# HIGHER ZARISKI GEOMETRY: AN OVERVIEW

GIOVANNI ROSSANIGO

Abstract. These are the notes for the first talk in the seminar *Higher Zariski Geometry*.

## 1. Introduction

The goal of this talk is to motivate the study of higher Zariski geometry by understanding how geometric objects can be encoded in categorical terms. A guiding principle in modern mathematics is that geometric objects can be studied through increasingly refined invariants. Among these, derived and categorical invariants play a central role. A fundamental example is the assignment

$$X \mapsto \mathrm{Perf}(X),$$

which associates to a scheme $X$ its category of perfect complexes. This construction is functorial and encodes a substantial amount of the geometry of $X$. It is therefore natural to ask:

(1) What kind of category is $\mathrm{Perf}(X)$?
(2) What geometric information can be recovered from $\mathrm{Perf}(X)$?
(3) How can one organize such categories into a geometric theory?

The first question leads to the theory of tensor triangulated categories, which appear naturally in algebraic geometry, homotopy theory, and representation theory. The second question motivates the construction of the Balmer spectrum, which associates a topological space to a tensor triangulated category. In many cases, this space recovers the underlying topological space of a scheme. However, this construction only produces a topological space, and not yet a fully-fledged geometric object. In particular, the associated structure sheaf is obtained by a rather indirect procedure, and does not arise from a genuine sheaf of categories.

The main goal of this seminar is to explain how higher geometric methods provide a natural framework to overcome these limitations. In this setting, tensor triangular geometry can be embedded into a broader geometric theory, in which both the underlying space and the structure sheaf arise from a conceptual and functorial construction.

## 2. Tensor triangular geometry and 2-rings

We now address the first question raised in the introduction: what kind of category is $\mathrm{Perf}(X)$? Historically, before the advent of $\infty$-categories, one studied derived categories in the sense of Grothendieck and Verdier. Given a quasi-compact quasi-separated scheme $X$, one considers the derived category of quasi-coherent sheaves $D(\mathrm{QCoh}(X))$, together with its full subcategory of perfect complexes $\mathrm{Perf}(X)$.

A fundamental insight of Verdier [Ver96] (that has also appeared in the work of Puppe) is that these categories carry a triangulated structure, encoding homological information via exact triangles. Moreover, they are equipped with a symmetric monoidal structure given by the derived tensor product.

**Definition 2.1** (Informal)**.** A triangulated category is an additive category $\mathcal{K}$ equipped with an automorphism $\Sigma\colon \mathcal{K} \to \mathcal{K}$, called the *suspension*, and a class of *distinguished triangles*

$$x \to y \to z \to \Sigma x,$$

satisfying a list of axioms. A *tensor triangulated category* is a triangulated category together with a compatible symmetric monoidal structure. We will refer to tensor triangulated categories simply as *tt-categories.*

Tensor triangulated categories arise in several areas of mathematics.

**Example 2.2** (Stable homotopy theory)**.** The stable homotopy category of finite spectra is a *tt*-category, where the tensor product is given by the smash product. A fundamental result of Hopkins and Smith classifies its thick tensor ideals.

**Example 2.3** (Representation theory)**.** Let $G$ be a finite group and $k$ a field of characteristic $p > 0$ dividing $|G|$. The stable module category $\mathrm{stab}(kG)$ is a *tt*-category, whose tensor product is induced by the tensor product over $k$ with diagonal $G$-action.

Despite their success, triangulated categories have important limitations. The main issue is that their structure is not functorial: for instance, cones are not uniquely defined. More generally, triangulated categories do not behave well under basic categorical constructions.

The modern solution is to replace triangulated categories with stable $\infty$-categories.

**Definition 2.4.** An $\infty$-category is *stable* if it admits a zero object, finite limits and colimits, and a square is a pullback if and only if it is a pushout.

Stable $\infty$-categories provide a better framework: stability is a property rather than additional structure, and constructions such as limits, colimits, and functor categories behave well.

