

## 2-RINGS

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ABSTRACT. These are the notes for the third talk in the seminar *Higher Zariski Geometry*.

### 1. PRESENTABLE (STABLE) CATEGORIES

**Remark 1.1.** Recall that a cardinal  $\kappa$  is called *regular* if the category of sets of cardinality  $< \kappa$  has all colimits of size  $< \kappa$ . All the cardinals appearing in these notes are assumed to be regular. Recall that a category is called  $\kappa$ -*small* if its underlying space is  $\kappa$ -small and every homotopy group of its mapping spaces are  $\kappa$ -small. Recall also that a category  $I$  is  $\kappa$ -*filtered* if every diagram  $K \rightarrow I$  from a  $\kappa$ -small category can be extended to a cocone diagram  $K^\triangleright \rightarrow I$ .

**Construction 1.2.** Let  $\mathcal{C} \in \text{Cat}$  be a category. Let  $\text{PSh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$  be the presheaf category and let  $\mathfrak{y}_{\mathcal{C}} : \mathcal{C} \rightarrow \text{PSh}(\mathcal{C})$  be the Yoneda embedding. The  $\text{Ind}_{\kappa}$ -*completion* of  $\mathcal{C}$  is defined as the full subcategory

$$\text{Ind}_{\kappa}(\mathcal{C}) \subseteq \text{PSh}(\mathcal{C}),$$

on those presheaves which are  $\kappa$ -filtered colimits of representables. Notice that the Yoneda embedding factors through a fully-faithful functor  $\mathfrak{y} : \mathcal{C} \rightarrow \text{Ind}_{\kappa}(\mathcal{C})$ . One can show, by restricting the universal property of the presheaf category, that the  $\text{Ind}_{\kappa}$ -completion is such that restriction along the Yoneda embedding induces an equivalence

$$\mathfrak{y}^* : \text{Fun}^{\kappa\text{-filtered}}(\text{Ind}_{\kappa}(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}),$$

for every category  $\mathcal{D}$  with  $\kappa$ -filtered colimits. Here the left hand side denotes the full subcategory of  $\text{Fun}(\text{Ind}_{\kappa}(\mathcal{C}), \mathcal{D})$  spanned by those functors preserving  $\kappa$ -filtered colimits.

**Remark 1.3.** Let  $\mathcal{C} \in \text{Cat}^{\kappa\text{-filtered}}$  be a category with  $\kappa$ -filtered colimits. Recall that an object  $x \in \mathcal{C}$  is called  $\kappa$ -*compact* if mapping out of it preserve  $\kappa$ -filtered colimits. The collection of  $\kappa$ -compact objects organize into a full subcategory  $\mathcal{C}^{\kappa} \subseteq \mathcal{C}$  which is in general closed under  $\kappa$ -small colimit, as  $\kappa$ -small limits commute with  $\kappa$ -filtered colimits in  $\text{Spc}$ . If  $\mathcal{C} = \text{Ind}_{\kappa}(\mathcal{D})$  for a small category  $\mathcal{D} \in \text{Cat}$ , then there is an inclusion  $\mathcal{D} \subseteq (\text{Ind}_{\kappa}(\mathcal{D}))^{\kappa}$  which is not an equality in general.

**Definition 1.4.** We say that a category is  $\kappa$ -*presentable* if it is of the form  $\text{Ind}_{\kappa}(\mathcal{C})$  for  $\mathcal{C} \in \text{Cat}^{\text{rex}(\kappa)}$  a small category with  $\kappa$ -small colimits. We let  $\text{Pr}_{\kappa}^{\text{L}} \subseteq \widehat{\text{Cat}}$  be the non full subcategory of  $\kappa$ -presentable categories and colimit preserving functors that preserve the  $\kappa$ -compact object.

