GROUP COHOMOLOGY

Abstract. These are lecture notes for the course on group cohomology given by Kevin Li at the University of Regensburg during the winter semester 2023–2024. They were transcribed and edited by Giovanni Rossanigo, who is solely responsible for any errors or omissions.

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1. Motivations

This course concerns the relation between two notions of dimension for groups. We begin with the geometric notion. Let X be a CW-complex equipped with a filtration

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq \bigcup_{n\in\mathbb{N}} X^{(n)} = X.$$

Recall that this means that the n-skeleton $X^{(n)}$ of X is constructed by attaching n-cells to $X^{(n-1)}$ via a pushout

$$\begin{array}{ccc} \coprod_{\alpha\in A} S^{n-1} & \longrightarrow & X^{(n-1)} \\ \downarrow & \text{(po)} & \downarrow \\ \coprod_{\alpha\in A} D^n & \longrightarrow & X^{(n)} \end{array}$$

and that the topology on X is the weak one ($A \subseteq X$ is closed $\Leftrightarrow A \cap X^{(n)}$ is closed for every $n \in \mathbb{N}$). The dimension of a CW-complex records the highest dimension in which cells occur.

Definition 1.1. Let X be a CW-complex. We define its *geometric dimension* to be

$$\mathrm{gd}(X) = \inf\{n \in \mathbb{N} \mid X = X^{(n)}\}.$$

The value $\mathrm{gd}(X)$ lies in $\mathbb{N} \cup \{\infty\}$ and is infinite when X has cells in arbitrarily high dimensions. If $\mathrm{gd}(X) < \infty$, we say that X is *finite-dimensional*. We can also consider the number of cells in each dimension. Geometric dimension records the dimensions of the cells, but not their number. We say that X has a *finite n-skeleton* if it has finitely many cells in dimensions at most n.

Example 1.2. The CW-complex $\bigvee_{\mathbb{N}} S^1$ has dimension 1 and infinitely many 1-cells. Conversely, $\mathbb{RP}^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{RP}^n$ has one cell in each dimension, but has infinite dimension.

We now relate these notions to groups.

Definition 1.3. Let G be a discrete group and let X be a connected CW-complex with a basepoint $x \in X$. We say that X is a *model for* $BG = K(G, 1)$ if $\pi_1(X, x) \cong G$ and the universal cover of X is contractible (that is, $\pi_n(X, x) \cong 0$ for $n \geq 2$).

As we will see in Section 2.5, a model for BG exists and is unique up to homotopy equivalence. The uniqueness follows from Proposition 2.31 and Theorem 4.21.

Example 1.4. A model for $B\mathbb{Z}$ is given by S^1, since $\pi_1(S^1) \cong \mathbb{Z}$ and the universal cover $\mathbb{R} \to S^1$ is contractible. Similarly, a model for $B(\mathbb{Z} \times \mathbb{Z})$ is given by the torus T_1. Categorically, this also follows from the adjunction

$$\pi_1 : h\mathrm{CW}_* \rightleftarrows \mathrm{Grp} : B$$

between π_1 and the B-construction. Here $h\mathrm{CW}_*$ denotes the homotopy category of pointed connected CW-complexes, with pointed homotopy classes of maps. This follows from the universal property in Proposition 2.31.

Example 1.5. A model for $B(\mathbb{Z} * \mathbb{Z})$ is $S^1 \vee S^1$: its fundamental group is $\mathbb{Z} * \mathbb{Z}$, and its universal cover is the contractible Cayley tree on two generators. The functor B is a right adjoint and preserves products. The wedge example follows from Lemma 3.13.

Example 1.6. Do we have a model for $B\mathbb{Z}/2$? We have $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$, but the universal cover $S^2 \to \mathbb{RP}^2$ is not contractible, so $\mathbb{RP}^2$ is not a model for $B(\mathbb{Z}/2)$. In Example 2.32 we will see that $\mathbb{RP}^\infty \simeq B(\mathbb{Z}/2\mathbb{Z})$, since $S^\infty \to \mathbb{RP}^\infty$ is a universal cover.

The dimension of a CW-model for BG leads to the following definition.

Definition 1.7. Let G be a discrete group. The *geometric dimension* of G is

$$\mathrm{gd}(G) = \inf\{n \in \mathbb{N} \mid \text{there exists a model of } BG \text{ of dimension } n\}.$$

To compute the geometric dimension of G, we take the least dimension among all CW-models for BG.

Example 1.8. Let us compute $\mathrm{gd}(\mathbb{Z} \times \mathbb{Z})$. We know that T_1 is a model for $B(\mathbb{Z} \times \mathbb{Z})$, and that it has dimension 2, giving us $\mathrm{gd}(\mathbb{Z} \times \mathbb{Z}) \leq 2$. Can $\mathrm{gd}(\mathbb{Z} \times \mathbb{Z})$ be 1? If so, then $\pi_1(S^1 \times S^1)$ would be free, since the fundamental group of a connected one-dimensional CW-complex is free. This is impossible, so $\mathrm{gd}(\mathbb{Z} \times \mathbb{Z}) = 2$.

Example 1.9. Let us compute $\mathrm{gd}(\mathbb{Z}/2)$. The space $\mathbb{RP}^\infty$ is a model for $B(\mathbb{Z}/2)$ and has nonzero cohomology in arbitrarily high degrees. Since no finite-dimensional CW-complex has this property, we must have $\mathrm{gd}(\mathbb{Z}/2) = \infty$.

This first notion of dimension is geometric. We next introduce a notion whose algebraic nature will become clear from the construction of group cohomology.

Definition 1.10. Let G be a discrete group and let BG be a CW-model for its classifying space. We define the *cohomological dimension* to be

$$\mathrm{cd}(G) = \sup\{n \in \mathbb{N} \mid H^n(BG; M) \neq 0 \text{ for } M \in {}_G\mathrm{Mod}\}.$$

Here M ranges over left G-modules, and $H^n(BG; M)$ is cohomology with the associated local coefficients, as in Definition 2.34.

We will explain this definition in Section 2.5; for now, note the inequality of Theorem 4.12

$$\mathrm{cd}(G) \leq \mathrm{gd}(G)$$

which expresses the possibility that algebraic invariants do not fully determine geometric dimension.

The following theorem shows how closely the two dimensions are related. A reference for this theorem is [Dav08, Theorem F.3.2].

Theorem 1.11 (Eilenberg–Ganea)**.** Let G be a discrete group. We have $\mathrm{gd}(G) \leq \max(\mathrm{cd}(G), 3)$.

Thus, if $\mathrm{cd}(G) \geq 3$, then $\mathrm{cd}(G) = \mathrm{gd}(G)$. We prove Theorem 1.11 in Theorem 4.23. It remains to study the cases where $\mathrm{cd}(G) < 3$. For $\mathrm{cd}(G) = 1$ we have an answer, which we will not prove. See [Dav08, Theorem F.3.3].

Theorem 1.12 (Stallings–Swan)**.** Let G be a discrete group. If $\mathrm{cd}(G) = 1$, then G is a free group.

Indeed, if $\mathrm{cd}(G) = 1$, then Theorem 1.12 gives $G \cong \mathrm{Free}(S)$ for some set S. Thus a model for BG is the one-point union $\bigvee_S S^1$ of copies of S^1 indexed by S, giving us $\mathrm{gd}(G) = 1$. Hence the remaining case is $\mathrm{cd}(G) = 2$. The assertion that the two dimensions also agree in this case is the Eilenberg–Ganea conjecture.

Conjecture 1.13 (Eilenberg–Ganea, 1957)**.** Let G be a discrete group with $\mathrm{cd}(G) = 2$. Then $\mathrm{gd}(G) = 2$.

The conjecture was open when this course was given in the winter semester 2023–2024. It was subsequently disproved in the September 2026 OpenAI preprint [OAI26]. The counterexample is a finitely generated residually finite group G satisfying

$$\mathrm{cd}(G) = 2, \qquad \mathrm{gd}(G) = 3.$$

We will discuss proposed examples in Section 4.4 and Section 5.5. We will also study finiteness conditions. These concern the number of cells in each skeleton of a CW-complex. The first such condition is the following.

Definition 1.14. Let G be a discrete group and let $n \geq 0$ be an integer. We say that G is of *type* F_n if there exists a model for BG with finite n-skeleton.

These conditions reflect familiar group properties. For example, a group G is of type F_1 if and only if it is finitely generated, and of type F_2 if and only if it is finitely presented. As with dimension, these geometric conditions have algebraic counterparts.