**Remark 2.5.** Two additional features of stable $\infty$-categories are particularly important for us.

(1) First, one often restricts to *idempotent-complete* stable $\infty$-categories, that is, stable $\infty$-categories in which every idempotent splits. These form a subcategory $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{st}}$ of stable categories and exact functors and one can show [BGT13] that $\mathrm{Cat}^{\mathrm{perf}}$ carries a closed monoidal structure characterized by a universal property.
(2) Second, stable $\infty$-categories admit a well-behaved theory of localization. If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a sequence of exact functors, one says that it is a *Verdier sequence* if $\mathcal{C}$ identifies with the Verdier quotient of $\mathcal{B}$ by $\mathcal{A}$. After passing to idempotent completions, one obtains the corresponding notion of *Karoubi sequence.*

These constructions will play an important role later, since localizations of stable categories are among the basic building blocks of higher Zariski geometry.

From this perspective, tensor triangulated categories should be viewed as shadows of more structured objects. In particular, symmetric monoidal stable $\infty$-categories can be regarded as a categorified analogue of commutative rings.

**Definition 2.6** ([Mat16])**.** A 2-*ring* is a commutative algebra object in the symmetric monoidal $\infty$-category of small idempotent-complete stable $\infty$-categories. The $\infty$-category of 2-rings is denote by 2CAlg.

**Remark 2.7.** The homotopy category of a 2-ring is a *tt*-category, although not every *tt*-category arises in this way. In practice, most examples of interest admit such enhancements.

We will discuss stable $\infty$-categories and 2-rings in more detail in Talks 2 and 3.

## 3. Balmer spectrum

We now address the second question: what geometric information can be extracted from a tensor triangulated category?

The guiding idea is to view a *tt*-category as a categorified analogue of a commutative ring. In commutative algebra, one studies a ring $R$ by means of its prime ideals and the associated topological space $\mathrm{Spec}(R)$. The goal is to develop an analogous construction for tensor triangulated categories.

**Definition 3.1.** Let $\mathcal{K}$ be a *tt*-category. A *thick tensor ideal* $\mathcal{I} \subseteq \mathcal{K}$ is a full subcategory which is:

(1) Thick (that is, closed under triangles, shifts, and direct summands).
(2) Closed under tensoring with arbitrary objects of $\mathcal{K}$.

A thick tensor ideal $\mathcal{P} \subsetneq \mathcal{K}$ is *prime* if for all $x, y \in \mathcal{K}$, if $x \otimes y \in \mathcal{P}$ then $x \in \mathcal{P}$ or $y \in \mathcal{P}$.

As in classical algebraic geometry, we can construct a spectrum out of prime thick tensor ideals.

**Definition 3.2** ([Bal04])**.** Let $\mathcal{K}$ be a *tt*-category. The *Balmer spectrum* of $\mathcal{K}$ is the set

$$\operatorname{Spc}(\mathcal{K}) := \{\text{prime thick tensor ideals of } \mathcal{K}\},$$

endowed with the topology generated by the subsets

$$\operatorname{supp}(x) := \{\mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\}.$$

The assignment $x \mapsto \operatorname{supp}(x)$ should be understood as part of a more general notion of *support theory.*

**Definition 3.3** ([Bal04])**.** A *support datum* on a *tt*-category $\mathcal{K}$ consists of a topological space $X$ together with an assignment $x \in \mathcal{K} \mapsto \operatorname{supp}(x) \subseteq X$ of closed subsets such that

(1) $\operatorname{supp}(0) = \varnothing$ and $\operatorname{supp}(\mathbb{1}) = X$;
(2) $\operatorname{supp}(x \oplus y) = \operatorname{supp}(x) \cup \operatorname{supp}(y)$;
(3) $\operatorname{supp}(\Sigma x) = \operatorname{supp}(x)$;
(4) $\operatorname{supp}(x \otimes y) = \operatorname{supp}(x) \cap \operatorname{supp}(y)$.

Support data provide a way to translate categorical properties into geometric ones. For instance, thick tensor ideals can often be described in terms of subsets of the space $X$, and localizations correspond to restricting to open subsets.

**Remark 3.4.** A fundamental result of Balmer [Bal04] shows that the spectrum $\operatorname{Spc}(\mathcal{K})$ is universal among support data compatible with the tensor structure. In other words, any reasonable notion of support factors uniquely through $\operatorname{Spc}(\mathcal{K})$.

