} & \mathcal{K}/\mathcal{J} \\
{\scriptstyle \text{よ}_{\mathcal{K}}}\downarrow & & \downarrow{\scriptstyle \text{よ}_{\mathcal{K}/\mathcal{J}}} \\
\mathrm{Ind}(\mathcal{K}) & \xrightarrow{\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})} & \mathrm{Ind}(\mathcal{K}/\mathcal{J})
\end{array}$$

We now show that $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ preserves finite limits. Assume that J is finite and I is filtered, we compute

$$\begin{aligned}
\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})(\lim_{j\in J} \operatorname*{colim}_{i\in I} \text{よ}_{\mathcal{K}}(x_{ij})) &\simeq \mathrm{Ind}(\mathrm{L}_{\mathcal{J}})(\operatorname*{colim}_{i\in I} \lim_{j\in J} \text{よ}_{\mathcal{K}}(x_{ij})) \\
&\simeq \mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \left(\operatorname*{colim}_{i\in I} \text{よ}_{\mathcal{K}}(\lim_{j\in J} x_{ij}) \right) \\
&\simeq \operatorname*{colim}_{i\in I} \left((\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \circ \text{よ}_{\mathcal{K}})(\lim_{j\in J} x_{ij}) \right) \\
&\simeq \operatorname*{colim}_{i\in I} \left((\text{よ}_{\mathcal{K}/\mathcal{J}} \circ \mathrm{L}_{\mathcal{J}})(\lim_{j\in J} x_{ij}) \right) \\
&\simeq \operatorname*{colim}_{i\in I} \lim_{j\in J} \left(\text{よ}_{\mathcal{K}}/\mathcal{J} \circ \mathrm{L}_{\mathcal{J}}(x_{ij}) \right) \\
&\simeq \lim_{j\in J} \operatorname*{colim}_{i\in I} \left(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \circ \text{よ}_{\mathcal{K}}(x_{ij}) \right) \\
&\simeq \lim_{j\in J} \mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \left(\operatorname*{colim}_{i\in I} \text{よ}_{\mathcal{K}}(x_{ij}) \right).
\end{aligned}$$

Therefore, the functor $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ is exact. Since the functor also preserves filtered colimits, it preserves all small colimits. By the adjoint functor theorem, the functor admits a right adjoint R.

We will first show that R is the unique colimit preserving functor extending the canonical map $\mathcal{K}/\mathcal{J} \to \mathrm{Ind}(\mathcal{K})$. In order to show that R preserves all colimits, it suffices to show that R preserves filtered colimits, since R is exact.

We will show that the left adjoint $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ preserves compact objects. Since $\mathcal{K}$ is idempotent complete, we have $\mathrm{Ind}(\mathcal{K})^{\omega} \simeq \mathcal{K}$, hence the compact objects of $\mathrm{Ind}(\mathcal{K})$ and $\mathrm{Ind}(\mathcal{K}/\mathcal{J})$ are precisely the objects in the essential image of $\text{よ}_{\mathcal{K}}$ and $\text{よ}_{\mathcal{K}/\mathcal{J}}$ respectively. Since the functor $\mathrm{Ind}(\mathcal{K}) \to \mathrm{Ind}(\mathcal{K}/\mathcal{J})$ restricts to a functor $\mathcal{K} \to \mathcal{K}/\mathcal{J}$, this implies that $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ preserves compact objects.

Since the category $\mathrm{Ind}(\mathcal{K})$ is freely generated by the representable presheaves (that is, the compact objects) under filtered colimits, $\mathrm{Ind}(\mathcal{K})$ is compactly generated and there exists a jointly conservative collection of compact objects $\yo_{\mathcal{K}}(x_i) \in \mathrm{Ind}(\mathcal{K})$. For $\mathrm{colim}_{j\in J} \yo_{\mathcal{K}}(y_j)$ in $\mathrm{Ind}(\mathcal{K})$, where J is filtered,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left(\yo_{\mathcal{K}} x_i, R(\operatorname*{colim}_{j\in J} \yo_{\mathcal{K}}(y_i))\right) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left(\mathrm{L}_{\mathcal{J}}(\yo_{\mathcal{K}}(x_i)), \operatorname*{colim}_{j\in J} \yo_{\mathcal{K}}(y_i)\right) \\
&\simeq \operatorname*{colim}_{j\in J} \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}/\mathcal{J})}\left(\mathrm{L}_{\mathcal{J}}(\yo_{\mathcal{K}}(x_i)), \yo_{\mathcal{K}}(y_i)\right) \\
&\simeq \operatorname*{colim}_{j\in J} \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left(\yo_{\mathcal{K}}(x_i), R(\yo_{\mathcal{K}}(y_i))\right) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left(\yo_{\mathcal{K}}(x_i), \operatorname*{colim}_{i\in I} R(\yo_{\mathcal{K}}(y_i))\right).
\end{aligned}$$