**Remark 1.5.** Let  $\lambda \geq \kappa$  be regular cardinal. Then every  $\lambda$ -filtered category is  $\kappa$ -filtered, so that every  $\kappa$ -compact object is a  $\lambda$ -compact object. Furthermore, every  $\kappa$ -presentable category is  $\lambda$ -presentable. Indeed, if  $\mathcal{C} \simeq \text{Ind}_{\kappa}(\mathcal{C}^{\kappa})$  with  $\mathcal{C}^{\kappa}$  small and admitting  $\kappa$ -small colimits, then every object of  $\mathcal{C}^{\kappa}$  is  $\kappa$ -compact, hence  $\lambda$ -compact. Thus  $\mathcal{C}^{\kappa} \subseteq \mathcal{C}^{\lambda}$ . Since  $\mathcal{C}^{\kappa}$  generates  $\mathcal{C}$  under  $\kappa$ -filtered colimits, and every  $\kappa$ -filtered colimit may be refined by  $\lambda$ -filtered colimits after enlarging the compact subcategory to  $\mathcal{C}^{\lambda}$ , the canonical functor

$$\text{Ind}_{\lambda}(\mathcal{C}^{\lambda}) \rightarrow \mathcal{C}$$

is an equivalence, making  $\mathcal{C}$  is  $\lambda$ -presentable.

**Remark 1.6.** One can work with all  $\kappa$ -presentable categories at the same time by defining the category of *presentable categories* as the colimit

$$\mathrm{Pr}^{\mathrm{L}} = \mathrm{colim}_{\kappa} \mathrm{Pr}_{\kappa}^{\mathrm{L}}$$

along the inclusion  $\mathrm{Pr}_{\kappa}^{\mathrm{L}} \hookrightarrow \mathrm{Pr}_{\lambda}^{\mathrm{L}}$  for  $\lambda \geq \kappa$ .

Arrows of presentable categories are well behaved:

**Remark 1.7.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between categories in  $\mathrm{Pr}^{\mathrm{L}}$ . Then  $f$  is a left adjoint if and only if it preserves small colimits, and it is a right adjoint if and only if it preserves small limits and is accessible, that is it preserves  $\kappa$ -filtered colimits for some regular cardinal  $\kappa$ .

We now turn to the commutative algebra of  $\mathrm{Pr}^{\mathrm{L}}$ .

**Remark 1.8.** Lurie constructed in [Lur17, Section 4.8] a symmetric monoidal structure on  $\mathrm{Pr}^{\mathrm{L}}$  characterized by the following universal property: given two presentable categories  $\mathcal{C}$  and  $\mathcal{D}$ , their Lurie tensor product is a presentable category  $\mathcal{C} \otimes \mathcal{D}$  with a functor  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes \mathcal{D}$  such that for every presentable category  $\mathcal{E}$ , precomposition with it induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \rightarrow \mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Here the superscript  $\mathrm{L}, \mathrm{L}$  denotes the full subcategory spanned by those functors  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  which preserve colimits separately in each variable. It follows from this universal property that the symmetric monoidal structure is also closed. To be precise, for every pair of presentable categories  $\mathcal{C}$  and  $\mathcal{D}$ , there exists a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}))$$

in  $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ , so that  $\mathrm{Fun}^{\mathrm{L}}(-, -)$  exhibits the internal hom. Since  $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Spc}, \mathcal{C}) \simeq \mathcal{C}$ , it follows that the category of spaces  $\mathrm{Spc}$  is the neutral element for the Lurie tensor product. Furthermore, one can show that  $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ .

We will need a stable version of this observation.

**Construction 1.9.** Let  $\mathrm{Sp}$  denote the category of spectra, defined as the colimit

$$\mathrm{Sp} = \mathrm{colim}(\mathrm{Spc}_{*} \xrightarrow{\Sigma} \mathrm{Spc}_{*} \xrightarrow{\Sigma} \dots)$$

in  $\mathrm{Pr}^{\mathrm{L}}$ , or dually, as the limit

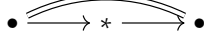
$$\mathrm{Sp} \simeq \lim(\dots \xrightarrow{\Omega} \mathrm{Spc}_{*} \xrightarrow{\Omega} \mathrm{Spc}_{*})$$

in  $\mathrm{Pr}^{\mathrm{R}}$ , the category of presentable categories and right adjoints. Here  $\mathrm{Spc}_{*}$  denotes the category of pointed spaces, whereas  $\Sigma$  and  $\Omega$  denote the suspension and loop functor, respectively. By adjoining a point to an space and then “infinitely suspending it” we obtain a colimit preserving functor  $\Sigma_{+}^{\infty} : \mathrm{Spc} \rightarrow \mathrm{Sp}$ .