Definition 1.15. Let G be a discrete group and let $n \geq 0$ be an integer. We say that G is of *type* FP_n if there exists a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ in which P_i is finitely generated as a G-module for every $0 \leq i \leq n$.

One can show that type $F_n \Rightarrow$ type FP_n. Does the converse also hold? Actually, it does not. The examples are due to Bestvina and Brady [BB97, Examples 6.3(3)].

Theorem 1.16 (Bestvina–Brady)**.** For every integer $n \geq 2$, there exists a discrete group G of type FP_n but not of type F_n.

This follows from Example 5.36, using Theorem 5.34.

2. Definition of group cohomology

2.1. Group rings and cohomology.

Definition 2.1. Let G be a discrete group. The *group ring* $\mathbb{Z}G$ has the free abelian group on G as its underlying additive group, with multiplication extending the group operation linearly.

More explicitly, the elements of $\mathbb{Z}G$ are formal sums $\sum_{g \in G} n_g \cdot g$ with $n_g \in \mathbb{Z}$, all but finitely many of which are zero. Addition is defined coefficientwise, and multiplication extends the group operation linearly. The ring $\mathbb{Z}G$ is unital, with unit e_G, and is commutative if and only if G is abelian.

Example 2.2. Let us see some examples. If $G = \{e_G\}$ is the trivial group, then $\mathbb{Z}\{e_G\} \cong \mathbb{Z}$ via $e_G \mapsto 1$.

Example 2.3. If $G = \mathbb{Z}$ is the group of integers, then $\mathbb{Z}G \cong \mathbb{Z}[T, T^{-1}]$ is the ring of Laurent polynomials. Writing $G = \langle t \rangle$, we obtain the isomorphism

$$\sum_{n \in \mathbb{Z}} a_n t^n \mapsto \sum_{n \in \mathbb{Z}} a_n T^n,$$

where only finitely many a_n are nonzero. In a similar way, if $n \geq 1$ is an integer and $G = \mathbb{Z}/n\mathbb{Z}$, then $\mathbb{Z}G \cong \mathbb{Z}[T]/(T^n - 1)$.

The group ring construction extends to a functor $\mathbb{Z} : \mathrm{Grp} \to \mathrm{Ring}$. Indeed, every group homomorphism $\varphi : G \to H$ extends linearly to a ring homomorphism $\mathbb{Z}\varphi : \mathbb{Z}G \to \mathbb{Z}H$. The inclusion $G \hookrightarrow \mathbb{Z}G$, given by $g \mapsto g$, lands in the group of units $(\mathbb{Z}G)^\times$, since every $g \in G$ is invertible. This is an instance of the following universal property.

Proposition 2.4. Let G be a discrete group, let R be a unital ring, not necessarily commutative, and let $f : G \to R^\times$ be a group homomorphism. Then there exists a unique unital ring homomorphism $\overline{f} : \mathbb{Z}G \to R$ such that

$$\begin{array}{ccc} G & \xrightarrow{f} & R^\times \\ \downarrow & & \downarrow \\ \mathbb{Z}G & \xrightarrow[\overline{f}]{} & R \end{array}$$

commutes as a diagram of sets.

Proof. Given $f : G \to R^\times$, its unique linear extension is

$$\overline{f}\left(\sum_{g\in G} n_g \cdot g\right) := \sum_{g\in G} n_g f(g).$$

Since f is a group homomorphism, $\overline{f}$ is clearly a ring map. □

By using Proposition 2.4, we get a bijection

$$\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}G, R) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times)$$

for every group G and ring R. Hence we have an adjunction

$$\mathbb{Z} : \mathrm{Grp} \rightleftarrows \mathrm{Ring} : (-)^\times,$$

where the unit is a monomorphism.

Remark 2.5. We can also form group rings with coefficients in another ring. Categorically, for every commutative unital ring R we get an adjunction

$$R[-] : \mathrm{Grp} \rightleftarrows R\mathrm{Alg} : (-)^\times$$

where $R[-]$ sends a group to its group ring over R, by the same construction.

We now consider modules over $\mathbb{Z}G$. Throughout, rings need not be commutative, and modules are left modules unless stated otherwise. We will denote by ${}_G\mathrm{Mod}$ the category of left $\mathbb{Z}G$-modules, and call it the category of left G-modules. Equivalently, a G-module is an abelian group with a compatible G-action. More precisely, we use the following definition.

Definition 2.6. Let G be a discrete group. A *G-module* is an abelian group A with an action $G \times A \to A$ such that

(1) $e_G \cdot a = a$ for every $a \in A$;
(2) $g \cdot (a + b) = g \cdot a + g \cdot b$ for every $a, b \in A$ and $g \in G$;
(3) $h \cdot (g \cdot a) = (hg) \cdot a$ for every $a \in A$ and $g, h \in G$.

Every abelian group can be given the trivial G-action. This defines a functor ${}_\mathbb{Z}\mathrm{Mod} \to {}_G\mathrm{Mod}$, sending an abelian group to the same group with trivial action.

Example 2.7. Let G be a discrete group. The trivial G-module $\mathbb{Z}$ will be central to Definition 2.10. The module structure of $\mathbb{Z}$ already detects a property of G. We claim that $\mathbb{Z}$ is projective as a G-module if and only if $G = \{1\}$. The reverse implication is immediate; for the forward implication, suppose that $\mathbb{Z}$ is projective and $G \neq \{1\}$. Then the lifting problem

$$\begin{array}{ccccc} & & & & \mathbb{Z} \\ & & \overset{\widetilde{\mathrm{id}}}{\swarrow} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}G & \xrightarrow[\varepsilon]{} & \mathbb{Z} & \longrightarrow & 0 \end{array} \qquad (\text{where } \varepsilon : g \mapsto 1)$$

has a solution $\widetilde{\mathrm{id}}$. Since $\mathbb{Z}$ has a trivial G-action, $\widetilde{\mathrm{id}}$ must land in $(\mathbb{Z}G)^G$. By Lemma 2.12, if $|G| = \infty$, then $(\mathbb{Z}G)^G = 0$, so $\widetilde{\mathrm{id}} = 0$, a contradiction. If G is finite, then Lemma 2.12 gives $(\mathbb{Z}G)^G = \langle N \rangle$ for $N = \sum_{g\in G} g$. Then $\widetilde{\mathrm{id}}(1) = aN$ for some $a \in \mathbb{Z}$. Thus $\varepsilon(\widetilde{\mathrm{id}}(1)) = a|G| = 1$, forcing $|G| = 1$.

Example 2.8. Let G be a discrete group. $\mathbb{Z}G$ is a free G-module. More generally, let S be a G-set. Then the action of G on the basis S makes $\mathbb{Z}[S] := \bigoplus_{s\in S} \mathbb{Z} \cdot s$ a G-module, and if S is a free G-set, then $\mathbb{Z}[S]$ is a free G-module. Indeed, $S = \coprod_{S/G} G$ can be written in terms of its orbits and $\mathbb{Z}(\coprod_{S/G} G) \cong \bigoplus_{S/G} \mathbb{Z}G$, since $\mathbb{Z}[-]$ is a left adjoint. The construction applies in particular to $S = G/H$, the set of cosets of $H \leq G$. The freeness assertion applies to this G-set only when H is trivial.

Example 2.9. Consider a connected CW-complex X with fundamental group G, and let $\widetilde{X}$ be its universal cover. Let $n \geq 0$ be an integer. The group G acts freely on $\widetilde{X}$, hence also on its n-cells and singular n-simplices. These actions make $C_n^{\mathrm{cell}}(\widetilde{X};\mathbb{Z})$ and $C_n^{\mathrm{sing}}(\widetilde{X};\mathbb{Z})$ free G-modules.

Having introduced G-modules and seen some examples, we can give the definition of group cohomology.

Definition 2.10. Let G be a discrete group and let M be a left G-module. The *group cohomology* of G with values in M is

$$H^\bullet(G,M) = \mathrm{Ext}^\bullet_{\mathbb{Z}G}(\mathbb{Z},M)$$

in nonnegative degrees, where $\mathbb{Z}$ is the trivial G-module.

Note that

$$H^n(\{1\},M) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z},M) \cong \begin{cases} M & \text{for } n=0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

The degree-zero derived functor gives the following result.

Lemma 2.11. Let G be a discrete group and let M be a left G-module. Then $H^0(G,M) \cong M^G$, where $M^G = \{m \in M \mid gm = m \text{ for every } g \in G\}$.