By the jointly conservativity of the collection of compact objects $(x_i)_{i\in I}$, we conclude that

$$R(\operatorname*{colim}_{j\in J} \yo_{\mathcal{K}}(y_i)) \simeq \operatorname*{colim}_{i\in I} R(\yo_{\mathcal{K}}(y_i)),$$

hence R preserves filtered colimits. Now, we will show that R extends the given functor $\mathcal{K} \longrightarrow \mathrm{Ind}(\mathcal{K})$ in the statement. By the Yoneda lemma, it suffices to compute

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left(\yo_{\mathcal{K}}(x_i), R(\yo_{\mathcal{K}/\mathcal{J}}(y))\right) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left((\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}) \circ \yo_{\mathcal{K}})(x_i), \yo_{\mathcal{K}/\mathcal{J}}(y)\right) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}\left((\yo_{\mathcal{K}/\mathcal{J}} \circ \mathrm{L}_{\mathcal{J}})(x_i), \yo_{\mathcal{K}/\mathcal{J}}(y)\right) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}(\mathrm{L}_{\mathcal{J}}(x_i), y).
\end{aligned}$$

We will now show that the counit of the adjunction is an equivalence. Since $\mathrm{L}_{\mathcal{J}}$ is essentially surjective, $\left(\yo_{\mathcal{K}/\mathcal{J}}(\mathrm{L}_{\mathcal{J}}(k))\right)_{k\in\mathcal{K}}$ is a jointly conservative collection of compact objects of $\mathcal{K}/\mathcal{J}$. We have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{K}/\mathcal{J})}\left(\yo_{\mathcal{K}/\mathcal{J}}(\mathrm{L}_{\mathcal{J}}(k)), \mathrm{Ind}(\mathrm{L}_{\mathcal{J}})R(x)\right) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}/\mathcal{J})}\left(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})\left(\yo_{\mathcal{K}}(k)\right), \mathrm{Ind}(\mathrm{L}_{\mathcal{J}})R(x)\right) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}(\yo_{\mathcal{K}}(k), R\,\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})R(x)) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K})}(\yo_{\mathcal{K}}(k), R(x)) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}/\mathcal{J})}\left(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})\left(\yo_{\mathcal{K}}(k)\right), x\right) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{K}/\mathcal{J})}\left(\yo_{\mathcal{K}/\mathcal{J}}(\mathrm{L}_{\mathcal{J}}(k)), x\right),
\end{aligned}$$

for every $x \in \mathrm{Ind}(\mathcal{K}/\mathcal{J})$. Therefore the counit of the adjunction is an equivalence. By a standard fact from category theory, the right adjoint is then fully faithful.

Finally, we will show that the kernel of the functor $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ is $\mathrm{Ind}(\mathcal{J})$. We will first show that $\mathrm{Ind}(\mathcal{J}) \subseteq \ker(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}))$. For $x \in \mathcal{J}$, since $\mathrm{cofib}(0 \longrightarrow x) \simeq x \in \mathcal{J}$, the unique map $0 \longrightarrow x$ lies in the class of morphism W. Therefore, in the localization $\mathcal{K}[W^{-1}]$, we have $x \simeq 0$. Since the Yoneda embedding $\yo_{\mathcal{K}/\mathcal{J}}$ preserves the zero object, x is sent to 0 in $\mathcal{K}/\mathcal{J}$ under $\mathrm{L}_{\mathcal{J}}$. By (b), we have shown that $\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$ preserves all colimits, hence for a filtered colimit $\mathrm{colim}_{i\in I} \yo_{\mathcal{K}}(x_i)$ in $\mathrm{Ind}(\mathcal{J})$, we have

$$\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})\left(\operatorname*{colim}_{i\in I} \yo_{\mathcal{K}}(x_i)\right) \simeq \operatorname*{colim}_{i\in I} \mathrm{Ind}\,(\mathrm{L}_{\mathcal{J}})\,(\yo_{\mathcal{K}}(x_i)) \simeq (\yo_{\mathcal{K}/\mathcal{J}} \circ \mathrm{L}_{\mathcal{J}})(x_i) \simeq \yo_{\mathcal{K}/\mathcal{J}}(0) \simeq 0.$$