**Remark 1.10.** Let  $\Sigma_{+}^{\infty} : \mathrm{Spc} \rightarrow \mathrm{Sp}$  be the suspension spectrum functor. Notice that a presentable category  $\mathcal{C}$  is stable if and only if the canonical map  $\mathrm{id}_{\mathcal{C}} \otimes \Sigma_{+}^{\infty} : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathrm{Sp}$  is an equivalence. In particular, since  $\mathrm{Sp}$  is a stable category, the inverse of the equivalence  $\mathrm{Sp} \rightarrow \mathrm{Sp} \otimes \mathrm{Sp}$  makes  $\mathrm{Sp}$  into a commutative algebra of  $\mathrm{Pr}^{\mathrm{L}}$ , whose unit is given by the sphere spectrum  $\mathbb{S}$ . It follows that the category of stable presentable categories and left adjoints can be realized as  $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}} \simeq \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})$  and that taking spectrum objects  $- \otimes \mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}} \rightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$  defines a symmetric monoidal functor.

## 2. RETRACTS AND IDEMPOTENT COMPLETION

**Notation 2.1.** We let  $\text{Idem}^+$  denote the following category



We also let  $\text{Idem}$  to be the full subcategory of  $\text{Idem}^+$  spanned by the object lying in the middle (so that  $\text{Idem}$  consists of an object  $*$  together with a map  $e : * \rightarrow *$  such that  $e \circ e \simeq e$ ).

Given a category  $\mathcal{C} \in \text{Cat}$  we will refer to a functor  $\text{Idem} \rightarrow \mathcal{C}$  as an *idempotent*. An idempotent  $\text{Idem} \rightarrow \mathcal{C}$  is called *effective* if it extends to a map  $\text{Idem}^+ \rightarrow \mathcal{C}$ .

**Definition 2.2.** A category is called *idempotent-complete* if every idempotent is effective.

**Remark 2.3.** A retract closed full subcategory of an idempotent-complete category is idempotent-complete. Indeed, if  $\mathcal{C} \subseteq \mathcal{D}$  is a retract closed full subcategory of an idempotent one, and  $e : x \rightarrow x$  is an idempotent in  $\mathcal{C}$ , then the idempotent  $e$  is effective in  $\mathcal{D}$ , being the latter idempotent-complete.

Thus there exist an object  $y \in \mathcal{D}$  and maps  $y \xrightarrow{i} x \xrightarrow{p} y$  in  $\mathcal{D}$  such that  $pi \simeq \text{id}_y$  and  $ip \simeq e$ . In particular,  $y$  is a retract of  $x$  in  $\mathcal{D}$ . Since  $x \in \mathcal{C}$  and  $\mathcal{C}$  is closed under retracts, we have  $y \in \mathcal{C}$ . Since  $\mathcal{C} \subseteq \mathcal{D}$  is full, the maps  $i : y \rightarrow x$  and  $p : x \rightarrow y$  belong to  $\mathcal{C}$ . Therefore the above retraction data lie in  $\mathcal{C}$ , and hence the idempotent  $e$  is effective in  $\mathcal{C}$ .

**Remark 2.4.** Every category admitting telescopes is idempotent-complete. Indeed, let  $e : x \rightarrow x$  be an idempotent. Consider the sequential diagram  $x \xrightarrow{e} x \xrightarrow{e} x \xrightarrow{e} \dots$  and let

$$y := \text{colim}(x \xrightarrow{e} x \xrightarrow{e} x \rightarrow \dots).$$

The colimit maps induce a morphism  $p : x \rightarrow y$ . On the other hand, the maps  $e : x \rightarrow x$  at every stage are compatible with the transition maps, because  $e^2 \simeq e$ , and therefore induce a morphism  $i : y \rightarrow x$ . Then one has

$$ip \simeq e, \quad pi \simeq \text{id}_y.$$

Thus  $e$  is split. Hence every idempotent is effective.

We now show how to turn a category into an idempotent one.

**Construction 2.5.** Let  $\mathcal{C} \in \text{Cat}$  be a category. Let  $\text{PSh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{op}, \text{Spc})$  be the presheaf category and let  $\mathfrak{y}_{\mathcal{C}} : \mathcal{C} \rightarrow \text{PSh}(\mathcal{C})$  be the Yoneda embedding. Define  $\mathcal{C}^{\natural} \subseteq \text{PSh}(\mathcal{C})$  to be the full subcategory spanned by retracts of representables, that is by objects  $P$  for which there exists  $X \in \mathcal{C}$  and maps  $P \rightarrow \mathfrak{y}_{\mathcal{C}}(X) \rightarrow P$  whose composite is equivalent to  $\text{id}_P$ . We call  $\mathcal{C}^{\natural}$  the *idempotent completion* of  $\mathcal{C}$ .

