HIGHER ZARISKI GEOMETRY

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Contents

1. Introduction

It is a well known fact that the derived 1-category of sheaves with quasi-coherent cohomology cannot be recovered from the limit of all the derived categories on its affine subsets [Lur17]. This obstruction has led to the development of higher algebra approaches to derived algebraic geometry. In this seminar we will cover the basic definitions and theorems needed to setup the language of this modern theory. To be more detailed: we will introduce stable categories and give some key examples and some basic properties (we will recover classic results for derived categories), we will also define symmetric monoidal categories and their algebras, commutative ring spectra and modules of ring spectra. We will conclude with the definition of the derived category of sheaves with quasi-coherent cohomology and their perfect modules. The main reference used for this work is [Cno26], integrated with [Lur17] and [nLa25]. [Nee96] has just been used for the last statement.

2. Stable categories

Notation 2.1. Though these notes the term category will denote $(\infty, 1)$-categories and ordinary category will denote 1-categories. We will denote the category of small categories with Cat and the category of small animae (or groupoids) with An (An_* if pointed).

2.1. Pointed categories.

Definition 2.2. Let $\mathcal{C}$ be a category with initial object $\varnothing$ and terminal object $*$. We say that $\mathcal{C}$ is pointed if the unique map $\varnothing \to *$ is an equivalence and we define $\varnothing \simeq * =: 0$.

Remark 2.3. Let $\mathcal{C}$ be a pointed category. Let X, Y any two objects in $\mathcal{C}$. We denote with 0 the unique map $0 : X \to 0 \to Y$.

Definition 2.4. Let $\mathcal{C}$ be a pointed category. Let $X \to Y \to Z$ be a sequence of morphisms in $\mathcal{C}$.

(1) $X \to Y \to Z$ is a null sequence if there is an equivalence $h : g \circ f \simeq 0$, equivalently if there is a square in $\mathcal{C}$ of the form:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

(2) We define the following pushout (if it exists) as the cofiber of f

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{Cof}(f) \end{array}$$

(3) We define the following pullback (if it exists) as the fiber of f

$$\begin{array}{ccc} \mathrm{Fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & Y \end{array}$$

(4) $X \to Y \to Z$ is a (co-)fiber sequence if there exists a pullback (respectively pushout) square of the form:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Definition 2.5. Let $\mathcal{C}$ be a pointed category. Let X be an object of $\mathcal{C}$.

(1) If $\mathcal{C}$ has fibers we define the loop space of X to be the following pullback:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

(2) If $\mathcal{C}$ has cofibers we define the suspension of X to be the following pushout:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Lemma 2.6. Let $\mathcal{C}$ be pointed. Assume $\Omega, \Sigma : \mathcal{C} \to \mathcal{C}$ are defined. Then $\Sigma \dashv \Omega$.

Proof. Since $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ commutes with limits in $\mathcal{C}^{op}$ we know that the following square is a pullback:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(\Sigma X, Y) & & \\ \wr & & \\ \mathrm{Hom}_{\mathcal{C}}(X, \Omega Y) & \longrightarrow & \mathrm{Hom}(0, Y) \simeq * \simeq \mathrm{Hom}_{\mathcal{C}}(X, 0) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(X, 0) \simeq * \simeq \mathrm{Hom}_{\mathcal{C}}(0, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X, Y) \end{array}$$

Since $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with limits $\mathrm{Hom}_{\mathcal{C}}(X, \Omega Y)$ is pullback for the same diagram. We conclude by uniqueness of the pullback (up to equivalence). □

2.2. Equivalent definitions of stable categories.

Definition 2.7. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is stable if:

(1) $\mathcal{C}$ is pointed
(2) $\mathcal{C}$ has finite limits and colimits
(3) any square in $\mathcal{C}$ is pullback if and only if is a pushout (we call it exact square).

Definition 2.8. Let $\mathcal{C}$ be stable, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

a sequence of morphisms in $\mathcal{C}$, we say it is an exact sequence if there is an exact square of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \lrcorner \quad \ulcorner & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

Theorem 2.9. Let $\mathcal{C}$ be a pointed category, the followings are equivalent:

(1) $\mathcal{C}$ is stable
(2) $\mathcal{C}$ has both fibers and cofibers and any sequence is a fiber sequence if and only if it is a cofiber sequence
(3) $\mathcal{C}$ has fibers and the loop space functor Ω is an equivalence
(4) $\mathcal{C}$ has co-fibers and the suspension functor Σ is an equivalence