Now, we will show that $\ker(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}})) \subseteq \mathrm{Ind}(\mathcal{J})$. Let $x \in \ker(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}))$. Then we can write x as a filtered colimit $x \simeq \mathrm{colim}_{i\in I} \yo_{\mathcal{K}}(x_i)$, where $x_i \in \mathcal{K}$. Since the right adjoint of $\mathrm{L}_{\mathcal{J}}$ preserves filtered colimits, one can consider

$$T(x) \simeq R(\operatorname*{colim}_{i\in I} \mathrm{L}_{\mathcal{J}}(x_i)) \simeq \operatorname*{colim}_{i\in I} T(x_i),$$

where $T = R \circ \mathrm{Ind}(\mathrm{L}_{\mathcal{J}})$. We consider the fiber $F_i = \mathrm{fib}(x_i \longrightarrow T(x_i))$. Since in a stable category, filtered colimits commutes with finite limits, therefore, we have the following fiber sequence

$$\operatorname*{colim}_{i\in I} F_i \longrightarrow x \longrightarrow \operatorname*{colim}_{i\in I} T(x_i).$$

Since $x_i \in \ker(\mathrm{Ind}(\mathrm{L}_{\mathcal{J}}))$, we have $T(x_i) \simeq 0$, hence in the above fiber sequence, the right hand side is zero. In other words, we have $\mathrm{colim}_{i\in I} F_i \to x$. By [NS18, Theorem I.3.3][2], we have $F_i \in \mathrm{Ind}(\mathcal{I})$ and hence $\mathrm{colim}_{i\in I} F_i \in \mathrm{Ind}(\mathcal{I})$, this shows that $x \in \mathrm{Ind}(\mathcal{I})$.

(c) Since the tensor product of $\mathcal{K}$ is exact in both variables, the class of morphisms W is closed under tensor products. By the basic theory of monoidal localization (see [Lur17, Proposition 2.2.1.9]), there is a unique way to simultaneously endow the category $\mathcal{K}$ and $\mathcal{K}[W^{-1}]$ and the functor $\mathcal{K} \to \mathcal{K}[W^{-1}]$ with a symmetric monoidal structure and for any other symmetric monoidal category $\mathcal{L}$, the restriction induces a fully faithful functor between functor categories of symmetric monoidal functors

$$\mathrm{Fun}^{\otimes}(\mathcal{K}[W^{-1}], \mathcal{L}) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathcal{K}, \mathcal{L}),$$

whose essential image consists of those symmetric monoidal functors $\mathcal{K} \to \mathcal{L}$ which inverts the morphisms in W. Since $\mathcal{K} \to \mathcal{K}[W^{-1}]$ is an exact functor between stable categories, the above inclusion restricts to an fully faithful functor between categories of symmetric monoidal exact functors

$$\mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}[W^{-1}], \mathcal{L}) \hookrightarrow \mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}, \mathcal{L}),$$

with essential image consisting of functors that send objects in $\mathcal{I}$ to 0. By [Lur17, Proposition 4.8.1.10], the idempotent completion $\mathcal{K}[W^{-1}] \to \mathcal{K}/\mathcal{I}$ admits a uniquely symmetric structure such that the functor

$$\mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}/\mathcal{I}, \mathcal{L}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}[W^{-1}], \mathcal{L})$$

is an equivalence for any $\mathcal{L} \in 2\,\mathrm{CAlg}$. Assume that there is another way to simultaneously enhance $\mathcal{K} \to \mathcal{K}/\mathcal{I}$ to a symmetric monoidal functor, then by the description of $\mathrm{L}_{\mathcal{J}}$ in (a), the functor must factor through a symmetric monoidal functor $\mathcal{K}[W^{-1}] \to \mathcal{K}/\mathcal{I}$. By the above equivalence, this functor corresponds to the identity $\mathrm{id}_{\mathcal{K}/\mathcal{I}} \colon \mathcal{K}/\mathcal{I} \to \mathcal{K}/\mathcal{I}$. Therefore, the uniqueness follows.

□

Remark 2.3 (Locally small localization). In Proposition 2.2, we also need to know that the localization $\mathcal{K}[W^{-1}]$ is locally small. But this can be shown in terms of calculus of fractions.

Proposition 2.4. *Let $\mathcal{K} \in 2\,\mathrm{CAlg}$.*

(a) There exists an adjunction

$$\mathcal{K}/- \colon \mathrm{Idl}(\mathcal{K}) \rightleftarrows 2\,\mathrm{CAlg}_{\mathcal{K}/} : \ker(-),$$

where the left adjoint is fully faithful and the right adjoint preserves filtered colimits.

(b) For $\mathcal{I} \in \mathrm{Idl}(\mathcal{K})$, the Karoubi quotient $\mathcal{K}/\mathcal{I} \in (2\,\mathrm{CAlg}_{\mathcal{K}/})^{\omega}$ if and only if $\mathcal{I}$ is principal.