**Proposition 2.6.** Let  $\mathcal{C} \in \text{Cat}$  be a category. Then inclusion  $\mathfrak{y}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^{\natural}$  is fully faithful and the category  $\mathcal{C}^{\natural}$  is idempotent complete. Furthermore, precomposition with the Yoneda embedding induces an equivalence

$$(\mathfrak{y}_{\mathcal{C}})^* : \text{Fun}(\mathcal{C}^{\natural}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$$

for every idempotent complete  $\mathcal{D}$ .

*Proof.* The Yoneda embedding is fully faithful, hence so is its restriction to  $\mathcal{C}^{\natural}$ . The presheaf category  $\text{PSh}(\mathcal{C})$  admits all small colimits, hence is idempotent complete by [Remark 2.4](#). Since  $\mathcal{C}^{\natural}$  is, by definition, closed under retracts inside  $\text{PSh}(\mathcal{C})$ , it is itself idempotent complete by [Remark 2.3](#).

For the universal property, let  $\mathcal{D}$  be idempotent complete and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be any functor. By left Kan extension along  $\mathfrak{y}_{\mathcal{C}}$  we obtain  $\overline{F} : \text{PSh}(\mathcal{C}) \rightarrow \mathcal{D}$ , and since  $\mathcal{D}$  is idempotent complete the functor  $\overline{F}$  carries retracts of representables to retracts of the corresponding values of  $F$ . Thus  $\overline{F}$  factors essentially uniquely through  $\mathcal{C}^{\natural}$ , yielding an extension  $\mathcal{C}^{\natural} \rightarrow \mathcal{D}$ . Conversely, any functor  $\mathcal{C}^{\natural} \rightarrow \mathcal{D}$  is determined by restriction to  $\mathcal{C}$  because every object of  $\mathcal{C}^{\natural}$  is a retract of some  $\mathfrak{y}_{\mathcal{C}}(X)$ .  $\square$

**Remark 2.7.** Let  $\mathcal{C} \in \text{Cat}^{\text{add}}$  be an additive category. The functor  $\mathfrak{y}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^{\natural}$  exhibits  $\mathcal{C}$  as a *dense* subcategory of  $\mathcal{C}^{\natural}$ , that is, every object of  $\mathcal{C}^{\natural}$  is a direct summand of an object of  $\mathcal{C}$ .

## 3. 2-RINGS

We now study idempotent-completeness related to stable categories.

**Definition 3.1.** We let  $\text{Cat}^{\text{perf}} \subseteq \text{Cat}^{\text{st}}$  denote the full subcategory spanned by the idempotent-complete stable categories.

**Definition 3.2.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  in  $\text{Cat}^{\text{st}}$  be an exact functor. We will say that  $f$  is a *Karoubi equivalence* if the induced functor  $f^{\natural} : \mathcal{C}^{\natural} \rightarrow \mathcal{D}^{\natural}$  is an equivalence.

In other terms,  $f$  is a Karoubi equivalence if it is fully faithful and its essential image is dense, in the sense of [Remark 2.7](#), that is every object of  $\mathcal{D}$  is a retract of some object in the essential image of  $f$ .

**Proposition 3.3.** There exists a Bousfield localization  $(-)^{\natural} : \text{Cat}^{\text{st}} \rightleftarrows \text{Cat}^{\text{perf}} : \text{incl}$  at the class of Karoubi equivalences.