Proof. The implication $1 \Rightarrow 2$ is trivial. [Notice that 2 is self-dual so the implication $2 \Rightarrow 3$ also implies $2 \Rightarrow 4$]
For implication $2 \Rightarrow 3$, assuming 2 We have a cartesian square as the following:

$$\begin{array}{ccc} \Omega X & \xrightarrow{f} & 0 \\ \downarrow & \lrcorner \quad \ulcorner & \downarrow g \\ 0 & \longrightarrow & X \end{array}$$

by uniqueness of pullbacks (up to equivalence) $\Sigma\Omega X \simeq X$. Analogously with the definition of suspension, we obtain the following exact square:

$$\begin{array}{ccc} \Omega\Sigma X \simeq X & \xrightarrow{f} & 0 \\ \downarrow & \lrcorner \quad \ulcorner & \downarrow g \\ 0 & \longrightarrow & \Sigma X \end{array}$$

from which we conclude that $X \simeq \Omega\Sigma X$. Since a natural transformation is an isomorphism if and only if it is equivalence objectwise, we can conclude that $\Omega\Sigma \simeq id$ and $\Sigma\Omega \simeq id$ therefore Ω is an equivalence.

For the implication $3 \Rightarrow 1$, let $\mathcal{P} \subseteq \mathrm{Fun}([1] \times [1], \mathcal{C})$ be the subcategory spanned by pullback squares. We can consider the restriction

$$\mathcal{P} \to \mathrm{Fun}(\ulcorner, \mathcal{C})$$

where $\ulcorner$ is the category $* \leftarrow * \rightarrow *$. We claim it is an equivalence. To prove the claim consider $Y \leftarrow X \rightarrow Z$ in $\mathcal{C}$. We construct:

$$\begin{array}{ccccccc}
\Omega X & \longrightarrow & \Omega Y & \longrightarrow & 0 & & \\
\downarrow & \lrcorner\ \ 6 & \downarrow & \lrcorner\ \ 4 & \downarrow & & \\
\Omega Z & \longrightarrow & a & \longrightarrow & b & \longrightarrow & 0 \\
\downarrow & \lrcorner\ \ 5 & \downarrow & \lrcorner\ \ 3 & \downarrow & \lrcorner\ \ 1 & \downarrow \\
0 & \longrightarrow & c & \longrightarrow & X & \longrightarrow & Z \\
 & & \downarrow & \lrcorner\ \ 2 & \downarrow & & \\
 & & 0 & \longrightarrow & Y & &
\end{array}$$

Where 4,5 and 6 are obtained by pullback pasting. If we restrict to 6 and apply the inverse Ω^{-1} we get

$$\begin{array}{ccccc}
X & & \longrightarrow & & Y \\
\downarrow & \overset{\simeq}{\searrow} & & & \downarrow \simeq \\
 & & \Omega^{-1}\Omega X & \longrightarrow & \Omega^{-1}\Omega Y \\
 & & \downarrow & \lrcorner\ \ \Omega^{-1}6 & \downarrow \\
Z & \xrightarrow{\simeq} & \Omega^{-1}\Omega Z & \longrightarrow & \Omega^{-1}a
\end{array}$$

which defines an inverse to the restriction map above (by functoriality of pullbacks). We would like to show that for any $Y \leftarrow X \rightarrow Z$ in $\mathcal{C}$, we may complete it to a pullback square:

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow \\
Z & \longrightarrow & W
\end{array}$$

We claim it is a pushout square.
Let any a in $\mathcal{C}$

$$\mathrm{Hom}_{\mathrm{Fun}(\ulcorner,\mathcal{C})}\left(\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \\ Z & & \end{array}, \begin{array}{ccc} a & = & a \\ \| & & \\ a & & \end{array}\right) \simeq \mathrm{Hom}_{\mathrm{Fun}([1]\times[1],\mathcal{C})}\left(\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & W \end{array}, \begin{array}{ccc} a & = & a \\ \| & & \| \\ a & = & a \end{array}\right)$$

The restriction to W is an equivalence because for any map $W \rightarrow a$, by functoriality of the pullback, we recover a map between the whole squares, therefore:

$$\mathrm{Hom}_{\mathrm{Fun}([1]\times[1],\mathcal{C})}\left(\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & W \end{array}, \begin{array}{ccc} a & = & a \\ \| & & \| \\ a & = & a \end{array}\right) \simeq \mathrm{Hom}_{\mathcal{C}}(W, a)$$

We are left to prove the existence of arbitrary pullbacks, but we may construct arbitrary pushouts, therefore we can construct all cofibers therefore $\Sigma \dashv \Omega$ and, since Ω is an equivalence, so is Σ. We proved 4 and the opposite argument which shows the existence of arbitrary pullbacks. □

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two pointed categories. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. We say it is:

(1) *pointed* if it preserves terminal objects.
(2) *left exact* if it is pointed and preserves fiber sequences.
(3) *right exact* if it is pointed and preserves cofiber sequences.
(4) *exact* if it is both left and right exact.