Proof. The degree-zero derived functor is naturally isomorphic to the original left exact functor. Hence $H^0(G,M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},M)$. It therefore suffices to construct a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},-) \cong (-)^G$. To do that, define

$$\begin{aligned} \varphi_M : \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},M) &\to M^G \\ (f:\mathbb{Z}\to M) &\mapsto f(1). \end{aligned}$$

□

Using Lemma 2.11, the following computation gives the degree-zero cohomology of $\mathbb{Z}G$.

Lemma 2.12. Let G be a discrete group, and equip $\mathbb{Z}G$ with the left G-action by left multiplication. Then

$$(\mathbb{Z}G)^G \cong \begin{cases} 0 & \text{if } G \text{ is infinite}, \\ \mathbb{Z} & \text{if } G \text{ is finite}. \end{cases}$$

More precisely, if G is finite, we define the *norm element*

$$N = \sum_{g\in G} 1 \cdot g \in \mathbb{Z}G,$$

and $(\mathbb{Z}G)^G = \langle N \rangle$.

Proof. (1) Assume $|G| = \infty$. Then $a \in (\mathbb{Z}G)^G$ if and only if $g \cdot a = a$ for every $g \in G$. Write $a = \sum_{h\in G} n_h \cdot h$ as a finite linear combination. Then $ga = a$ implies that

$$g \cdot a = \sum_{h\in G} n_h \cdot gh = \sum_{h'\in G} n_{h'} \cdot h' = a,$$

Thus $n_h = n_{gh}$ for every $g, h \in G$, so all coefficients agree. Since G is infinite and a has finite support, these coefficients are zero.

(2) Assume that G is finite. We show that $(\mathbb{Z}G)^G = \mathbb{Z}\langle N \rangle$. First of all, $g \cdot N = \sum_{h\in G} g \cdot h = \sum_{h\in G} h$ since G is finite. Moreover, if $a \in (\mathbb{Z}G)^G$, then by writing $a = \sum_{h\in G} n_h \cdot h$ we have that $g \cdot a = \sum_{h\in G} n_h \cdot gh = \sum_{h'\in G} n_{h'} h' = a$. This implies that $n_h = n_{gh}$, which means that all coefficients are the same; hence $a = n \cdot N$.

□

By Lemma 2.11 and Lemma 2.12 we have that

$$G \text{ is infinite} \Leftrightarrow H^0(G;\mathbb{Z}G) = 0.$$

This illustrates the dependence of cohomology on the coefficient module. To compute $H^\bullet(G;M) = \mathrm{Ext}^\bullet_{\mathbb{Z}G}(\mathbb{Z},M)$, we choose a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. We begin with a surjection onto $\mathbb{Z}$. We have a canonical choice:

Definition 2.13. Let G be a discrete group, equip $\mathbb{Z}G$ with its left multiplication action, and equip $\mathbb{Z}$ with the trivial G-action. The *augmentation map* $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is defined by $g \mapsto 1$.

Clearly, the augmentation map is a surjective G-homomorphism. To extend this to a projective resolution of $\mathbb{Z}$, we consider its kernel

$$0 \to IG \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

which in general is not projective. We have the following description of this kernel.

Lemma 2.14. Let G be a discrete group and let $IG = \ker(\varepsilon : \mathbb{Z}G \to \mathbb{Z})$, where ε is the augmentation map of Definition 2.13. We regard IG as a left G-module by left multiplication.

(1) IG is generated, as an abelian group, by $\{g-1 \in \mathbb{Z}G \mid g \in G\}$. Here 1 is the unit of $\mathbb{Z}G$.
(2) Let $S \subseteq G$ be a subset. Then S generates G if and only if $\{s-1 \mid s \in S\}$ generates IG as a G-module.
(3) The group G is finitely generated if and only if IG is finitely generated as a G-module.

Proof. (1) First of all, $g-1 \in IG$ for every $g \in G$. Let $x = \sum n_g \cdot g \in IG = \ker(\varepsilon)$. Then $\sum n_g = 0$ and so

$$x = x - 0 = \sum n_g \cdot g - \sum n_g \cdot 1 = \sum n_g \cdot (g-1).$$

(2) ($\Rightarrow$) If S generates G, we want to show that $g - 1 = \sum \lambda_s (s-1)$ for $\lambda_s \in \mathbb{Z}G$. Allowing inverses of elements of S, we write

$$g - 1 = s_1 \cdots s_k - 1 = (s_1 \cdots s_{k-1})(s_k - 1) + \cdots$$

as a telescoping sum. Note that $s^{-1} - 1 = -s^{-1}(s-1)$. ($\Leftarrow$) Let $H = \langle S \rangle \leq G$. We show $H = G$. Consider the basis element $eH \in \mathbb{Z}[G/H]$. Now for all $s \in S$

$$(s-1) \cdot eH = sH - eH = eH - eH = 0$$

since $s \in H$. As IG is generated by the elements $s-1$, we obtain $(IG) \cdot eH = 0$. Thus $(g-1)eH = 0$ for every $g \in G$, so $gH = H$ and hence $g \in H$.

(3) It follows from (2). The reverse implication requires a little care.

□

For more explicit computations, we now specialize to cyclic groups.

2.2. **Cohomology of cyclic groups.** We begin by looking at the case of cyclic groups $G = \langle t \rangle$. In this case, Lemma 2.14(2) ensures that IG is generated, as a G-module, by one element. Hence we can continue our sequence as

$$\begin{array}{ccccccc} \mathbb{Z}G & \xrightarrow{-\cdot(t-1)} & \mathbb{Z}G & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\ & \searrow & \uparrow & & & & \\ & & IG = (t-1) & & & & \end{array}$$

which is clearly exact. To go further, we need to compute

$$\ker(- \cdot (t-1)) = (\mathbb{Z}G)^G.$$

We distinguish two cases using Lemma 2.12. If $G \cong \mathbb{Z}$ is infinite cyclic, then $(\mathbb{Z}G)^G \cong 0$, and so

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \xrightarrow{-\cdot(t-1)} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is the short exact sequence of $\mathbb{Z}[\mathbb{Z}]$-modules we were looking for. Hence, for every $M \in {}_{\mathbb{Z}[\mathbb{Z}]}\mathrm{Mod}$ we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}], M) & \xrightarrow{t-1} & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}], M) & \longrightarrow & 0 \\ & & \cong & & \cong & & \\ & & M & & M & & \end{array}$$

giving us

$$H^\bullet(\mathbb{Z}; M) \cong \begin{cases} M^{\mathbb{Z}} & \bullet = 0, \\ M_{\mathbb{Z}} & \bullet = 1, \\ 0 & \bullet \neq 0, 1. \end{cases}$$

Here $M_{\mathbb{Z}}$ denotes the coinvariants. In particular, note that the group cohomology

$$H^\bullet(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z} \cdot 0 \oplus \mathbb{Z} \cdot 1 + 0$$

is identical to the integral cohomology $H^\bullet(S^1; \mathbb{Z})$ of S^1. The topological interpretation in Section 2.5 explains this agreement. The second case is when $G \cong \mathbb{Z}/n$ is a finite cyclic group. In this case Lemma 2.12 implies that

$$\begin{array}{ccccccccc} \mathbb{Z}[G] & \xrightarrow{-\cdot(t-1)} & \mathbb{Z}[G] & \xrightarrow{-\cdot N} & \mathbb{Z}[G] & \xrightarrow{-\cdot(t-1)} & \mathbb{Z}[G] & \xrightarrow{\varepsilon} & \mathbb{Z} \longrightarrow 0 \\ & & & \searrow & \uparrow & \searrow & \uparrow & & \\ & & & & (\mathbb{Z}G)^G \cong \mathbb{Z}\langle N \rangle & & IG = (t-1) & & \end{array}$$

since $\ker(- \cdot (t-1)) = (\mathbb{Z}G)^G \cong \mathbb{Z}\langle N \rangle$ is generated by the norm element. By going further, we have

$$\ker(- \cdot N) = \cdots = \mathrm{Im}(t-1).$$

This gives a periodic free resolution of $\mathbb{Z}$. We obtain the following computation.