*Proof.* Let  $\mathcal{C} \in \text{Cat}^{\text{st}}$ . Its idempotent completion  $\mathcal{C}^{\natural}$  is again stable (and this can be shown by noting that it is a full subcategory of  $\text{Fun}^{\text{ex}}(\mathcal{C}^{\text{op}}, \text{Sp})$  closed under finite limits and finite colimits). Let  $\mathcal{D} \in \text{Cat}^{\text{perf}}$ . By the universal property of idempotent completion, restriction along the Yoneda embedding induces an equivalence

$$\mathfrak{y}^* : \text{Fun}(\mathcal{C}^{\natural}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Since the Yoneda embedding  $\mathcal{C} \rightarrow \mathcal{C}^{\natural}$  is exact and  $\mathcal{D}$  is stable, this equivalence restricts to an equivalence

$$\text{Fun}^{\text{ex}}(\mathcal{C}^{\natural}, \mathcal{D}) \simeq \text{Fun}^{\text{ex}}(\mathcal{C}, \mathcal{D}).$$

Thus  $(-)^{\natural}$  is left adjoint to the inclusion  $\text{Cat}^{\text{perf}} \hookrightarrow \text{Cat}^{\text{st}}$ . Finally, a functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  is inverted by this localization if and only if  $f^{\natural} : \mathcal{C}^{\natural} \rightarrow \mathcal{D}^{\natural}$  is an equivalence. By definition, these are precisely the Karoubi equivalences.  $\square$

We record a convenient strengthening of the above (proved in [\[CDH<sup>+</sup>25, Proposition A.3.4\]](#)).

**Remark 3.4.** Let  $\mathcal{C} \in \text{Cat}^{\text{st}}$  be a stable category and let  $K_0(\mathcal{C})$  be its Grothendieck group. Define the *minimalisation*  $\mathcal{C}_{\min} \subseteq \mathcal{C}$  to be the full stable subcategory spanned by objects  $X$  with  $[X] = 0$  in  $K_0(\mathcal{C})$ . We call  $\mathcal{C}$  *minimal* if  $\mathcal{C}_{\min} = \mathcal{C}$ .

Then localising  $\text{Cat}^{\text{st}}$  at the class of Karoubi equivalences yields both a left and a right Bousfield localisation. The left-local objects are the minimal stable categories, and the right-local objects are the idempotent complete stable categories. Moreover, the left and right adjoints of this localisation are given by  $\mathcal{C} \mapsto \mathcal{C}_{\min}$  and  $\mathcal{C} \mapsto \mathcal{C}^{\natural}$ , respectively. In particular, an exact functor is a Karoubi equivalence if and only if it induces an equivalence on minimalisations, and if and only if it induces an equivalence on idempotent completions. Consequently, the functor  $(-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Cat}^{\text{perf}}$  preserves both limits and colimits.

We now show that  $\text{Cat}^{\text{perf}}$  may be recovered inside  $\text{Pr}_{\text{st}}^{\text{L}}$ .

**Proposition 3.5.** There is an equivalence  $\text{Ind} : \text{Cat}^{\text{perf}} \rightleftarrows \text{Pr}_{\text{st}}^{\text{L}, \omega} : (-)^{\omega}$ .

*Proof.* It is clear that the ind-completion and taking compact objects sit into an adjunction  $\text{Ind} : \text{Cat}^{\text{perf}} \rightleftarrows \text{Pr}_{\text{st}}^{\text{L}, \omega} : (-)^{\omega}$ , thus it suffices to show that unit and counit are equivalences. Now:

- (1) Let  $\mathcal{C} \in \text{Cat}^{\text{perf}}$  and consider the unit  $\mathcal{C} \rightarrow (\text{Ind}(\mathcal{C}))^{\omega}$ . It suffices to show that every compact object of  $\text{Ind}(\mathcal{C})$  lies in the essential image of  $\mathfrak{y}_{\mathcal{C}}$ . Let  $X \in \text{Ind}(\mathcal{C})$  be compact. By definition of  $\text{Ind}(\mathcal{C})$ , the object  $X$  can be written as a filtered colimit  $X \simeq \text{colim}_{i \in I} \mathfrak{y}_{\mathcal{C}}(c_i)$  with  $c_i \in \mathcal{C}$ . Consider the identity map  $\text{id}_X : X \rightarrow X \simeq \text{colim}_i \mathfrak{y}_{\mathcal{C}}(c_i)$ . Since  $X$  is compact,  $\text{id}_X$  factors through some stage, so there exists  $i \in I$  and a map  $u : X \rightarrow \mathfrak{y}_{\mathcal{C}}(c_i)$  such that the composite

$X \xrightarrow{u} \mathfrak{K}_{\mathcal{C}}(c_i) \rightarrow \operatorname{colim}_i \mathfrak{K}_{\mathcal{C}}(c_i) \simeq X$  is equivalent to  $\operatorname{id}_X$ . Thus  $X$  is a retract of  $\mathfrak{K}_{\mathcal{C}}(c_i)$  in  $\operatorname{Ind}(\mathcal{C})$ .