Remark 2.11. Notice that a functor between stable categories is left exact if and only if it is right exact if and only if it is exact.

Definition 2.12. We denote with Cat^{st} the subcategory of Cat whose objects are stable categories in Cat and its morphisms are exact functors between them.

Lemma 2.13. Let $\mathcal{C}$, $\mathcal{D}$ be two pointed categories with finite colimits. Let $F : \mathcal{C} \to \mathcal{D}$ be a pointed functor. F sends pushouts to pullbacks (it is reduced and 1-excisive) if and only if for all $x \in \mathcal{C}$ the map $\eta_x : F(x) \to \Omega F \Sigma(x)$ is an equivalence.

Proof. The implication $\Rightarrow$ Is trivial since applying F to the left cartesian square produces the right cocartesian square:

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma x \end{array} \qquad \begin{array}{ccc} F(x) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & F(\Sigma x) \end{array}$$

For the implication $\Leftarrow$, for all $x \in \mathcal{C}, \eta_x : F(x) \to \Omega F \Sigma(x)$ is an equivalence. Consider any pushout:

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & \ulcorner & \downarrow \\ z & \longrightarrow & y \end{array}$$

and construct the following diagram by pushout pasting:

$$\begin{array}{ccccccc} w & \longrightarrow & x & \longrightarrow & 0 & & \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow & & \\ z & \longrightarrow & y & \longrightarrow & \mathrm{Cof}(x \to y) & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{Cof}(z \to y) & \longrightarrow & \Sigma w & \longrightarrow & \Sigma z \\ & & \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ & & 0 & \longrightarrow & \Sigma x & \longrightarrow & \Sigma y \end{array}$$

Applying F to it gives a map

$$F(w) \to F(x) \times_{F(y)} F(z) \to \Omega F(\Sigma w) \to \Omega F(\Sigma x) \times_{\Omega F(\Sigma y)} \Omega F(\Sigma z)$$

The middle arrow is, by hypothesis, both retraction of the first map and section of the third and is therefore and equivalence. We deduce that the other two are equivalences by 2-out of 3 (the whole property is called 2-out of 6). □

Corollary 2.14. Let $\mathcal{C}$ and $\mathcal{D}$ be two stable categories Let $F : \mathcal{C} \to \mathcal{D}$ be a pointed functor. The following are equivalent:

(1) F preserves suspensions
(2) F preserves finite colimits
(3) F preserves the loop spaces
(4) F preserves finite limits

Proof. The implications $2 \Rightarrow 1$ and $4 \Rightarrow 3$ are obvious. Proving the implication $1 \Rightarrow 2$ would dually prove $3 \Rightarrow 4$.
The implication $1 \Rightarrow 2$ follows by the previous Theorem 2.9 and Lemma 2.13 since we have:

$$F(x) \simeq \Omega\Sigma F(x) \simeq \Omega F(\Sigma x)$$

so F sends pushouts to pullbacks which are again pushouts in a stable category.
The implications $1 \Leftrightarrow 3$ are trivial from the definition of stable category. □

2.3. **Stable categories are additive.**

Definition 2.15. Let $\mathcal{C}$ be a pointed category with finite products and coproducts. $\mathcal{C}$ is

(1) semi-additive if, for any X, Y objects of $\mathcal{C}$, $Id : X \amalg Y \to X \times Y$ obtained by

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{(0,1_X)} & \\ X \amalg Y & \xrightarrow{Id} & X \times Y \\ \uparrow & \nearrow_{(0,1_Y)} & \\ Y & & \end{array}$$

is an equivalence. We define $X \oplus Y := X \amalg Y$, the biproduct of X and Y.

(2) additive if $\mathcal{C}$ is semi-additive and, for any X object of $\mathcal{C}$, the shear map:

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{(1_X,1_X)} & \\ X \oplus X & \xrightarrow{shear} & X \oplus X \\ \uparrow & \nearrow_{(0,1_X)} & \\ X & & \end{array}$$

is an equivalence.