Corollary 2.15. Let $n \geq 1$ be an integer, let $G = \langle t \mid t^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$, and let M be a left G-module. Write $N = 1 + t + \cdots + t^{n-1} \in \mathbb{Z}G$, and interpret N and $t - 1$ as endomorphisms of the underlying abelian group of M through its G-action. Then

$$H^\bullet(\mathbb{Z}/n\mathbb{Z}, M) \cong \begin{cases} M^{\mathbb{Z}/n\mathbb{Z}} & \bullet = 0, \\ \ker(N)/\mathrm{Im}(t-1) & \bullet \text{ odd}, \\ \ker(t-1)/\mathrm{Im}(N) & \bullet \text{ even}, \ \bullet \geq 2. \end{cases}$$

In particular, for the trivial G-module $M = \mathbb{Z}$ we have

$$H^\bullet(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \bullet = 0, \\ 0 & \bullet \text{ odd}, \\ \mathbb{Z}/n\mathbb{Z} & \bullet \text{ even}, \ \bullet \geq 2. \end{cases}$$

Remark 2.16. Let $n \geq 1$ be an integer, and give $\mathbb{Z}$ and $\mathbb{Q}$ the trivial $\mathbb{Z}/n\mathbb{Z}$-actions. For $n = 2$, this is the integral cohomology of $\mathbb{RP}^\infty$. Similarly, $H^\bullet(\mathbb{Z}/n, \mathbb{Q}) \cong \mathbb{Q}$ for $\bullet = 0$, and zero otherwise.

2.3. **The standard and bar resolutions.** Computing group cohomology requires projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, which can be difficult to construct explicitly. In this section we construct two canonical free resolutions. We begin by constructing the standard resolution. Take a group $G \in \mathrm{Grp}$ and let us consider the G-set G^{n+1} with the diagonal action on the left

$$g \cdot (g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n).$$

With this definition we see immediately that G^{n+1} is a free G-set. In particular

$$\mathbb{Z}[G^{n+1}] \cong \mathbb{Z}\left[\coprod_{G^n} G\right] \cong \bigoplus_{G^n} \mathbb{Z}G$$

is a free G-module. We can therefore make the following definition.

Definition 2.17. Let G be a discrete group and give $\mathbb{Z}$ the trivial G-action. The *standard resolution* $C_\bullet(G)$ of $\mathbb{Z}$ over $\mathbb{Z}G$ has $C_n(G) = \mathbb{Z}[G^{n+1}]$ for $n \geq 0$, with the diagonal left G-action:

$$\cdots \to \mathbb{Z}[G^3] \xrightarrow{\partial} \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where, for every integer $n \geq 1$, the boundary map is given by

$$\partial_n : C_n(G) \to C_{n-1}(G)$$

$$(g_0, \ldots, g_n) \mapsto \partial_n(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

The map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map of Definition 2.13.

A direct computation gives $\partial^2 = 0$. We can also show that we indeed have a free resolution of $\mathbb{Z}$:

Lemma 2.18. Let G be a discrete group, and let $(C_\bullet(G), \partial)$ be the complex of Definition 2.17, with its augmentation to $\mathbb{Z}$. Then $C_\bullet(G)$ is a free resolution of the trivial left G-module $\mathbb{Z}$.

Proof. We prove exactness by constructing a contraction of the augmented complex over $\mathbb{Z}$. This suffices, since exactness of a sequence of $\mathbb{Z}G$-modules can be checked on the underlying abelian groups. Hence we have to show that its identity is chain homotopic to zero. We define the homotopy $s_n : C_n(G) \to C_{n+1}(G)$ by

$$s_n(g_0, \ldots, g_n) = (e_G, g_0, \ldots, g_n).$$

In degree -1, we set $C_{-1}(G) = \mathbb{Z}$, $\partial_0 = \varepsilon$, and $s_{-1}(1) = (e_G)$. This is clearly a $\mathbb{Z}$-map, but not a G-map. A calculation shows that $s_{n-1} \circ \partial_n + \partial_{n+1} \circ s_n = \mathrm{id}_{C_n(G)}$, and $\varepsilon \circ s_{-1} = \mathrm{id}_{\mathbb{Z}}$. □

We now define for a G-module M the cochain complex

$$C^\bullet(G, M) := \mathrm{Hom}_{\mathbb{Z}G}(C_\bullet(G), M).$$

By Lemma 2.18, we obtain the following computation.

Proposition 2.19. Let G be a discrete group and let M be a left G-module. Let $C_\bullet(G)$ be the standard resolution of Definition 2.17, and set $C^\bullet(G, M) = \mathrm{Hom}_{\mathbb{Z}G}(C_\bullet(G), M)$. Then $H^\bullet(G, M) = H^\bullet(C^\bullet(G, M))$.

This shows how we can compute the group cohomology of a group with coefficients in a module M.

We now introduce the bar resolution. This amounts to a change of coordinates in the standard resolution. The idea is that

$$\mathbb{Z}[G^{n+1}] \cong_G \mathbb{Z}G[G^n].$$

Here $\mathbb{Z}G[G^n]$ is free over $\mathbb{Z}G$ on the basis G^n, written $[g_1|\cdots|g_n]$. This leads to the following definition.

Definition 2.20. Let G be a discrete group and give $\mathbb{Z}$ the trivial G-action. The *bar resolution* $\overline{C}_\bullet(G)$ of $\mathbb{Z}$ consists of

(1) $\overline{C}_n(G) = \mathbb{Z}G[G^n]$, the free left $\mathbb{Z}G$-module with basis G^n, for each integer $n \geq 0$;
(2) the boundary maps $\overline{\partial}_n : \overline{C}_n(G) \to \overline{C}_{n-1}(G)$, defined by

$$\begin{aligned}\overline{\partial}_n([g_1|\cdots|g_n]) &= g_1[g_2|\cdots|g_n] \\ &\quad + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &\quad + (-1)^n[g_1|\cdots|g_{n-1}]\end{aligned}$$

for $n \geq 1$, together with the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ of Definition 2.13.

The following lemma identifies the two resolutions.

Lemma 2.21. Let G be a discrete group, and let $C_\bullet(G)$ and $\overline{C}_\bullet(G)$ be the complexes of Definition 2.17 and Definition 2.20, respectively. There is a G-chain isomorphism $C_\bullet(G) \cong \overline{C}_\bullet(G)$ whose forward and inverse maps are given, in every degree $n \geq 0$, by the $\mathbb{Z}G$-linear extensions of

$$\begin{aligned}(g_0, \ldots, g_n) &\mapsto g_0[g_0^{-1}g_1|g_1^{-1}g_2|\cdots|g_{n-1}^{-1}g_n], \\ [g_1|\cdots|g_n] &\mapsto (e_G, g_1, g_1g_2, \ldots, g_1\cdots g_n).\end{aligned}$$

By Lemma 2.21 and Proposition 2.19, for every left G-module M we get that

$$H^\bullet(G; M) \cong H^\bullet(\mathrm{Hom}_{\mathbb{Z}G}(\overline{C}_\bullet(G), M)).$$

We now apply the bar resolution to compute low-degree cohomology. We now compute the first cohomology group $H^1(G; M)$ for a module $M \in {}_G\mathrm{Mod}$. Indeed, we have the cochain complex

$$\overline{C}^2(G, M) \xleftarrow{\delta^1} \overline{C}^1(G, M) \xleftarrow{\delta^0} \overline{C}^0(G, M)$$

where

$$\begin{aligned}\delta^0(f : \mathbb{Z}G \to M)([g_1]) &= f(\overline{\partial}_1[g_1]) = f(g_1 \cdot [\,] - [\,]) \\ &= g_1 f([\,]) - f([\,])\end{aligned}$$

where $\overline{C}_0(G) = \mathbb{Z}G = \mathrm{dom}(f)$, and

$$\begin{aligned}\delta^1(\varphi)([g_1|g_2]) &= \varphi(\overline{\partial}_2[g_1|g_2]) \\ &= \varphi(g_1[g_2] - [g_1g_2] + [g_1]) \\ &= g_1\varphi([g_2]) - \varphi([g_1g_2]) + \varphi([g_1]).\end{aligned}$$

Hence

Proposition 2.22. Let G be a discrete group and let M be an abelian group with the trivial G-action. Then

$$H^1(G, M) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, M).$$

Proof. Using the bar resolution of Definition 2.20, we get $H^1(G, M) = \ker(\delta^1)/\operatorname{Im}(\delta^0)$. But

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[G], M) \cong \mathrm{Hom}_{\mathrm{Set}}(G, M)$$

on basis elements, and since

$$\begin{aligned}\varphi \in \ker(\delta^1) &\Leftrightarrow g_1\varphi([g_2]) - \varphi([g_1g_2]) + \varphi([g_1]) = 0 \quad \text{for all } g_1, g_2 \in G \\ &\Leftrightarrow \varphi([g_2]) - \varphi([g_1g_2]) + \varphi([g_1]) = 0 \\ &\Leftrightarrow \varphi \in \mathrm{Hom}_{\mathrm{Grp}}(G, M)\end{aligned}$$

and

$$\varphi \in \mathrm{Im}(\delta^0) \Leftrightarrow \cdots \Leftrightarrow \varphi = 0$$

we conclude. □

Corollary 2.23. Let G be a finite group and let M be a torsion-free abelian group with the trivial G-action. Then $H^1(G, M) = 0$.