Since  $\mathfrak{K}_{\mathcal{C}}$  is fully faithful, a retract data on  $\mathfrak{K}_{\mathcal{C}}(c_i)$  correspond to retract data on  $c_i$  in  $\mathcal{C}$ . Since  $\mathcal{C}$  is idempotent-complete, this retract is represented by an object  $c \in \mathcal{C}$  and an equivalence  $X \simeq \mathfrak{K}_{\mathcal{C}}(c)$ . Therefore every compact object of  $\operatorname{Ind}(\mathcal{C})$  lies in  $\mathfrak{K}_{\mathcal{C}}(\mathcal{C})$ , and we conclude that the induced functor  $\mathcal{C} \rightarrow (\operatorname{Ind}(\mathcal{C}))^\omega$  is essentially surjective. It is fully faithful since  $\mathfrak{K}_{\mathcal{C}}$  is fully faithful, hence it is an equivalence.

- (2) For the counit, let  $\mathcal{D} \in \operatorname{Pr}_{\operatorname{st}}^{\mathbb{L}, \omega}$  and consider  $\operatorname{Ind}(\mathcal{D}^\omega) \rightarrow \mathcal{D}$ . The inclusion  $i : \mathcal{D}^\omega \hookrightarrow \operatorname{Ind}(\mathcal{D}^\omega)$  extends (by the universal property of ind-completion) to a colimit-preserving functor  $\varepsilon : \operatorname{Ind}(\mathcal{D}^\omega) \rightarrow \mathcal{D}$  whose restriction to  $\mathcal{D}^\omega$  is  $i$ . It suffices to show that  $\varepsilon$  is an equivalence. First,  $\varepsilon$  is essentially surjective since  $\mathcal{D}$  is compactly generated by  $\mathcal{D}^\omega$ , and  $\varepsilon$  preserves colimits and contains  $\mathcal{D}^\omega$  in its image. For full faithfulness, it suffices to check it on compact generators: given  $x, y \in \mathcal{D}^\omega$  the equivalence  $\operatorname{Hom}_{\operatorname{Ind}(\mathcal{D}^\omega)}(\mathfrak{K}(x), \mathfrak{K}(y)) \simeq \operatorname{Hom}_{\mathcal{D}^\omega}(x, y) \simeq \operatorname{Hom}_{\mathcal{D}}(x, y)$  shows that  $\varepsilon$  is fully faithful on the subcategory of compact objects. Since both source and target are generated under colimits by these compact objects and  $\varepsilon$  preserves colimits, it follows that  $\varepsilon$  is fully faithful on all objects. Hence  $\varepsilon$  is an equivalence.

Hence the claim.  $\square$

Using this result, we can equip  $\operatorname{Cat}^{\operatorname{perf}}$  with a monoidal structure.

**Remark 3.6.** Proceed as in [Remark 1.8](#): the category  $\operatorname{Cat}^{\operatorname{st}}$  admits a symmetric monoidal structure characterized as follows. If  $\mathcal{C}, \mathcal{D} \in \operatorname{Cat}^{\operatorname{st}}$ , then  $\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D}$  is the stable category equipped with an exact functor  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D}$  which is exact separately in each variable and such that, for every stable category  $\mathcal{E}$ , precomposition induces an equivalence

$$\operatorname{Fun}^{\operatorname{ex}}(\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D}, \mathcal{E}) \rightarrow \operatorname{Fun}^{\operatorname{ex}, \operatorname{ex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Here  $\operatorname{Fun}^{\operatorname{ex}, \operatorname{ex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  denotes the full subcategory of  $\operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  spanned by those functors which are exact separately in each variable. Using idempotent completion, this gives a symmetric monoidal structure on  $\operatorname{Cat}^{\operatorname{perf}}$  by setting

$$\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{perf}}} \mathcal{D} := (\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D})^{\natural}.$$

Then:

**Proposition 3.7.** The equivalence  $\operatorname{Ind} : \operatorname{Cat}^{\operatorname{perf}} \simeq \operatorname{Pr}_{\operatorname{st}}^{\mathbb{L}, \omega} : (-)^\omega$  is symmetric monoidal.