Remark 2.16. Let $\mathcal{C}$ be is a semi-additive category. Let $f, g : X \to Y$ be a two morphisms in $\mathcal{C}$. We may define:

$$f + g : X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\binom{1_Y}{1_Y}} Y$$

which induces a structure of commutative monoid on $\pi_0 Hom_{\mathcal{C}}(X, Y)$. If moreover $\mathcal{C}$ is additive we may define the inverse of a map $f : X \to Y$ as follows:

$$X \xrightarrow{f} Y \xrightarrow{(1_Y,0)} Y \oplus Y \xrightarrow{shear^{-1}} Y \oplus Y \xrightarrow{pr_2} Y$$

Euristically we are defining an operation on Y and the last map is just f postcomposed with

$$g \mapsto (g, e) \mapsto (g, -g) \mapsto -g$$

therefore we conclude that the homotopy category of $\mathcal{C}$ is an additive 1-category (in the classical sense).

Lemma 2.17. Let $\mathcal{C}$ be a stable category. $\mathcal{C}$ is additive.

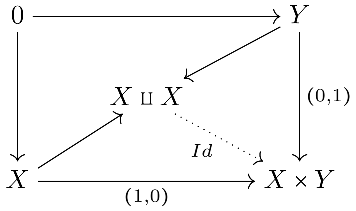
Proof. We first notice that, since $\mathcal{C}$ is stable, it has pullbacks, pushouts and 0, therefore we get all finite products and coproducts. Let X and Y be any two objects in $\mathcal{C}$. We have two squares, as the following, where the left is cartesian and the right is cocartesian:

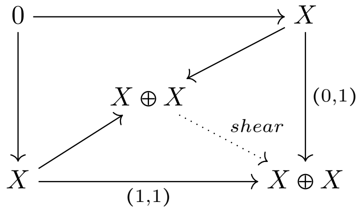
$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & X \amalg Y \end{array}$$

the thesis is equivalent to the following squares being cartesian:

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (0,1)} \\ X & \xrightarrow[(1,0)]{} & X \times Y \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (0,1)} \\ X & \xrightarrow[(1,1)]{} & X \oplus X \end{array}$$

(the second square makes sense only after we've proved the existence of the first).
If they exist:





and both *shear* and Id are equivalences by uniqueness up to equivalence) of pushouts. As we've shown we have the following squares are cartesian:

$$\begin{array}{ccccc} Y & \xrightarrow{(0,1)} & X \times Y & \xrightarrow{pr_Y} & Y \\ \downarrow & \lrcorner & {\scriptstyle pr_X}\downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{(0,1)} & X \oplus X & \xrightarrow{pr_2} & X \\ \downarrow & \lrcorner & {\scriptstyle pr_1}\downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

We may pull back $(0,1)$ along $(1,0)$ and $(1,1)$ respectively, and, by pullback pasting, we get the desired diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & X & & \\ \downarrow & \lrcorner\ \ulcorner & \downarrow{\scriptstyle (1,0)} & & \\ Y & \xrightarrow{(0,1)} & X \times Y & \xrightarrow{pr_Y} & Y \\ \downarrow & \lrcorner & {\scriptstyle pr_X}\downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & X & & \\ \downarrow & \lrcorner\ \ulcorner & \downarrow{\scriptstyle (1,1)} & & \\ X & \xrightarrow{(0,1)} & X \oplus X & \xrightarrow{pr_2} & X \\ \downarrow & \lrcorner & {\scriptstyle pr_1}\downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

The top squares are also pushouts since pullbacks are also exact squares ☐

2.4. **Stable categories as triangulated categories.** Historically, the main tool for studying derived categories has been the theory of triangulated categories. As introduced in the abstract, stable categories were introduced to replace that language in the ∞-categorical case. The following theorem is serves as a formal statement of this.

Proposition 2.18. Let $\mathcal{C}$ be a stable category. Let $h\mathcal{C}$ denote its homotopy category. $h\mathcal{C}$ has a triangulated structure.

Proof. (This will be just a sketch because TR4) will not be proved.)
We already showed that $h(\mathcal{C})$ is an additive category in the classical sense.
We define $\forall n \in \mathbb{Z}$

$$[n] := \begin{cases} \Sigma^n & n \geq 0 \\ \Omega^{-n} & n \leq 0 \end{cases}$$

We choose as class of exact triangles all exact sequences (in the stable sense) $X \to Y \to Z$ such that there exists an exact square of the form:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner\ \ulcorner & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

TR1) we have to prove that $0 \to X = X$ is a triangle and that any $f : X \to Y$ may be completed to a triangle, that's trivially obtained by:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \lrcorner\ \ulcorner & \| \\ 0 & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \lrcorner\ \ulcorner & \downarrow \\ 0 & \longrightarrow & Cof(f) \end{array}$$

TR2) We have to prove that for any $X \to Y \to Z$ triangle, we may "rotate" it to get triangles $Z[-1] \to X \to Y$ and $Y \to Z \to X[1]$. To prove that we start with an exact square granted by the definition of triangle:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

by pushout pasting we get the following cocartesian square:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

The right square is the second thesis we wanted, the first is dual.