Proof. (a) Let $\mathcal{J} \in \mathrm{Idl}(\mathcal{K})$. For any map of 2-rings $f \colon \mathcal{K} \to \mathcal{L}$, we have the following pullback square:

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{ex},\otimes}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I}, \mathcal{L}) & \longrightarrow & \mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}/\mathcal{I}, \mathcal{L}) \\ \downarrow & & \downarrow {\scriptstyle (\mathrm{L}_{\mathcal{J}})^*} \\ \{f\} & \longrightarrow & \mathrm{Fun}^{\mathrm{ex},\otimes}(\mathcal{K}, \mathcal{L}) \end{array}$$

The pullback consists of functors $\mathcal{K}/\mathcal{J} \to \mathcal{L}$ such that the precomposition with $\mathrm{L}_{\mathcal{J}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{I}$ is f. By Proposition 2.2, under the precomposition with $\mathrm{L}_{\mathcal{J}}$, functors $\mathcal{K}/\mathcal{I} \to \mathcal{L}$ corresponds precisely to functor $\mathcal{K} \to \mathcal{L}$ such that objects in $\mathcal{I}$ are being sent to 0. Therefore,

$$\mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I}, \mathcal{L})^{\simeq} \simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I}, \ker(f)) \simeq \begin{cases} * & \mathcal{I} \subseteq \ker(f); \\ \varnothing & \text{otherwise.} \end{cases}$$

[2] Again, I didn't manage to find a way to bypass Theorem I.3.3

Now, we will show that the left adjoint is fully faithful. Let $\mathcal{I}$ and $\mathcal{J}$ be tt-ideals of $\mathcal{K}$. By the above,

$$\mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I},\mathcal{K}/\mathcal{J})^{\simeq} \simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I},\mathcal{J}) \simeq \begin{cases} * & \mathcal{I}\subseteq\mathcal{J}; \\ \varnothing & \text{otherwise,} \end{cases}$$

if we know that the kernel of $\mathrm{L}_{\mathcal{J}}\colon \mathcal{K}\longrightarrow\mathcal{K}/\mathcal{J}$ is $\mathcal{J}$. Since the Yoneda embedding is fully faithful, hence conservative, if $x\in\ker(\mathrm{L}_{\mathcal{J}})$, then the image of x in $\mathcal{K}[W^{-1}]$ is zero. This implies that the cofiber of the map $0\longrightarrow x$ lies in $\mathcal{J}$, but this cofiber is x, hence $x\in\mathcal{J}$. Conversely, assume that $x\in\mathcal{J}$, then the cofiber of $0\longrightarrow x$ lies in $\mathcal{J}$, hence $0\simeq x$ in $\mathcal{K}[W^{-1}]$, which implies that $\mathrm{L}_{\mathcal{J}}(x)\simeq 0$, since the Yoneda embedding preserves the zero object.

We will now show that the right adjoint $\ker(-)$ preserves filtered colimits. Let $F\colon I\longrightarrow 2\,\mathrm{CAlg}_{\mathcal{K}/-}$ be any filtered system. Since the forgetful functor $\mathcal{K}_{/-}$ preserves and creates filtered colimit, we obtain

$$f=\operatorname*{colim}_{i\in I} f_i\colon \mathcal{K}\longrightarrow\operatorname*{colim}_{i\in I}F(i).$$

It suffices to show that $\ker(f)=\mathrm{colim}_{i\in I}\ker(f_i)$. Let $x\in\mathcal{K}$. Recall from the last talk that there exists a Bousfield localization

$$\sharp\colon \mathrm{Cat}_{\mathrm{st}}\longrightarrow\mathrm{Cat}^{\mathrm{perf}}$$

Therefore, the forgetful functor $\mathrm{Cat}^{\mathrm{perf}}\longrightarrow\mathrm{Cat}_{\mathrm{st}}$ preserves and creates filtered colimits.

Recall that if $\mathcal{C}=\mathrm{colim}_{i\in I}\mathcal{C}_i$ is a filtered colimits of small categories, then the mapping spaces in $\mathcal{C}$ can be computed as

$$\mathrm{Map}_{\mathcal{C}}(x,y)\simeq\operatorname*{colim}_{j\in I_{(i_0,i_1)/}}\mathrm{Map}_{\mathcal{C}_j}(c_j,d_j),$$

where $x\in\mathcal{C}_{i_0}$, $y\in\mathcal{C}_{i_1}$, $c_j,d_j\in\mathcal{C}_j$ and $I_{(i_0,i_1)}$ is the category of diagram of the shape $i_0\longrightarrow j\longleftarrow i_1$. We will sketch a proof of this fact as follows. The conditions in the definition of a complete Segal anima is formulated in terms of finite limits, hence commutes with filtered colimits. Therefore, the filtered colimit of a collection of complete Segal animae is still complete Segal. But then filtered colimit is also computed levelwise.