Until now we have only constructed the Balmer spectrum as a topological space; to promote $\operatorname{Spc}(\mathcal{K})$ to a geometric object, we need a structure sheaf. The construction is due to Balmer [Bal04, Definition 6.1], following a result of Neeman-Thomason [Bal02, Theorem 2.13].

**Construction 3.5.** Let $\mathcal{K}$ be a *tt*-ring. Let $U \subseteq \operatorname{Spc}(\mathcal{K})$ be an open subset and let $U^c$ be its closed complement. Consider the thick $\otimes$-ideal

$$\mathcal{K}_{U^c} = \{x \in \mathcal{K} \mid \operatorname{supp}(x) \subseteq U^c\}$$

Then we may form the localization sequence $\mathcal{K}_{U^c} \to \mathcal{K} \to \mathcal{K}/\mathcal{K}_{U^c}$ and consider the presheaf of *tt*-categories

$$\tilde{\mathcal{O}}_{\mathcal{K}} : U \mapsto \mathcal{K}/\mathcal{K}_{U^c} \tag{1}$$

where an inclusion of open sets $U \subseteq V$ is sent to the induced map K/KV c 7→ K/KUc. The presheaf $\widetilde{\mathcal{O}}_{\mathcal{K}}$ should be thought of as the categorical analogue of the structure sheaf: it assigns to an open set a localized category of objects supported on it. However, this presheaf is not a sheaf in any reasonable sense. The issue is twofold:

(1) It takes values in tensor triangulated categories, for which there is no well-behaved notion of gluing.
(2) More concretely, descent fails: objects defined locally on an open cover do not glue uniquely (or even at all) to a global object.

To remedy this issue, one passes to endomorphisms of the unit object. This yields a presheaf of commutative rings

$$U \mapsto \mathrm{End}_{\mathcal{K}(U)}(\mathbb{1}),$$

which can be sheafified to obtain a sheaf of rings on $\mathrm{Spc}(\mathcal{K})$. This sheaf of commutative rings $\mathcal{O}_{\mathcal{K}}$ turns $\mathrm{Spc}(\mathcal{K})$ into a ringed space $\mathrm{Spec}(\mathcal{K})$.

Then:

**Theorem 3.6** ([Bal04])**.** We have the following reconstruction results.

(1) Let $X$ be a topologically noetherian scheme. Then $\mathrm{Spec}(\mathrm{Perf}(X)) \cong X$ as schemes.
(2) Let $G$ is a finite group and $k$ a field of positive characteristic $p$ (dividing the order of $G$). Then $\mathrm{Spec}(\mathrm{stab}(kG)) \cong \mathrm{Proj}(H^\bullet(G,k))$ as schemes.

We will discuss the Balmer spectrum in Talk 4 and prove (a part of) the reconstruction theorem in Talk 5.

## 4. Interlude on Lurie's geometry

In [Lur09], Lurie introduces a general framework to formalize the basic structures of derived geometry. The idea is to start from a small Grothendieck site $\mathcal{G}$ and construct an $\infty$-category of $\infty$-topoi equipped with a local $\mathcal{G}$-structure, together with notions of affine objects, schemes, and functors of points, analogous to those of classical algebraic geometry.

**Definition 4.1.** A *geometry* is a triple $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ consisting of:

(1) An idempotent-complete small $\infty$-category $\mathcal{G}$ admitting finite limits.
(2) A class of morphisms $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, called *admissible morphisms*, which is closed under pullbacks, retracts, and composition (equivalently, left cancellation).
(3) A Grothendieck topology $\tau$ on $\mathcal{G}$ generated by admissible morphisms.

Admissible morphisms are meant to model basic open immersions, while the topology $\tau$ encodes coverings by such opens. The deifnition should be more clear by having in mind the following:

**Example 4.2** (Classical Zariski geometry)**.** The classical Zariski geometry is given by the following data:

(1) The underlying category $\mathcal{G}_{\mathrm{clZar}} := (\mathrm{CAlg}^{\omega})^{\mathrm{op}}$ is given by the opposite of the category of finitely presented commutative rings.
(2) The class of admissible morphisms consists of the localization maps $R \to R[x^{-1}]$ for $x \in R$ where $R$ is is a finitely presented commutative ring.
(3) A finite collection of admissible morphisms $\{R \to R[x_i^{-1}]\}_{i\in I}$ is declared to generate a covering sieve if the elements $\{x_i\}_{i\in I} \subseteq R$ generate the unit ideal in $R$.