TR3) We have to prove that for anyy two maps u, v as below between the triangles $X \to Y \to Z$ and $X' \to Y' \to Z'$, there exists a dotted map as follows:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & X'[1] \end{array}$$

but that's obtained by the universal property of pushouts.

TR4) Follows by pushout pasting.

□

An important tool for the study of triangulated categories are t-structures. There is an analogous definition for stable categories.

Definition 2.19. Let $\mathcal{C}$ be a stable category. A t-structure on $\mathcal{C}$ is a couple of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that:

(1) The pair is *orthogonal*: $\forall x \in \mathcal{C}_{\geq 0}, y \in \mathcal{C}_{\leq 0}$, $\mathrm{Hom}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq *$.
(2) The pair is *stable under shifts*: $\Sigma\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$ and $\Sigma^{-1}\mathcal{C}_{\leq 0} \subseteq \mathcal{C}_{\leq 0}$.
(3) The pair *admits truncations*: $\forall x \in \mathcal{C}$ there exists an exact sequence $x_{\geq 1} \to x \to x_{\leq 0}$ with $x_{\geq 1} \in \Sigma\mathcal{C}_{\geq 0}$ and $x_{\leq 0} \in \mathcal{C}_{\leq 0}$.

Definition 2.20. Let $\mathcal{C}$ be a stable category. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on $\mathcal{C}$. We define the heart of $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ as $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$.

Remark 2.21. The heart of a t-structure of a stable category is an abelian 1-category.

Lemma 2.22. Let $\mathcal{C}$ be a stable category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The following are equivalent:

(1) f is an isomorphism
(2) $\mathrm{Cof}(f) \simeq 0$
(3) $\mathrm{Fib}(f) \simeq 0$

Proof. We show the implications $1 \iff 2$ the other are dual.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Cof}(f) \end{array}$$

is an exact square. We conclude since isomorphisms are stable under pullback. □

Example 2.23. Not all triangulated categories arise as the homotopy category of a stable category.

2.5. **Examples of stable categories.** We end this section with a couple of examples that will be core to the next section.

Example 2.24. Let $\mathcal{A}$ be an abelian category. Its derived category $\mathscr{D}(\mathcal{A}) := \mathscr{C}(\mathcal{A})[Q.iso^{-1}]$ is a stable category (as a localization of ∞−category whose homotopy category is the classical derived category). It is not hard to show that Σ is the the shift $[1]$ which is clearly an equivalence.

Definition 2.25. Let Pr^R be the category of presentable categories and right adjoints. We can define:

$$\mathrm{Sp} := \lim(\quad \cdots \quad \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \quad \cdots \quad)$$

In practice a spectrum E is a sequence of pointed animae $(E_n)_{n\in\mathbb{N}}$ together with specified equivalences in An_*, $\sigma_n : E_n \to \Omega E_{n+1}$. For any two spectra X, Y

$$\mathrm{Hom}_{Sp}(X,Y) = \lim_{\mathbb{N}} \mathrm{Hom}_{\mathrm{An}_*}(X_n, Y_n)$$

Remark 2.26. The construction that we see in Definition 2.25 is the formal way of inverting $\Omega : \mathrm{An}_* \to \mathrm{An}_*$ and therefore produces a stable category by Theorem 2.9.

3. Symmetric monoidal categories and algebras

3.1. Monoids.

Definition 3.1. Let $\mathcal{C}$ be a category with finite products. A monoid in $\mathcal{C}$ is a functor:

$$M : \Delta^{op} \to \mathcal{C}$$
$$[n] \mapsto M_n$$

satisfying the Segal condition: $\forall n \in \mathbb{N}$, $\forall i \leq n$ the maps $e_i : [1] \cong \{i-1, i\} \hookrightarrow [n]$ produce an equivalence:

$$(M(e_i))_{i=1}^n : M_n \to \prod_{i=1}^n M_1$$

Remark 3.2. We observe that the Segal condition provides an equivalence $M_0 \simeq *$ and

$$d_1 : [1] \to [2]$$
$$0 \mapsto 0, 1 \mapsto 2$$

induces a map

$$d_1^* \circ ((e_0, e_1)^*)^{-1} : M_1 \times M_1 \simeq M_2 \to M_1$$

Definition 3.3. Let $\mathcal{C}$ be a category with finite products. We define $\mathrm{Mon}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ as the full subctegory spanned by monoids in $\mathcal{C}$.