Proof. This follows from Proposition 2.22. □

More generally, for any G-module $M \in {}_G\mathrm{Mod}$, there is a description

$$H^1(G, M) = \{\text{derivations}\}/\{\text{principal derivations}\}.$$

We conclude this section by studying the universality of $H^1(G, IG)$.

Lemma 2.24. Let G be a discrete group and let $IG = \ker(\varepsilon : \mathbb{Z}G \to \mathbb{Z})$, with its left multiplication action, where ε is the augmentation map of Definition 2.13. There exists a class $\alpha \in H^1(G; IG)$ such that for every G-module M and $\beta \in H^1(G; M)$ there exists a G-map $f : IG \to M$ such that $f_* : H^1(G; IG) \to H^1(G; M)$ sends α to β.

Proof. The map f_* is defined in Section 2.6. Consider now an element $\beta \in H^1(G; M)$. Since $H^1(G; M) = H^1(\mathrm{Hom}_{\mathbb{Z}G}(C_\bullet(G), M))$, we can represent β by a map $b : C_1(G) \to M$ such that $\delta^1(b) = b\partial_2 = 0$. Hence

$$\begin{array}{ccccccccc} \mathbb{Z}[G^3] & \xrightarrow{\partial_2} & \mathbb{Z}[G^2] & \xrightarrow{\partial_1} & \mathbb{Z}G & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle b}\downarrow & \searrow & \uparrow & & & & \\ & & M & \xleftarrow[f]{} & IG & & & & \end{array}$$

there exists a unique map $f : IG \to M$ through which b factors. Take $\alpha = [\partial_1] \in H^1(G; IG)$, where ∂_1 is regarded as a map into IG; then

$$f_*(\alpha) = [f \circ \partial_1] = [b] = \beta.$$

□

By Lemma 2.24, if $H^1(G; M) \neq 0$ for some $M \in {}_G\mathrm{Mod}$, then $H^1(G; IG) \neq 0$.

2.4. $H^2(G, M)$ **and central extensions.** In this section we relate $H^2(G, M)$ to central extensions of G by M. We begin with the definition of a central extension.

Definition 2.25. Let G be a discrete group and let M be an abelian group. A *central extension* of G by M is an exact sequence of groups

$$1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

where $i(M) \subseteq Z(E)$.

Central extensions, as suggested by the name, impose some "abelian condition". Extensions of abelian groups are classified by Ext^1, whereas here G need not be abelian. Therefore the condition $i(M) \subseteq Z(E)$ allows us to apply this theory. Different extensions can have isomorphic middle groups, so we must specify when two extensions are equivalent. We use the following notion of equivalence.

Definition 2.26. Let G be a discrete group, let M be an abelian group, and let E and E' be central extensions of G by M in the sense of Definition 2.25. These extensions are *equivalent* if there exists a group homomorphism $\varphi : E \to E'$ such that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes.

Note that we cannot immediately argue by the five lemma that φ is an isomorphism, since Grp is not an additive category (hence not an abelian category). However, $\varphi : E \to E'$ is an isomorphism. A direct diagram chase proves both injectivity and surjectivity. This defines an equivalence relation on central extensions. We are interested in the set

$$\mathcal{E}(G, M) = \{\text{central extensions } 1 \to M \to E \to G \to 1\}/\sim$$

of central extensions of G by M, modulo equivalence.

Example 2.27. We can describe some extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$. Indeed (in additive notation),

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0,$$

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

are distinct elements of $\mathcal{E}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$, since $\mathbb{Z}$ is torsion-free and $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not. Are there any other elements of $\mathcal{E}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$?

As seen in Example 2.27, we can easily produce at least one element of $\mathcal{E}(G, M)$, but can we describe it completely? Group cohomology allows us to describe completely the set of central extensions of G by M, by the following theorem. For the general classification of extensions, see [Bro82, Chapter IV].

Theorem 2.28. Let G be a discrete group and let M be an abelian group with the trivial G-action. Let $\mathcal{E}(G, M)$ denote the set of central extensions of G by M, modulo the equivalence relation of Definition 2.26. Then there exists a bijection $\varphi : \mathcal{E}(G, M) \to H^2(G, M)$ of sets.

Proof. We construct the bijection φ and its inverse. We begin by constructing φ. Let $1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ be a representative of an equivalence class of central extensions. We identify M with $i(M)$, writing its operation additively when computing cocycles. Choose a section $s : G \to E$ of π: it exists as a function by the axiom of choice, but need not be a group homomorphism. Using this section, define $f_s : G \times G \to E$ by

$$f_s(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

This map need not be trivial, since s need not be a group map. Note that $\mathrm{Im}(f_s) \subseteq M$ since $M = \ker(\pi)$ and $\pi f_s(g_1, g_2) = 1$ for every $g_1, g_2 \in G$. Extending linearly gives the $\mathbb{Z}G$-map

$$\begin{aligned} f_s : \overline{C}_2(G) = \mathbb{Z}G[G^2] &\to M \\ g[g_1|g_2] &\mapsto f_s(g_1, g_2) \end{aligned}$$

where M has the trivial action. Associativity of $s(g_1)s(g_2)s(g_3)$ and centrality of M give

$$f_s(g_1, g_2) + f_s(g_1g_2, g_3) = f_s(g_2, g_3) + f_s(g_1, g_2g_3).$$

Thus $\delta^2(f_s) = 0$, since

$$\begin{aligned} \delta^2(f_s)([g_1|g_2|g_3]) &= f_s(\overline{\partial}_3[g_1|g_2|g_3]) \\ &= f_s\big(g_1[g_2|g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2]\big) \\ &= f_s(g_2, g_3) - f_s(g_1g_2, g_3) + f_s(g_1, g_2g_3) - f_s(g_1, g_2) = 0. \end{aligned}$$

There is a bijection $E \cong M \times G$,

$$h \mapsto \big(hs(\pi(h))^{-1}, \pi(h)\big),$$
$$(m, g) \mapsto ms(g).$$

Under this bijection, multiplication in E becomes

$$(m, g) \cdot (m', g') = (m + m' + f_s(g, g'), gg').$$

Hence we have a class $\gamma_E := [f_s] \in H^2(G, M)$. If s' is another section, there is a function $u : G \to M$ with $s'(g) = i(u(g))s(g)$. The associated cocycles satisfy

$$f_{s'}(g, h) = f_s(g, h) + u(g) + u(h) - u(gh) = f_s(g, h) + (\delta^1 u)(g, h),$$

so their cohomology classes agree. An equivalence of extensions carries s to a section of the equivalent extension with exactly the same cocycle, so γ_E depends only on the equivalence class of E. ($\Leftarrow$) Let $\gamma \in H^2(G, M)$ be a cohomology class and $f \in [\gamma]$ a representative. Then $f : \overline{C}_2(G) \to M$ is a cocycle and we can define $E_f :=_{\text{Set}} M \times G$ via

$$(m, g) \cdot (m', g') := (m + m' + f(g, g'), gg').$$

The cocycle identity

$$f(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0$$

is precisely the associativity condition for this multiplication. Putting $c = f(e_G, e_G)$, the same identity gives $f(e_G, g) = f(g, e_G) = c$ and $f(g, g^{-1}) = f(g^{-1}, g)$ for every $g \in G$. Consequently, the identity and inverses are

$$1_{E_f} = (-c, e_G), \qquad (m, g)^{-1} = \big(-m - f(g, g^{-1}) - c, g^{-1}\big).$$

This proves that E_f is a group without requiring the cocycle to be normalized. We define

$$1 \to M \xrightarrow{i} E_f \xrightarrow{\pi} G \to 1$$

by $i(m) = (m - f(e_G, e_G), e_G)$ and $\pi(m, g) = g$. The map i is an injective homomorphism with central image equal to $\ker \pi$, and π is surjective, so this is a central extension. If $f' = f + \delta^1 u$, the map

$$E_f \to E_{f'}, \qquad (m, g) \mapsto (m - u(g), g)$$

is an equivalence of extensions; its compatibility with the kernel inclusions uses $f'(e_G, e_G) = c + u(e_G)$. Thus $[E_f]$ depends only on $[f]$. Starting with E and a section s, the bijection $E_{f_s} \to E$ given by $(m, g) \mapsto i(m)s(g)$ is an equivalence of extensions. Conversely, the section $g \mapsto (0, g)$ of $E_f \to G$ has associated cocycle f, when the kernel is identified with M through i. The two constructions are therefore inverse. □

In the bijection of Theorem 2.28, the zero class corresponds to the trivial extension $M \times G$. More generally, for any G-module M, the cohomology group $H^2(G, M)$ can be identified with extensions

$$1 \to M \to E \to G \to 1$$

such that E acts by conjugation on M, and $G = E/M \curvearrowright M$ is the action of G on M, modulo equivalence.