The above shows that the filtered colimit of stable categories with exact transition maps is again stable. This follows from the above fact and that finite diagrams must be contained in some finite stage in the filtered colimit.

Returning back to the proof of the statement, the above implies that $\mathrm{colim}_{i\in I}F(i)$ is stable. In order to prove the statement, we need to show that $x\in\ker(f)$ if and only if $x\in\ker(f_i)$ for some $i\in I$. Assume that $x\in\ker(f_i)$ for some $i\in I$, then $f_i(x)=0$. Since the transition maps are exact, this implies that $f(x)=\mathrm{colim}_{i\in I}f_i(x)=0$. Conversely, assume that $f(x)=0$, we want to show that $f_i(x)=0$ for some $i\in I$. Since π_0 is a left adjoint, the functor $\pi_0\colon\mathrm{An}\longrightarrow\mathrm{Cat}$ preserves filtered colimits. Therefore,

$$\pi_0\,\mathrm{Map}(f(x),f(x))\simeq\operatorname*{colim}_{i\in I_{i_0/}}\pi_0\,\mathrm{Map}_{F(i)}(f_j(x),f_j(x))$$

Recall that in a pointed category, an object is zero if and only if the identity map is the zero map. If $[\mathrm{id}_{f(x)}]\simeq[0_{f(x)}]$, then by the explicit description of filtered colimits of sets, there exists $j\in I$ such that $[\mathrm{id}_{f_j}(x)]\simeq[0_{f_j(x)}]$, hence $f_j(x)\simeq 0$.

(b) Let $\mathcal{I}$ be a tt-ideal of $\mathcal{K}$ and consider a filtered colimit $\mathrm{colim}_{i\in I}\mathcal{K}_i$ in $2\,\mathrm{CAlg}_{\mathcal{K}/}$, where $f_i\colon\mathcal{K}\longrightarrow\mathcal{K}_i$. Assume that $\mathcal{I}$ is compact in $\mathrm{Idl}(\mathcal{K})$, then we have

$$\begin{aligned}\mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I},\operatorname*{colim}_{i\in I}\mathcal{K}_i)^{\simeq} &\simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I},\ker(\operatorname*{colim}_{i\in I}f_i))\\ &\simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I},\operatorname*{colim}_{i\in I}\ker(f_i))\\ &\simeq \operatorname*{colim}_{i\in I}\mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I},\ker(f_i))\\ &\simeq \operatorname*{colim}_{i\in I}\mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I},\mathcal{K}_i)^{\simeq}.\end{aligned}$$

Conversely, assume that $\mathcal{K}/\mathcal{I}$ is compact in $2\,\mathrm{CAlg}_{\mathcal{K}/}$, then for a filtered family of tt-ideals $\mathcal{I}_i$, recall that $\mathcal{I}_i$ is the kernel of the functor $f_i\colon \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_i$ in $2\,\mathrm{CAlg}_{\mathcal{K}/}$, therefore

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Idl}}(\mathcal{I}, \operatorname*{colim}_{i\in I}(\mathcal{I}_i)) &\simeq \mathrm{Map}_{\mathrm{Idl}}(\mathcal{I}, \operatorname*{colim}_{i\in I}\ker(f_i)) \\
&\simeq \mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}\left(\mathcal{I}, \ker\left(\operatorname*{colim}_{i\in I} f_i\right)\right) \\
&\simeq \mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I}, \operatorname*{colim}_{i\in I}\mathcal{K}/\mathcal{I}_i)^{\simeq} \\
&\simeq \operatorname*{colim}_{i\in I}\mathrm{Fun}^{\otimes,\mathrm{ex}}_{\mathcal{K}/}(\mathcal{K}/\mathcal{I}, \mathcal{K}/\mathcal{I}_i)^{\simeq} \\
&\simeq \operatorname*{colim}_{i\in I}\mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I}, \ker(f_i)) \\
&\simeq \operatorname*{colim}_{i\in I}\mathrm{Map}_{\mathrm{Idl}(\mathcal{K})}(\mathcal{I}, \mathcal{I}_i).
\end{aligned}$$

In the above, we have used the fact that the core functor $(-)^{\simeq}$ preserves filtered colimits (this can be shown by detecting the filtered colimit by a jointly conservative collection of finite animae, and use the fact that $(-)^{\simeq}$ is the right adjoint of the inclusion $\mathrm{An} \longrightarrow \mathrm{Cat}$). The above shows that $\mathcal{I}$ is compact in $\mathrm{Idl}(\mathcal{K})$ if and only if $\mathcal{K}/\mathcal{I}$ is compact in $2\,\mathrm{CAlg}_{\mathcal{K}/}$. The statement then follows from Proposition 1.12. □

3. The Balmer spectrum

We will review some basic facts about *Stone duality*, following [Joh82, §II.3] and [Hoy25, §V].