We would now introduce commutative monoids in order to define symmetric monoidal categories.

Definition 3.4. Let Span(Fin) be the category with objects finite sets and, for any two S, T finite sets,

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(S,T) := \left\{ \begin{array}{ccc} & U & \\ {}^{g}\swarrow & \downarrow \cong & \searrow^{f} \\ S & & T \\ {}_{g'}\nwarrow & & \nearrow_{f'} \\ & U' & \end{array} \right\}$$

We should to associate an ∞−category to what now is a priori a $(2,1)$-category but I'll not cover these details.

Definition 3.5. Let $\mathcal{C}$ be a category with finite products. A commutative monoid in $\mathcal{C}$ is a functor:

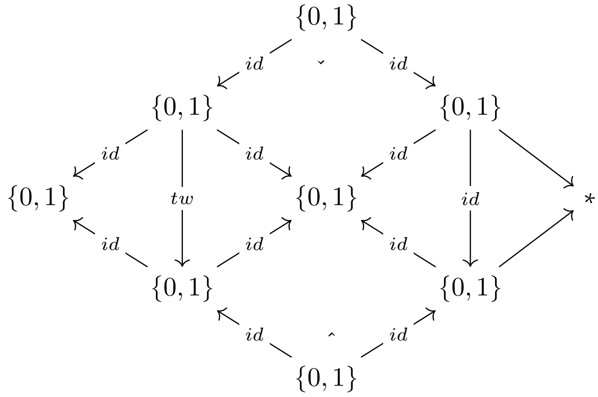
$$M : \mathrm{Span}(\mathrm{Fin}) \to \mathcal{C}$$

that preserves products. We denote with

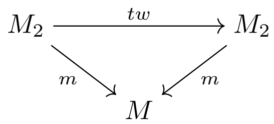
$$\mathrm{CMon}(\mathcal{C}) = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Span}(Fin), \mathcal{C})$$

the full subcategory spanned by commutative monoids.

Remark 3.6. If we set $M := M(*)$ and $M_2 := C(\{0,1\})$, by the following diagram of sets:



We deduce that there is a map $m : M_2 \to M$ due to the right square and a map $tw : M_2 \to M_2$ and a triangle:



We will not write the details showing that commutative monoids are, indeed, monoids.

3.2. **Commutative Algebras and Modules.**

Definition 3.7. A symmetric monoidal category is a commutative monoid in Cat. We will often denote a symmetric monoidal category $\mathcal{C}_\bullet$ as $(\mathcal{C}, \otimes, \mathbb{1})$ where $\mathcal{C} := \mathcal{C}_1$, $\mathbb{1} := (s_0)^* : * \to \mathcal{C}$ and $\otimes := d_1^*$ where d_1 is the map previously mentioned in Remark 3.2 and is the only map $s_0 : [1] \to [0]$.

Definition 3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal categories. We denote with $\mathrm{Map}^{\otimes}(\mathcal{C}, \mathcal{D}) := \mathrm{Hom}_{\mathrm{CMon(Cat)}}(\mathcal{C}, \mathcal{D})$ the collection of symmetric monoidal functors. By evaluating on $*$ we can embed this anima into $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We define $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by $\mathrm{Map}^{\otimes}(\mathcal{C}, \mathcal{D})$.

Remark 3.9. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. By Definition 3.7 we know there are functors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathbb{1} : * \to \mathcal{C}$ that induce functors: $h\otimes : h\mathcal{C} \times h\mathcal{C} \to h\mathcal{C}$ and $h\mathbb{1} : * \to h\mathcal{C}$. They satisfy the usual identities due to the segal condition and the structure is also symmetric due to Remark 3.6.

Definition 3.10. Let Fin be the 1-category of finite sets with the co-catesian monoidal structure.

Definition 3.11. Let $\mathcal{C}$ be a symmetric monoidal category. An algebra over $\mathcal{C}$ is an object of $\mathrm{Fun}^{\otimes}(\mathrm{Fin}, \mathcal{C}) =: \mathrm{CAlg}(\mathcal{C})$.