Example 2.29. Since $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ by Corollary 2.15, Theorem 2.28 shows that Example 2.27 lists all central extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$. We can also try to describe $\mathcal{E}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$, which has cardinality 3. Its elements are

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/3 \to \mathbb{Z}/3 \to 0,$$
$$0 \to \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\pi_1} \mathbb{Z}/3 \to 0, \qquad \pi_1(1) = 1 \bmod 3,$$
$$0 \to \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z}/3 \to 0, \qquad \pi_2(1) = 2 \bmod 3.$$

Indeed, the two other extensions are not isomorphic, since

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 3} & \mathbb{Z} & \xrightarrow{\pi_1} & \mathbb{Z}/3 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 3} & \mathbb{Z} & \xrightarrow{\pi_2} & \mathbb{Z}/3 & \longrightarrow & 0 \end{array}$$

if such a φ exists, then $\varphi(1) = \pm 1$. But $\varphi(1) = -1$ is excluded by the commutativity of the first square, and $\varphi(1) = 1$ is excluded by the second one.

2.5. **Resolutions via topology.** Explicit computations can remain difficult even with the standard and bar resolutions. We now relate group cohomology to topology by constructing a space BG whose cohomology with local coefficients computes $H^\bullet(G, M)$. We begin with a connected CW-complex X with fundamental group $G = \pi_1(X)$. By the theory of covering spaces, G acts freely on the universal cover $\widetilde{X}$, via monodromy. For example, if $X = S^1$ is the circle, then $\pi_1(S^1) \cong \mathbb{Z}$ acts by translation on the universal cover $\mathbb{R} \to S^1$, and it is clear that $\mathbb{Z}$ acts on the set of n-cells of $\mathbb{R}$. More generally, G acts freely on the set of n-cells of $\widetilde{X}$. So if we define

$$C_n^{\mathrm{cell}}(\widetilde{X}) = \mathbb{Z}\{n\text{-cells of } \widetilde{X}\}$$

we obtain a free, hence projective, G-module. The cellular boundary maps give a chain complex of free G-modules

$$C_n^{\mathrm{cell}}(\widetilde{X}) \to C_{n-1}^{\mathrm{cell}}(\widetilde{X}) \to \cdots \to C_0^{\mathrm{cell}}(\widetilde{X}) \to \mathbb{Z} \to 0.$$

If $\widetilde{X}$ is contractible, the augmented complex is exact and hence is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Definition 2.30. Let G be a discrete group and let X be a connected CW-complex with a basepoint $x \in X$ and universal cover $\widetilde{X}$. We say that X is a *model for* BG if $\pi_1(X, x) \cong G$ and $\widetilde{X}$ is contractible.

In other words, X is an Eilenberg–MacLane space of type $K(G, 1)$. The space BG is also called the *classifying space* for G. The universal property is stated in Proposition 2.31. See also [Hat02, Proposition 1B.9].

Proposition 2.31. Let G be a discrete group, let (Y, y) and (X, x) be pointed connected CW-complexes, and suppose that the universal cover of X is contractible. Fix an isomorphism $\pi_1(X, x) \cong G$. Then

$$\pi_1 : [Y, X]_* \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(Y, y), G)$$

is bijective, where $[Y, X]_*$ denotes the set of pointed homotopy classes of pointed maps.

Hence maps into BG are determined by what they do on π_1. In particular, every two models for BG are homotopy equivalent. Thus BG determines a homotopy type and gives an adjunction

$$\pi_1 : \mathrm{hoCW}_* \rightleftarrows \mathrm{Grp} : B.$$

Here hoCW_* denotes the homotopy category of pointed connected CW-complexes. We give a sketch of the proof of Proposition 2.31.

Proof. We sketch the proof of surjectivity. Let $\varphi : \pi_1(Y) \to \pi_1(X) \cong G$ be a group homomorphism. We construct a map $f : Y \to X$ inducing φ on fundamental groups. We proceed inductively over the skeleta of Y, constructing maps $f^{(n)} : Y^{(n)} \to X$. By collapsing a maximal tree in $Y^{(1)}$, we may replace Y by a homotopy equivalent CW-complex with a single 0-cell. For $n = 0$, the map $f^{(0)} : Y^{(0)} \to X$ sends the basepoint y_0 to x_0. For $n = 1$, a 1-cell γ determines an element $\gamma \in \pi_1(Y)$,

and so we can define $f^{(1)}$ by mapping it to a loop representing $\varphi(\gamma)$. For $n = 2$, we assume we have $f^{(1)} : Y^{(1)} \to X$. We need to check that

$$\begin{array}{ccc} Y^{(1)} & \xrightarrow{f^{(1)}} & X \\ {\scriptstyle\sigma}\uparrow & \nearrow_{\sim 0} & \\ S^1 & & \end{array}$$

but

$$\begin{array}{ccc} \pi_1(Y^{(1)}) & \xrightarrow{f^{(1)}_*} & \pi_1(X) \\ \downarrow & \nearrow_{\varphi} & \\ \pi_1(Y) & & \end{array} \qquad \begin{array}{ccc} \sigma & \longmapsto & 0 \\ \downarrow & \nearrow & \\ 0 & & \end{array}$$

so the map extends. For $n \geq 3$ we do the same, but we lift to

$$\begin{array}{ccc} Y^{(n-1)} & \xrightarrow{f^{(n-1)}} & X \\ \uparrow & & \uparrow \\ S^{n-1} & \dashrightarrow & \widetilde{X} \simeq * \end{array}$$

since S^{n-1} is simply connected. The universal cover $\widetilde{X}$ is contractible, so the composition is nullhomotopic. □

Example 2.32. (1) A model for the trivial group $\{1\}$ is given by $B\{1\} = *$, since it has trivial fundamental group, and its universal cover $* \to *$ is contractible.

(2) A model for $B\mathbb{Z}$ is given by the circle $S^1 \simeq B\mathbb{Z}$, since its universal cover $\mathbb{R} \to S^1$ is contractible. Since B is a right adjoint, it preserves products, giving $B\mathbb{Z}^n \simeq T^n$, the n-dimensional torus, for every $n \geq 0$.

(3) Let $k \geq 0$ be an integer and let $F_k = *_k \mathbb{Z}$ be the free group on k generators. Then a model for BF_k is given by $BF_k \simeq \bigvee_k S^1$, since its universal cover is a $2k$-regular tree, which is contractible.

(4) A model for $B\mathbb{Z}/2$ is given by $\mathbb{RP}^\infty$, since its universal cover S^∞ is contractible. Moreover, for each integer $n \geq 1$, $B\mathbb{Z}/n$ is the infinite lens space $S^\infty/(\mathbb{Z}/n)$, with universal cover S^∞.

(5) Let $g \geq 1$ be an integer, let Σ_g be the closed connected orientable surface of genus g, and let $G = \pi_1(\Sigma_g)$, that is

$$G = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

Then $\Sigma_g \simeq B\pi_1(\Sigma_g)$ is a model, since the universal cover is homeomorphic to $\mathbb{R}^2$.

Example 2.32 gives models for several familiar groups. We now show that for any group G there exists a standard model for BG. It is defined as the geometric realization of the simplicial nerve $\mathbb{B}G_n := G^n$ where face maps are induced by the product operation of G, that is

$$BG = |\mathbb{B}G|.$$

In particular, if we define the standard simplicial model of G by

$$EG = \left(\coprod_{n \in \mathbb{N}} \Delta^n \times G^{n+1} \right) / \sim \tag{1}$$

where $\sim$ is the equivalence relation generated by

$$\big(i_j(t), (g_0, \dots, g_n)\big) \sim \big(t, (g_0, \dots, \widehat{g_j}, \dots, g_n)\big)$$

for all $n \geq 1$ and $j \in \{0, \dots, n\}$, $t \in \Delta^{n-1}$ and $g_0, \dots, g_n \in G$, where $i_j : \Delta^{n-1} \to \Delta^n$ is the inclusion of the j-th face. The diagonal action on G^{n+1} induces an action $G \times EG \to EG$.