Notice that $\mathrm{Ind}(\mathcal{G}_{\mathrm{clZar}}^{\mathrm{op}}) = \mathrm{Ind}(\mathrm{CAlg}^{\omega}) = \mathrm{CAlg}$ is the category or rings.

From the example, it is clear that, given a geometry $\mathcal{G}$, one is interested in studying the category

$$\mathrm{Pro}(\mathcal{G}) \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})^{\mathrm{op}},$$

which plays the role of the category of affine objects. To build (affine) schemes we should build locally ringed spaces, but we replace spaces with $\infty$-topoi.

**Remark 4.3.** Recall that an $\infty$-topos is a left-exact localization of an $\infty$-category of presheaves on a small $\infty$-category. The main examples are given by $\infty$-categories of sheaves of spaces on a topological space or on a small Grothendieck site.

The theory developed in [Lur09] associates to a geometry $\mathcal{G}$ an $\infty$-category $\mathrm{LTop}(\mathcal{G})$ of $\infty$-topoi with local $\mathcal{G}$-structure. A description of this category is as follows:

(1) Objects are pairs $(X, \mathcal{O})$ where $X$ is an $\infty$-topos and $\mathcal{O}$ is an $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaf on $X$ satisfying a locality condition.
(2) Morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are given by geometric morphisms $f^* : X \to Y$ together with a compatible morphism of structure sheaves $f^*\mathcal{O}_Y \to \mathcal{O}_X$ satisfying a locality condition.

The key structural result is the existence of an affine spectrum functor.

**Theorem 4.4** (Lurie)**.** For any geometry $\mathcal{G}$, there is an adjunction

$$\mathrm{Spec}_{\mathcal{G}} : \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \rightleftarrows \mathrm{LTop}(\mathcal{G}) : \Gamma_{\mathcal{G}},$$

where $\Gamma_{\mathcal{G}}$ sends $(\mathcal{X}, \mathcal{O})$ to its global sections $\mathcal{O}(\mathcal{X})$. In particular, for every $(X, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G})$ there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{LTop}(\mathcal{G})}\big(\mathrm{Spec}_{\mathcal{G}}(A), (X, \mathcal{O})\big) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}\big(A, \Gamma_{\mathcal{G}}(X, \mathcal{O})\big).$$

The left adjoint $\mathrm{Spec}_{\mathcal{G}}$ is called the *absolute spectrum functor*.

We will give a more detailed account of this theory in Talks 6 and 7.

## 5. The geometry on 2-rings

In Section 3 we have seen that a tensor triangulated category $\mathcal{K}$ gives rise to a topological space $\mathrm{Spc}(\mathcal{K})$, together with a presheaf of triangulated categories which fails to satisfy descent. In Section 4 we introduced Lurie's formalism of geometries, which provides a general framework to construct geometric objects with a well-behaved theory of sheaves. The goal of this section is to explain how these two perspectives fit together.

More precisely, we will construct a Zariski geometry on 2-rings and show that the associated spectrum recovers the Balmer spectrum as its underlying topological space. In this way, tensor triangular geometry naturally fits into the framework of higher geometry.

The first step is to construct a Zariski geometry on 2-rings, which plays the role of the classical Zariski geometry on commutative rings. We will see this result in Talk 8.

**Theorem 5.1** ([ABC+25, Theorem A])**.** The following data:

(1) The opposite of the category of compact 2-rings $\mathcal{G}_{\mathrm{Zar}} := (2\mathrm{CAlg}^{\omega})^{\mathrm{op}}$;
(2) The class of admissible morphisms $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{adm}}$ given by principal Verdier localizations

$$\mathcal{K} \to \mathcal{K}/\langle a \rangle;$$

(3) A finite collection of admissible morphisms $\{f_i : \mathcal{K} \to \mathcal{K}_i\}_{i \in I}$ is declared to generate a covering sieve if the kernel of the induced map

$$\prod_{i \in I} f_i : \mathcal{K} \to \prod_{i \in I} \mathcal{K}_i$$

consists of $\otimes$-nilpotent objects;

define a geometry, called the *Zariski geometry* on the opposite of the category of 2-rings.