Remark 3.12. Let $\mathcal{C}$ be a symmetric monoidal category. Let $A(-) : Fin \to \mathcal{C}$ be a commutative algebra over $\mathcal{C}$. Fix $A := A(*)$. Since in Fin $\{0,1\} = * \amalg *$ we deduce that $A(\{0,1\}) \simeq A \otimes A$ and the map $\exists!\{0,1\} \to *$ provides $A \otimes A \to A$.

Definition 3.13. Let $\mathrm{Fin}^{\mathrm{Mod}}$ be the 1-category whose objects are couples of finite sets (S_A, S_M) and its arrows $(S_A, S_M) \to (T_A, T_M)$ are couples of maps of sets $f : S_A \amalg S_M \to T_A \amalg T_M$ such that the following diagram commutes:

$$\begin{array}{ccc} S_A \amalg S_M & \xrightarrow{f} & T_A \amalg T_M \\ \uparrow & & \uparrow \\ S_M & \xrightarrow[f_{|M}]{\cong} & T_M \end{array}$$

We define a monoidal structure on $\mathrm{Fin}^{\mathrm{Mod}}$ given by taking the coproduct componentwise. The unit of such structure is the couple $(\varnothing, \varnothing)$

Definition 3.14. Let $\mathcal{C}$ be a symmetric monoidal category. We define the category of modules in $\mathcal{C}$:

$$\mathrm{Mod}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{Mod}}, \mathcal{C})$$

Remark 3.15. Let $\mathcal{C}$ be a symmetric monoidal category. We could observe that an object of $\mathrm{Mod}(\mathcal{C})$ is a couple (A, M) where A is an algebra over $\mathcal{C}$ and we have a functor $A \otimes M \to M$ obtained by $(*, *) \to (\varnothing, *)$.

Remark 3.16. We may observe that there exists a map:

$$a : \mathrm{Fin} \to \mathrm{Fin}^{\mathrm{Mod}}$$

$$S \mapsto (S, \varnothing)$$

which defines a canonical functor:

$$a^* : \mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$$

which sends a module to its underlying algebra.

Definition 3.17. Let $\mathcal{C}$ be a symmetric monoidal category. Let A be a commutative algebra over $\mathcal{C}$. We define the category of modules in $\mathcal{C}$ over A as the following pullback:

$$\begin{array}{ccc} \mathrm{Mod}_A(\mathcal{C}) & \longrightarrow & \mathrm{Mod}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow a^* \\ * & \xrightarrow[A]{} & \mathrm{CAlg}(\mathcal{C}) \end{array}$$

3.3. **Examples.** We end this work with some examples and some theorems that justify our interest in such objects.

Example 3.18. Let R be a ring, the derived category $\mathscr{D}(\mathrm{Mod}_R)$ with the derived tensor is a symmetric monoidal category.

Notation 3.19. Let R be a ring. We fix $\mathscr{D}(R) \coloneqq \mathscr{D}(\mathrm{Mod}_R)$

Example 3.20. We will see in the next seminar that Sp is symmetric monoidal.

Definition 3.21. We say that a commutative algebra over Sp is a commutative ring spectrum (or $\mathbb{E}_\infty$–ring). In this case we omit Sp in Mod and CAlg. We denote $\mathrm{Mod}_R(\mathrm{Sp})$ as Mod_R.

Definition 3.22 ([nLa25],[Lur17])**.** Let R be a ring spectrum. We denote by $\mathrm{Perf}_R \subseteq \mathrm{Mod}_R$ the smallest stable subcategory closed under retracts and containing R, called the category of perfect modules.

Definition 3.23. Let R be an $\mathbb{E}_\infty$-ring. We can define a t-structure on Mod_R:

(1) $(\mathrm{Mod}_R)_{\geq 0} \coloneqq \langle R \rangle_{colimits,ext}$, the smallest subcategory of Mod_R containing R, closed under colimits and extentions.

(2) $(\mathrm{Mod}_R)_{\leq 0}$ is the full subcategory, $(\mathrm{Mod}_R)_{\geq 0})^{\perp}$, spanned by objects $y \in \mathrm{Mod}_R$ such that $\forall x \in (\mathrm{Mod}_R)_{\geq 0}$ $\mathrm{Hom}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq *$.

Theorem 3.24 (H.A. 7.1.2.13, [Lur17])**.** Let R be a commutative ring spectrum. Mod_R is a compactly generated stable category and its compact objects are the perfect modules. Furthermore, if R is an ordinary ring, if we regard it as a ring spectrum, there is a canonical equivalence of symmetric monoidal categories: $\mathrm{Mod}_R \to \mathscr{D}(A)$, where $A = \mathrm{Mod}_R^{\heartsuit}$ is the heart of the t-structure defined previously in Definition 3.23.