*Proof.* Let  $\mathcal{E} \in \operatorname{Pr}_{\operatorname{st}}^{\mathbb{L}}$ . Then

$$\begin{aligned} \operatorname{Fun}^{\mathbb{L}}(\operatorname{Ind}(\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{perf}}} \mathcal{D}), \mathcal{E}) &\simeq \operatorname{Fun}^{\operatorname{ex}}(\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{perf}}} \mathcal{D}, \mathcal{E}) \\ &\simeq \operatorname{Fun}^{\operatorname{ex}}(\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D}, \mathcal{E}) \\ &\simeq \operatorname{Fun}^{\operatorname{ex}, \operatorname{ex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}). \end{aligned}$$

The first equivalence is the universal property of Ind-completion, the second follows because  $\mathcal{E}$  is idempotent-complete and the third is the universal property of  $\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{st}}} \mathcal{D}$ . On the other hand,

$$\begin{aligned} \operatorname{Fun}^{\mathbb{L}}(\operatorname{Ind}(\mathcal{C}) \otimes \operatorname{Ind}(\mathcal{D}), \mathcal{E}) &\simeq \operatorname{Fun}^{\mathbb{L}, \mathbb{L}}(\operatorname{Ind}(\mathcal{C}) \times \operatorname{Ind}(\mathcal{D}), \mathcal{E}) \\ &\simeq \operatorname{Fun}^{\operatorname{ex}, \operatorname{ex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}), \end{aligned}$$

where the first equivalence is the universal property of the Lurie tensor product, and the second follows by applying the universal property of Ind-completion separately in the two variables. Thus both  $\operatorname{Ind}(\mathcal{C} \otimes_{\operatorname{Cat}^{\operatorname{perf}}} \mathcal{D})$  and  $\operatorname{Ind}(\mathcal{C}) \otimes \operatorname{Ind}(\mathcal{D})$  corepresent the same functor of  $\mathcal{E} \in \operatorname{Pr}_{\operatorname{st}}^{\mathbb{L}}$ . Hence they are equivalent by Yoneda.  $\square$

**Corollary 3.8.** The Lurie tensor product of two  $\omega$ -presentable stable presentable categories is again  $\omega$ -presentable. Since the category of spectra is  $\omega$ -presentable, it follows that  $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega} \hookrightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$  is symmetric monoidal.

We can finally take commutative algebra objects:

**Definition 3.9.** We denote by  $2\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$  the category of 2-rings.

Explicitly, a 2-ring consist of a small idempotent complete stable category  $\mathcal{A}$  together with a symmetric monoidal structure for which the tensor product  $-\otimes- : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is exact in each variable separately.

#### 4. EXAMPLES

We conclude by discussing some examples.

**Example 4.1.** Let  $X$  be a quasi-compact quasi separated scheme. Then the category of perfect complexes  $\mathrm{Perf}(X)$  defines a 2-ring.

**Example 4.2.** Let  $R \in \mathrm{CAlg}(\mathrm{Sp})$  be a commutative ring spectrum. We denote by  $\mathrm{Perf}_R$  the full subcategory of  $\mathrm{Mod}_R$  spanned by the compact  $R$ -modules. Since  $\mathrm{Mod}_R$  is a stable presentable symmetric monoidal category and the tensor product preserves compact objects,  $\mathrm{Perf}_R$  is a small idempotent-complete stable symmetric monoidal category. Hence  $\mathrm{Perf}_R$  defines a 2-ring.

In particular, for  $R = \mathbb{S}$  one obtains

$$\mathrm{Perf}_{\mathbb{S}} = \mathrm{Sp}^{\omega},$$

the category of finite spectra. This is the initial 2-ring.

**Example 4.3.** Let  $R \in \mathrm{CAlg}(\mathrm{Sp})$  be a commutative ring spectrum and let  $G$  be a finite group. Define

$$\mathrm{Rep}(G, R) := \mathrm{Fun}(BG, \mathrm{Perf}_R).$$

Then  $\mathrm{Rep}(G, R)$  is a small idempotent-complete stable category. Moreover, the pointwise tensor product endows  $\mathrm{Rep}(G, R)$  with a symmetric monoidal structure, which is exact in each variable. Hence  $\mathrm{Rep}(G, R)$  defines a 2-ring.

#### REFERENCES

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