Proposition 2.33. Let G be a discrete group and let EG be the space defined by Equation 1 and its face identifications, with the diagonal left G-action.

(1) EG is contractible (and locally path connected);
(2) the diagonal action $G \curvearrowright EG$ is properly discontinuous.

It follows that the canonical projection $EG \to BG := EG/G$ is the universal cover for BG, and moreover $\pi_1(BG) \cong G$. This constructs a model for BG. Having constructed the space BG, we can define the cohomology with twisted coefficients.

Definition 2.34. Let (X, x) be a pointed connected CW-complex, set $G = \pi_1(X, x)$, and let $\widetilde{X}$ be its universal cover with the left G-action by deck transformations. Let M be a left G-module. We define the *twisted cohomology* of X with coefficients in M by

$$H^\bullet_{\mathrm{tw}}(X; M) := H^\bullet\big(\mathrm{Hom}_G(C^{\mathrm{cell}}_\bullet(\widetilde{X}), M)\big) \in \mathrm{Ab}.$$

This makes sense, since the cellular complex $C^{\mathrm{cell}}_\bullet(\widetilde{X})$ carries a G-action. The relation with group cohomology is the following.

Proposition 2.35. Let G be a discrete group, let M be a left G-module, and let BG be a connected CW-complex with fundamental group identified with G and contractible universal cover. Then

$$H^\bullet(G; M) \cong H^\bullet_{\mathrm{tw}}(BG; M).$$

Proof. Since the universal cover of BG is contractible, its augmented cellular chain complex gives a free resolution $P_\bullet \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$, namely

$$\cdots \to C^{\mathrm{cell}}_n(\widetilde{BG}) \to \cdots \to C^{\mathrm{cell}}_0(\widetilde{BG}) \to \mathbb{Z} \to 0.$$

Therefore, we have

$$\begin{aligned} H^\bullet(G, M) &:= \mathrm{Ext}^\bullet_{\mathbb{Z}G}(\mathbb{Z}, M) \\ &= H^\bullet(\mathrm{Hom}_G(P_\bullet, M)) \\ &= H^\bullet\big(\mathrm{Hom}_G(C^{\mathrm{cell}}_\bullet(\widetilde{BG}), M)\big) =: H^\bullet_{\mathrm{tw}}(BG, M). \end{aligned}$$

□

Hence we can always compute the group cohomology of a group G with coefficients in any G-module M by using the twisted cohomology of BG. We can specialize Proposition 2.35 to the trivial module $M = \mathbb{Z}$. In this case there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{Ab}}(C^{\mathrm{cell}}_\bullet(X), \mathbb{Z}) \cong \mathrm{Hom}_G(C^{\mathrm{cell}}_\bullet(\widetilde{X}), \mathbb{Z}).$$

Indeed, let X be a connected CW-complex with fundamental group G and universal cover $\widetilde{X}$. Each cell of X corresponds to one G-orbit of lifted cells, and the cellular boundary maps respect the covering projection. Consequently, there is an isomorphism of chain complexes

$$\mathbb{Z} \otimes_{\mathbb{Z}G} C^{\mathrm{cell}}_\bullet(\widetilde{X}) \cong C^{\mathrm{cell}}_\bullet(X),$$

where the first factor is the trivial right $\mathbb{Z}G$-module. The tensor–Hom adjunction now gives

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{cell}}_\bullet(X), \mathbb{Z}) &\cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z} \otimes_{\mathbb{Z}G} C^{\mathrm{cell}}_\bullet(\widetilde{X}), \mathbb{Z}\big) \\ &\cong \mathrm{Hom}_{\mathbb{Z}G}(C^{\mathrm{cell}}_\bullet(\widetilde{X}), \mathbb{Z}). \end{aligned}$$

The same argument works with any abelian coefficient group carrying the trivial G-action.

In particular, we can compute

$$\begin{aligned} H^n_{\mathrm{tw}}(X;\mathbb{Z}) &= H^n\big(\mathrm{Hom}_G(C^{\mathrm{cell}}_\bullet(\widetilde{X}),\mathbb{Z})\big) \\ &\cong H^n\big(\mathrm{Hom}_{\mathrm{Ab}}(C^{\mathrm{cell}}_\bullet(X),\mathbb{Z})\big) \cong H^n(X;\mathbb{Z}) \qquad \text{(singular!)}. \end{aligned}$$

Hence, if we choose $X = BG$ we get that

$$H^\bullet(G;\mathbb{Z}) \cong H^\bullet_{\mathrm{tw}}(BG;\mathbb{Z}) \cong H^\bullet(BG;\mathbb{Z})$$

is the usual singular cohomology.

Example 2.36. We now compute $H^\bullet(G,\mathbb{Z})$ for all the groups listed in Example 2.32. We have

(1) $H^\bullet(\{1\},\mathbb{Z}) = H^\bullet(B\{1\},\mathbb{Z}) = H^\bullet(*,\mathbb{Z})$;
(2) $H^\bullet(\mathbb{Z},\mathbb{Z}) = H^\bullet(B\mathbb{Z},\mathbb{Z}) = H^\bullet(S^1,\mathbb{Z})$;
(3) $H^\bullet(\mathbb{Z}^n,\mathbb{Z}) = H^\bullet(B\mathbb{Z}^n,\mathbb{Z}) = H^\bullet(T^n,\mathbb{Z})$;
(4) $H^\bullet(F_k,\mathbb{Z}) = H^\bullet(BF_k,\mathbb{Z}) = H^\bullet(\bigvee_k S^1,\mathbb{Z})$;
(5) $H^\bullet(\pi_1(\Sigma_g),\mathbb{Z}) = H^\bullet(B\pi_1(\Sigma_g),\mathbb{Z}) = H^\bullet(\Sigma_g,\mathbb{Z})$.

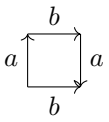
The cohomology groups on the right are familiar topological computations.

Example 2.37. $G = \langle a, b \mid abab^{-1} = 1\rangle$ is the fundamental group of the Klein bottle K. Its universal cover is $\mathbb{R}^2$: composing the universal cover of the torus with the two-sheeted covering $T \to K$ gives $\mathbb{R}^2 \to K$. Hence $BG \simeq K$ is a model for $G = \pi_1(K)$. Hence

$$H^\bullet(G,\mathbb{Z}) \cong H^\bullet(BG,\mathbb{Z})$$

but the cellular homology of $BG \simeq K$ is

$$C_\bullet(K): \qquad \underset{2}{\mathbb{Z}} \xrightarrow{(2,0)} \underset{1}{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \longrightarrow 0$$



and so by dualizing

$$H^\bullet(G,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \bullet = 0,1, \\ \mathbb{Z}/2 & \bullet = 2, \\ 0 & \text{otherwise.} \end{cases}$$

2.6. **Functoriality of group cohomology.** We conclude this chapter by studying functoriality. Fix a discrete group G and a nonnegative integer n. We first realize $H^n(G;M)$ as a covariant functor

$$H^n(G,-) : {}_G\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$$

and describe its contravariance in the group variable. For a homomorphism $\varphi : H \to G$, this takes the form of maps

$$\varphi^* : H^n(G;M) \to H^n(H;\mathrm{res}_\varphi M),$$

where res_φ is the restriction functor described below. We begin with a G-module homomorphism $f : M \to N$.