Proof. We give a sketch of the proof about the stability of Mod_R. $\mathrm{Sp} \in \mathrm{Pr}^L_{st}$ (the category of presentable stable categories with left adjoints as morphisms) and its monoidal structure preserves colimits moreover for any X spectrum, $X \simeq \mathrm{colim}\Sigma^n \mathbb{S}$. Let M be an object of Mod_R, we can see M as an object of Sp so $M \simeq \mathrm{colim}\Sigma^n\mathbb{S}$, $R \otimes M \simeq M$, therefore $R \otimes \mathrm{colim}\Sigma^n\mathbb{S} \simeq \mathrm{colim}\Sigma^n R \simeq M$, since the $\Sigma^n R$, $n \in \mathbb{N}$ are compact in Mod_R they generate it. Modules are clearly a pointed category and have all colimits since we can just form colimits in spectra and we know that they commute with the tensor product. Now let $M \simeq \mathrm{colim}_{\Sigma^n\mathbb{S}\to M}\Sigma^n \mathrm{R} \otimes \mathbb{S}$ any module. Since $\mathrm{Mod_R}$ is presentable and Σ preserves coproducts, there exists a functor Ω such that $\Sigma \dashv \Omega$. Now $\Omega\Sigma(\mathrm{R} \otimes -) \simeq \Omega\mathrm{R} \otimes \Sigma -$. We know that $\mathrm{R} \otimes -$ sends pullbacks to pushouts (since pullbacks in *Sp* coincide with pushouts). We conclude by Lemma 2.13 that the last term is equivalent to $\mathrm{R} \otimes -$. Let us apply this equivalence to $\mathrm{colim}\Sigma^n\mathbb{S}$: $\Omega\Sigma(\mathrm{R} \otimes \mathrm{colim}\Sigma^n\mathbb{S}) \simeq \Omega\mathrm{R} \otimes \Sigma\mathrm{colim}\Sigma^n\mathbb{S} \simeq \mathrm{R} \otimes \mathrm{colim}\Sigma^n\mathbb{S} \simeq M$ while the first term is $\Omega\Sigma M$. Now this proves that Σ is full and faithful. Since Σ commutes with colimits and tensor and Σ of Sp is essentially surjective we conclude that Σ on $\mathrm{Mod_R}$ is also essentially surjective. We conclude it is an equivalence by the fundamental theorem of category theory. □

We can extend these notions on rings to quasi-compact quasi-separated schemes.

Example 3.25. Let X be a quasi-compact quasi-separated scheme. We define:

$$\mathrm{QCoh}(X) \coloneqq \lim_{\mathrm{Spec}(A)\to X}\mathrm{Mod}_A$$

If X has affine diagonal (for example if X is separated) this definition coincides with the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology.

Remark 3.26. This definition is justified by the so-called Zariski's descent which states the following: let R be an ordinary ring. Let $f, g \in R$ two elements (non nihilpotent). Then the following square is a pullback:

$$\begin{array}{ccc} \mathscr{D}(R) & \longrightarrow & \mathscr{D}(R[1/g]) \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{D}(R[1/f]) & \longrightarrow & \mathscr{D}(R[1/fg]) \end{array}$$

which is the statement in Example 3.25 when reduced to open affine subsets of an affine scheme.

Remark 3.27. Let X be a quasi-compact, quasi-separated scheme. We can endow $\mathrm{QCoh}(X)$ with a tensor product that restricts to the tensor product on all Mod_A's (as we usually do for schemes).

Example 3.28. Let X be a quasi-compact, quasi-separated scheme. We can define $\mathrm{Perf}(X)$ as the subcategory of dualizable objects in $\mathrm{QCoh}(X)$. Notice that, by above remark Remark 3.27

$$\mathrm{Perf}(X)^{dual} \simeq (\lim_{\mathrm{Spec}(A)\to X}\mathrm{Mod}_A)^{dual} \simeq \lim_{\mathrm{Spec}(A)\to X}(\mathrm{Mod}_A)^{dual} \simeq \lim_{\mathrm{Spec}(A)\to X}\mathrm{Perf}_A$$

The last equivalence is proved by the fact that under the monodidal structure on Mod_A dualizable objects and compact objects coincide (Since Perf_A is full we just need to prove the statement on objects, on the homotopy category use corollary 2.3 of [Nee96])

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