Definition 3.1 (Ideals of a distributive lattice)**.** Let L be a distributive lattice.

(a) A subset $I \subseteq L$ is called *downward-closed* if $x \in I$ and $y \le x$ implies that $y \in I$.
(b) An *ideal* of L is a *downward-closed* subset $I \subset L$, which is closed under finite colimits.

We denote $\mathrm{Idl}(L)$ the poset of ideals of L ordered by inclusion.

Remark 3.2 (Idl as Ind-completion)*.* Let L be a distributive lattice. There exists a natural equivalence

$$F\colon \mathrm{Idl}(L) \xrightarrow{\sim} \mathrm{Ind}(L) \simeq \mathrm{Fun}^{\mathrm{lex}}(L^{\mathrm{op}}, \mathrm{An}).$$

Since both sides are 0-truncated, one can write down this functor by hands:

$$F(I)(x) = \begin{cases} * & x \in I, \\ \varnothing & x \notin I. \end{cases}$$

We will first show that the functor $F(I)\colon L^{\mathrm{op}} \longrightarrow \mathrm{An}$ preserves finite limits. For $x, y \in L$, we have to show

$$F(I)(x \vee y) = F(I)(x) \times F(I)(y).$$

If $x \vee y \in I$, then by downward-closure, $x \in I$ and $y \in I$, hence $F(I)(x) = F(I)(y) = *$ and both sides are $*$. Conversely, the case where $x \vee y \notin I$ but $x \in I$ and $y \in I$ cannot happen, since I is closed under finite colimits.

For essential surjectivity, assume that $G\colon L^{\mathrm{op}} \longrightarrow \mathrm{An}$ preserves finite limits. In particular, for every $x \in L$,

$$G(x) \simeq G(x \vee x) \simeq G(x) \times G(x).$$

This implies that $G(x)$ is (-1)-truncated, hence $G(x) \simeq *$ or $G(x) \simeq \varnothing$. We define

$$I = \{x \in L \mid G(x) \simeq *\}.$$

Then one can verify easily that G is given by the functor $F(I)$.

For fully faithfulness, one can check by hand that

$$\mathrm{Nat}(F(I), F(J)) \simeq \mathrm{Map}_{\mathcal{I}}(I, J) = \begin{cases} * & I \subseteq J, \\ \varnothing & \text{else.} \end{cases}$$

Proposition 3.3. *Let L be a distributive lattice. The poset of ideals* $\mathrm{Idl}(L)$ *is a coherent frame.*

Proof. We will first show that $\mathrm{Idl}(L)$ is a frame. We will first show completeness. The colimit $\bigvee_{i\in I} J_i$ is given smallest ideal containing the union $\bigcup_{i\in I} J_i$. Concretely, it is given by

$$\bigvee_{i\in I} J_i = \{x \in L \mid x \le y_{i_1} \vee \ldots \vee y_{i_n}, \text{ for some } y_{i_k} \in I_{i_k}\}.$$

The limit $\bigwedge_{i\in I} J_i$ is given by the intersection $\bigcap_{i\in I} J_i$. It is easy to verify that this is an ideal.

Moreover, we show that finite limits distributes over arbitrary colimits. In other words, we need to show

$$K \cap \bigvee_{i\in I} J_i = \bigvee_{i\in I} (K \cap J_i),$$

where K and J_i are ideals of L. The direction

$$\bigvee_{i\in I} (K \cap J_i) \subseteq K \cap \bigvee_{i\in I} J_i$$

is clear, since $K \cap J_i \subseteq K \cap \bigvee_{i\in I} J_i$ for every $i \in I$. Conversely, assume that $x \in K \cap \bigvee_{i\in I} J_i$, then $x \le y_{i_1} \vee \ldots \vee y_{i_n}$ for some $y_{i_k} \in J_{i_k}$ and $x \in K$. Since L is distributive, it is closed under finite limit, hence we have

$$x \le x \wedge (y_{i_1} \vee \ldots \vee y_{i_n}) = (x \wedge y_{i_1}) \vee \ldots \vee (x \wedge x_{y_{i_n}}),$$

since L is distributive. By downward-closure, we have $x \wedge y_{i_k} \in K \cap J_{i_k}$ and therefore $x \in \bigvee_{i\in I} (K \cap \mathcal{J}_i)$. We will now show that L is idempotent complete. By the description Remark 3.2, it suffices to show that $(\mathrm{Idl}(L))^\omega \simeq L$. We construct this equivalence by hand. Let $x \in L$, we define the *ideal generated by* x as $\langle x \rangle = \{y \in L \mid y \le x\}$. It is clear that this is the smallest ideal of L that contains x. We define the functor

$$L \longrightarrow (\mathrm{Idl}(L))^\omega, \quad x \mapsto \langle x \rangle.$$