We should note the following:

**Remark 5.2.** This result should be compared with the classical Zariski geometry. Principal Verdier localizations play the role of basic open immersions, while the covering condition encodes a form of nilpotence detection, generalizing the condition that elements generate the unit ideal.

Having constructed the geometry, we can apply Lurie's general formalism to obtain an affine spectrum functor and its adjoint.

**Corollary 5.3** ([ABC+25, Corollary B])**.** Let $(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$. Then there is a natural equivalence

$$\mathrm{Hom}_{2\mathrm{CAlg}}\big(\mathcal{K}, \Gamma_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{X}, \mathcal{O})\big) \simeq \mathrm{Hom}_{\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})}\big(\mathrm{Spec}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{K}), (\mathcal{X}, \mathcal{O})\big).$$

This adjunction should be viewed as a categorified version of the classical adjunction between affine schemes and commutative rings. Now we can go back to our discussion on the Balmer spectrum the language of geometries allows us to recover it, up to a sheaf category. We will see this result in Talk 9.

**Theorem 5.4** ([ABC+25, Theorem C]). Let $\mathcal{K} \in \mathrm{2CAlg}$. The underlying $\infty$-topos of the absolute spectrum $\mathrm{Spec}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{K})$ may be identified with the $\infty$-topos $\mathrm{Shv}(\mathrm{Spc}(\mathcal{K}))$ of sheaves on the Balmer spectrum of the underlying tt-category of $\mathcal{K}$. Moreover, this identification is natural in $\mathcal{K}$.

Concerning the Balmer spectrum, this categorification of Zariski geometry allows us to fix the issue of Construction 3.5. Before we were able to obtain a sheaf of rings on the Balmer sepctrum, by "decategorifying" a (want to be) pre(sheaf) of triangulated categories. Talk 11 will show us the following.

**Theorem 5.5** ([ABC+25, Theorem D]). Let $\mathcal{K}$ be a rigid[1] 2-ring. The equivalence of Theorem 5.4 identifies the natural $\mathcal{G}_{\mathrm{Zar}}$-structure on $\mathrm{Spec}(\mathcal{K})$ with a 2CAlg-valued sheaf on $\mathrm{Spc}(\mathcal{K})$, denoted $\mathcal{O}_{\mathcal{K}}$, which upon passage to homotopy categories agrees with the presheaf of Equation 1 on quasicompact open subsets.

There two other advantages that the abstract setup with geometries has. Namely, comparison with other geometries and descent theory. Let us start with the former one.

**Remark 5.6.** The comparison with classical Zariski geometry is mediated by the morphism of geometries $\mathrm{Perf} : \mathcal{G}_{\mathrm{cZar}} \to \mathcal{G}_{\mathrm{Zar}}$ induced by the assignment $R \mapsto \mathrm{Perf}(R)$. By Lurie's formalism, any morphism of geometries $\alpha : \mathcal{G} \to \mathcal{G}'$ induces a restriction functor

$$\mathrm{res}_\alpha : \mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G}), \qquad (X, \mathcal{O}) \mapsto (X, \mathcal{O} \circ \alpha).$$

In particular, a $\mathcal{G}_{\mathrm{Zar}}$-structured $\infty$-topos can be viewed as a $\mathcal{G}_{\mathrm{cZar}}$-structured $\infty$-topos by restricting its structure along Perf. On the other hand, the adjunction

$$\mathrm{Perf} : \mathrm{CAlg} \rightleftarrows \mathrm{2CAlg} : R(-)$$

identifies the associated classical affine object of a 2-ring $\mathcal{K}$ with the commutative ring $R(\mathcal{K}) = \mathrm{End}(\mathbb{1}_{\mathcal{K}}$. Using the affine spectrum-global sections adjunctions and the compatibility

$$\Gamma_{\mathcal{G}_{\mathrm{cZar}}} \circ \mathrm{res}_{\mathrm{Perf}} \simeq R(-) \circ \Gamma_{\mathcal{G}_{\mathrm{Zar}}},$$

one obtains by passing to mates a natural transformation

$$\rho_{\mathcal{K}} : \mathrm{res}_{\mathrm{Perf}}\big(\mathrm{Spec}_{\mathcal{G}_{\mathrm{Zar}}}(\mathcal{K})\big) \to \mathrm{Spec}_{\mathcal{G}_{\mathrm{cZar}}}(R(\mathcal{K})).$$