We will first show that $\langle x \rangle$ is compact. Let $\bigvee_{i\in I} J_i$ be a filtered colimit of ideals. We want to show that

$$\mathrm{Map}_{\mathrm{Idl}(L)} \left(\langle x \rangle, \bigvee_{i\in I} J_i \right) \simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Idl}(L)} \left(\langle x \rangle, J_i \right).$$

It suffices to show that $x \in \bigvee_{i\in I} J_i$ if and only if $x \in J_i$ for some $i \in I$. Assume that $x \in J_i$, it is obvious that $x \in \bigvee_{i\in I} J_i$. Conversely, assume that $x \in \bigvee_{i\in I} J_i$, then there exists $y_{i_k} \in J_{i_k}$ such that $x \le y_{i_1} \vee \ldots \vee y_{i_n}$. Since the family is filtered, there exists $i' \in I$ such that $y_{i_k} \in J_{i'}$ for all k. Since $J_{i'}$ is an ideal, hence closed finite colimits. This implies that $x \in J_{i'}$. Since $\langle x \rangle \subseteq \langle y \rangle$ implies that $x \in \langle y \rangle$, which implies that $x \le y$, the functor we defined above is fully faithful. We will now show that every compact object in $\mathrm{Idl}(L)$ is the principal ideal. Let I be compact in $\mathrm{Idl}(L)$, We can write $I = \bigvee_{x\in I} \langle x \rangle$. Consider $I \subseteq \bigvee_{i\in I} \langle x \rangle$, which implies that $I = \langle x_1 \rangle \vee \ldots \vee \langle x_n \rangle = \langle x_1, \ldots, x_n \rangle$ for some $x_1 \vee \ldots \vee x_n \in I$.

Therefore, the compact objects of $\mathrm{Ind}(L)$ is precisely the objects in the image of the Yoneda embedding. Since Yoneda embedding preserves small limits, compact objects are closed under finite limits. By definition of $\mathrm{Ind}(L)$, any object can be written as the filtered colimit of representables. Therefore, the frame L is coherent. □

Proposition 3.4. *Let $f : L \longrightarrow L'$ be a morphism of distributive lattices. One can define a morphism of coherent frames* $\mathrm{Idl}(f) : \mathrm{Idl}(L) \longrightarrow \mathrm{Idl}(L')$ *by sending an ideal $I \subseteq L$ to the smallest ideal containing $f(I)$ in L'. This assembles into a functor and there is an canonical equivalence*

$$\mathrm{Idl}(-) : \mathrm{DLat} \xrightarrow{\sim} \mathrm{Frm}^{\mathrm{coh}},$$

where the inverse is given by taking compact objects $(-)^\omega$.

Proof. Let L be a distributive lattice. Recall that L is idempotent complete and

$$(\mathrm{Idl}(L))^\omega \simeq \mathrm{Ind}(L)^\omega \simeq L.$$

Let F be a coherent frame. In particular, every object of F can be written as a small colimit of compact objects, hence F is compactly generated. Therefore, we have

$$\mathrm{Idl}(F^{\omega}) \simeq \mathrm{Ind}(F^{\omega}) \simeq F.$$

This proves the proposition. □

Example 3.5 (The frame of a topological space). Let T be a topological space. The poset $\mathrm{Open}(T)$ of open subsets of T is a frame:

- Given a family of open subsets $(U_i)_{i\in I}$, its supremum is the union $\bigcup_{i\in I} U_i$ and its infimum is the interior of the intersection $\bigcap_{i\in I} U_i$.
- The distributivity law follows from set-theoretic distributivity law and the fact that finite intersections of open subsets are open.

If $T \to S$ is a morphism of topological spaces, then $f^{-1}\colon \mathrm{Open}(S) \to \mathrm{Open}(T)$ is a morphism of posets that preserves colimits and finite limits. Hence it is a morphism of frames. This defines a functor

$$\mathrm{Open}\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Frm}.$$

Definition 3.6 (Point of a frame)**.** Let F be a frame. We define a topological space $|F|$ as follows:

- The elements of $|F|$ are the *points of* F, that is, the morphisms of frames $F \to [1]$, where $[1] = \{0 \le 1\}$.
- The open sets of $|F|$ are the subsets of the form $|u| = \{p\colon F \to [1] \mid p(u) = 1\}$, where $u \in F$.