Then:

**Theorem 5.7** ([ABC+25, Theorem 4.48]). Let $R \in \mathrm{CAlg}$ be an ordinary commutative ring. Then the comparison transformation

$$\mathrm{Spec}(\mathrm{Perf}(R)) \to \mathrm{Spec}_{\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}}}(R)$$

is an equivalence in $\mathrm{RTop}(\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}})$. Moreover, this result can be globalised to quasi-compact quasi-separated schemes [Che25].

We now turn to descent theory.

**Theorem 5.8** ([ABC+25, Theorem F]). Let $\mathcal{K}$ be a rigid 2-ring. The following assignments extend to $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$-valued sheaves on $\mathrm{Spec}(\mathcal{K})$:

(1) The assignment Mod sending a quasi-compact open subset $U$ to $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{O}_{\mathcal{K}}(U))}(\mathrm{Pr}^L)$.

[1] Recall that a 2-ring is rigid if every object is dualizable.

(2) The assignment $\mathrm{Mod}^{\omega}$ sending $U$ to the full subcategory of $\mathcal{M}_{\mathrm{dod}}(U)$ consisting of those objects whose underlying presentable $\infty$-categories are compactly generated.
(3) The assignment Mod sending $U$ to $\mathrm{Mod}_{\mathcal{O}_{\mathcal{K}}(U)}(\mathrm{Cat}^{\mathrm{perf}})$.

We are left to discuss the topics of Talks 13-14. Talk 13 studies support theory. In Section 3 we have seen that the Balmer spectrum provides a notion of support for objects of a *tt*-category. A natural question is whether this support theory is universal and whether it can be refined in a higher-categorical setting. This is indeed the case

**Theorem 5.9** ([ABC⁺25, Theorem A.14])**.** Let $\mathcal{K}$ be a rigid 2-ring. Then the assignment

$$x \mapsto \mathrm{supp}(x) \subseteq \mathrm{Spc}(\mathcal{K})$$

extends to a support theory compatible with the $\mathcal{G}_{\mathrm{Zar}}$-structure on $\mathrm{Spec}(\mathcal{K})$, and is universal among such support data.

In order to understand the topic of Talk 14, we briefly recall the relation between idempotent objects and localizations.

**Definition 5.10.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. An *idempotent algebra* of $\mathcal{C}$ is an object $A \in \mathcal{C}_{\mathbb{1}/}$ such that the induced map $A \otimes \mathbb{1} \to A \otimes A$ is an equivalence. Such an object admits a unique structure of a commutative algebra.

Idempotent algebras classify certain localizations of $\mathcal{C}$. In particular, an idempotent algebra $A$ determines a *smashing localization*, that is, a symmetric monoidal localization for which the localizing category is a category of modules over an idempotent algebra of $\mathcal{C}$. There is a huge class of smashing localizations:

**Example 5.11.** In the case of a rigid 2-ring $\mathcal{K}$ every *tt*-ideal $\mathcal{I} \subseteq \mathcal{K}$ induces a smashing localization $\mathrm{Ind}(\mathcal{K}) \to \mathrm{Ind}(\mathcal{K}/\mathcal{I})$.

The telescope conjecture asks for every smashing localization to arises in this way. It has been proved in many cases (for example, for noetherian rings [NB92]) and right heriditary rings by [K0]) but it fails in general, as shown by counterexamples in stable homotopy theory [BHLS23] and in ring theory [Kel94].

The following local to global principle simplifies the study of the conjecture:

**Theorem 5.12** ([ABC⁺25, Theorem B.2])**.** Let $\mathcal{K} \in 2\mathrm{CAlg}_{\mathrm{rig}}$. Then $\mathcal{K}$ satisfies the telescope conjecture if and only if $\mathcal{K}/\mathcal{P}$ satisfies the telescope conjecture for every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$.

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