Given a morphism of frames $f\colon F' \to F$, the induced map $|f|\colon |F| \to |F'|$ is continuous, since the preimage of $|u|$ is $|f(u)|$. The intuition of the above definition is that, the morphism $F \to [1]$ detects whether an element belongs to a certain open set. The open sets are precisely the points that is in an open set $u \in F$.

Remark 3.7 (The adjunction between topological spaces and frames). One can check by hand that the functor $\mathrm{Open}\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Frm}$ is left adjoint to the functor $|-|$. This adjunction is idempotent, that is, it restricts to an equivalence between the essential images of both functors.

Definition 3.8 (Sober spaces and spatial frames)**.** We have the following definitions:

(a) A topological space T is *sober* if its lies in the essential image of $|-|\colon \mathrm{Frm}^{\mathrm{op}} \to \mathrm{Top}$, or equivalently, the unit map $T \to |\mathrm{Open}(T)|$ is an isomorphism.
(b) A frame F is *spatial*, if it lies in the essential image of $\mathrm{Open}\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Frm}$, or equivalently if the counit map $\mathrm{Open}(|F|) \to F$ is an isomorphism.

We denote $\mathrm{Frm}^{\mathrm{spa}}$ the category of spatial frames and $\mathrm{Top}^{\mathrm{sob}}$ the category of sober spaces.

Remark 3.9. The adjunction Remark 3.7 restricts to an equivalence between the subcategory $\mathrm{Frm}^{\mathrm{spa}} \subseteq \mathrm{Frm}$ and $\mathrm{Top}^{\mathrm{sob}} \subseteq \mathrm{Top}$. It furthermore restricts to an equivalence between the subcategory $\mathrm{Frm}^{\mathrm{coh}} \subseteq \mathrm{Frm}^{\mathrm{spa}}$ and the opposite of $\mathrm{Top}^{\mathrm{spec}} \subseteq \mathrm{Top}^{\mathrm{sob}}$, the category of *spectral spaces*, with objects given by those sober spaces which are quasi-compact with a basis of quasi-compact open subsets closed under finite intersections, and morphisms given by quasicompact maps.

Remark 3.10 (Hochster duality). We have the following facts:

(a) The category DLat admits an involutive duality given by $L \mapsto L^{\mathrm{op}}$, that is, by flipping the partial order and interchanging joins and meets.
(b) By Proposition 3.4, there is an involutive duality $(-)^{\vee}$ on $\mathrm{Frm}^{\mathrm{coh}}$, given by $F^{\vee} \simeq \mathrm{Idl}\left((F^{\omega})^{\mathrm{op}}\right)$, where F is a coherent frame.
(c) By Remark 3.9, there is an involutive duality $(-)^{\vee}$ on $\mathrm{Top}^{\mathrm{spec}}$, where for a topological space X, the dual $X^{\vee}$ has the same underlying set as X, and the topology is generated by an open basis comprising the complements of quasi-compact open subsets of X.

The involutive duality in (b) and (c) are called *Hochster duality*.

Let $\mathcal{K} \in 2\,\mathrm{CAlg}$. By Proposition 1.12, the poset $\mathrm{Rad}(\mathcal{K})$ is a coherent frame, hence it is spatial. Therefore, we may form the following definiton:

Definition 3.11. Let $\mathcal{K} \in 2\,\mathrm{CAlg}$. The *Balmer spectrum* of $\mathcal{K}$, denoted by $\mathrm{Spc}(\mathcal{K})$, is the spectral space associated to the Hochster dual frame $\mathrm{Rad}(\mathcal{K})^\vee$.

Remark 3.12 (Comparison to the classical tensor triangulated geometry). Let $\mathcal{K} \in 2\,\mathrm{CAlg}$.

(a) Unwinding the definitions, $\mathrm{Spc}(\mathcal{K})$ is the topological space whose points are labelled by prime tt-ideals of $\mathcal{K}$, and whose topology is generated by open subsets $U(a) = \{\mathcal{P} \subseteq \mathcal{K} \text{ prime} \mid a \in \mathcal{P}\}$ for $a \in \mathcal{K}$

(b) The term tensor triangulated geometry originates from the fact that the Balmer spectrum was first introduced for triangulated categories rather than for 2-rings. Nevertheless, there is a canonical isomorphism

$$\mathrm{Spc}(\mathrm{h}(\mathcal{K})) \simeq \mathrm{Spc}(\mathcal{K}),$$

where $\mathcal{K} \in 2\,\mathrm{CAlg}$, and $\mathrm{h}(-)$ denotes the homotopy category functor. Indeed, all of the notions introduced in these notes are already visible at the level of the homotopy category.

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