To construct the induced map $f_* : H^n(G;M) \to H^n(G;N)$, we work with cochain complexes. Choose a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$ and consider the map of cochain complexes

$$f_* = f \circ - : \underline{\mathrm{Hom}}_G(P_\bullet, M[0]) \to \underline{\mathrm{Hom}}_G(P_\bullet, N[0])$$

and set

$$f_* := H^n(f_*) : H^n(G;M) \to H^n(G;N)$$

to be our derived map. These constructions are functorial, giving $H^n(G,-) : {}_G\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$. We next consider functoriality in the group variable. Consider a group homomorphism $\varphi : H \to G$. We

will construct the induced map φ^* on cohomology. The homomorphism φ first gives a functor between module categories

$$\begin{aligned} \mathrm{res}_\varphi : {}_G\mathrm{Mod} &\to {}_H\mathrm{Mod} \\ (M, a) &\mapsto \mathrm{res}_\varphi(M, a) =: \mathrm{res}_\varphi M \end{aligned}$$

which keeps the underlying abelian group M and gives it the action

$$\begin{aligned} \mathrm{res}_\varphi(a) : H \times \mathrm{res}_\varphi(M) &\to \mathrm{res}_\varphi(M) \\ (h, m) &\mapsto a(\varphi(h), m). \end{aligned}$$

This functor keeps the underlying maps of abelian groups. Now to get a map

$$\varphi^* : H^n(G, M) \to H^n(H, \mathrm{res}_\varphi M)$$

on cohomology, we compare a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$ with a projective resolution $Q_\bullet \to \mathbb{Z} \to 0$ over $\mathbb{Z}H$. Restriction first gives a map

$$\underline{\mathrm{Hom}}_G(P_\bullet, M[0]) \to \underline{\mathrm{Hom}}_H(\mathrm{res}_\varphi P_\bullet, \mathrm{res}_\varphi M[0])$$

by functoriality. The augmented complex $\mathrm{res}_\varphi P_\bullet \to \mathbb{Z} \to 0$ is exact, but its terms need not be projective over $\mathbb{Z}H$. To solve this problem, we take a projective resolution $Q_\bullet \to \mathbb{Z} \to 0$ of H-modules. Then

$$\begin{array}{ccc} Q_\bullet & \longrightarrow & \mathbb{Z}[0] \\ \vdots & & \| \\ \mathrm{res}_\varphi P_\bullet & \longrightarrow & \mathbb{Z}[0] \end{array}$$

and precomposition gives the map

$$\begin{aligned} \underline{\mathrm{Hom}}_G(P_\bullet, M[0]) &\to \underline{\mathrm{Hom}}_H(\mathrm{res}_\varphi P_\bullet, \mathrm{res}_\varphi M[0]) \\ &\to \underline{\mathrm{Hom}}_H(Q_\bullet, \mathrm{res}_\varphi M[0]). \end{aligned}$$

The comparison theorem for projective resolutions [Bro82, Chapter I] makes the map $Q_\bullet \to \mathrm{res}_\varphi P_\bullet$ unique up to chain homotopy, since $Q_\bullet$ is projective in each degree and the target is an exact augmented resolution. Hence the induced map on cohomology is independent of this lift. Comparing any two projective resolutions gives chain homotopy equivalences, so the map is also independent of the chosen resolutions. For $K \xrightarrow{\psi} H \xrightarrow{\varphi} G$, the composite of comparison maps is a comparison map for $\varphi\psi$; uniqueness up to chain homotopy gives $(\varphi\psi)^* = \psi^*\varphi^*$, with the corresponding restrictions of coefficients. The identity homomorphism induces the identity, and postcomposition with a homomorphism of coefficient modules commutes with these maps. This proves the asserted functoriality.

Remark 2.38. There is also a topological interpretation. Let H and G be discrete groups, let $\varphi : H \to G$ be a group homomorphism, and let M be an abelian group with trivial G- and H-actions. The classifying space construction $B : \mathrm{Grp} \to \mathrm{CW}_*$ determines a map $B\varphi : BH \to BG$ of pointed CW-complexes. With these trivial coefficients, we get a map

$$H^\bullet(B\varphi) = \varphi^* : H^\bullet(BG, M) \to H^\bullet(BH, M)$$

and so, via the comparison of Proposition 2.35,

$$\varphi^* : H^\bullet(G; M) \to H^\bullet(H; M).$$

Here $\mathrm{res}_\varphi M = M$.

3. Cohomology of group theoretic constructions

This chapter develops techniques for computing the cohomology of direct products, amalgamated products, and subgroups. These computations will be useful in the study of dimensions of groups. We use functoriality and the interpretation of group cohomology through classifying spaces.

3.1. **Direct products.** We begin with direct products. By Proposition 2.35, we can start with the corresponding topological result. For products, this is the following theorem. For the Künneth formula and its cohomological variants, see [Hat02, Section 3.B] and [Bro82, Chapter V].

Theorem 3.1 (Künneth)**.** Let X_1, X_2 be CW-complexes, and assume that the homology groups $H_n(X_1;\mathbb{Z})$ are finitely generated for every $n \geq 0$. Then, for every $n \geq 0$, there exists a split exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X_1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^j(X_2;\mathbb{Z}) \to H^n(X_1 \times X_2;\mathbb{Z}) \\ \to \bigoplus_{i+j=n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^i(X_1;\mathbb{Z}), H^j(X_2;\mathbb{Z})\big) \to 0.$$

For example, let us apply the coefficient variant of Theorem 3.1 explained in Remark 3.4, with coefficients in $\mathbb{Q}$ (but any other field would produce the same result). Under the corresponding finiteness hypothesis, we claim that there exists an isomorphism

$$H^n(X_1 \times X_2;\mathbb{Q}) \cong \bigoplus_{i+j=n} H^i(X_1;\mathbb{Q}) \otimes_{\mathbb{Q}} H^j(X_2;\mathbb{Q}).$$

This follows from the vanishing

$$\operatorname{Tor}_1^{\mathbb{Q}}\big(H^i(X_1;\mathbb{Q}), H^j(X_2;\mathbb{Q})\big) = 0.$$

Indeed, every $\mathbb{Q}$-module is free, so these Tor groups vanish. Note that the assumptions are clearly satisfied if X_1 is constructed by attaching a finite number of cells in each dimension. For discrete groups G_1, G_2, we seek a formula for $H^\bullet(G_1 \times G_2;\mathbb{Z})$. The idea is to use the isomorphisms

$$H^\bullet(G_i;\mathbb{Z}) \cong H^\bullet_{\mathrm{tw}}(BG_i;\mathbb{Z}) \cong H^\bullet(BG_i;\mathbb{Z})$$

that relate the group cohomology of G_i with the singular cohomology of the classifying space BG_i. To apply Theorem 3.1, we need an isomorphism

$$H^\bullet(BG_1 \times BG_2;\mathbb{Z}) \cong H^\bullet(B(G_1 \times G_2);\mathbb{Z}),$$

and so we need to prove the following lemma. In products of CW-complexes we use the weak CW topology; see [Hat02, Section 3.B].

Lemma 3.2. Let G_1, G_2 be discrete groups. Then $B(G_1 \times G_2) \simeq BG_1 \times BG_2$.

Proof. We need to show that $BG_1 \times BG_2$ is a model for $B(G_1 \times G_2)$, that is, it has fundamental group $G_1 \times G_2$, and its universal cover is contractible. However,

$$\pi_1(BG_1 \times BG_2) \cong \pi_1(BG_1) \times \pi_1(BG_2) \cong G_1 \times G_2,$$

since π_1 preserves products, and since the universal cover of a product is the product of the universal covers, we have

$$\widetilde{BG_1 \times BG_2} \cong \widetilde{BG_1} \times \widetilde{BG_2} \simeq * \times * \simeq *.$$

□

Hence we can apply Theorem 3.1 and use the isomorphism in the middle term to get the following theorem.

Theorem 3.3. Let G_1, G_2 be discrete groups and suppose that $H_n(G_1;\mathbb{Z})$ is finitely generated for every $n \geq 0$. Give $\mathbb{Z}$ the trivial action. Then, for every $n \geq 0$, there exists a split exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(G_1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^j(G_2;\mathbb{Z}) \to H^n(G_1 \times G_2;\mathbb{Z}) \\ \to \bigoplus_{i+j=n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^i(G_1;\mathbb{Z}), H^j(G_2;\mathbb{Z})\big) \to 0.$$

Remark 3.4. The Künneth formula also holds with coefficients in any principal ideal domain R. Indeed, if $H_n(X_1;R)$ is finitely generated for every $n \geq 0$, then there is a split short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X_1;R) \otimes_R H^j(X_2;R) \to H^n(X_1 \times X_2;R) \\ \to \bigoplus_{i+j=n+1} \operatorname{Tor}_1^{R}\big(H^i(X_1;R), H^j(X_2;R)\big) \to 0.$$

This statement can also be lifted to group cohomology when R has trivial group action, since by Proposition 2.35 we have

$$H^\bullet(G;R) \cong H^\bullet_{\mathrm{tw}}(BG;R) \cong H^\bullet(BG;R).$$

In particular, we can apply the formula with $\mathbb{Q}$-coefficients.

Example 3.5. Let $k \geq 1$ and give $\mathbb{Z}$ the trivial $\mathbb{Z}^k$-action. As an application of Theorem 3.3, we compute the cohomology of $\mathbb{Z}^k$ with integer coefficients. We claim that

$$H^n(\mathbb{Z}^k;\mathbb{Z}) \cong \mathbb{Z}^{\binom{k}{n}},$$

where $\binom{k}{n} = 0$ for $n > k$, and we prove this by induction. The case $k = 1$ is known, so assume the claim for $k-1$ and prove it for k. Since $\mathbb{Z}^k \cong \mathbb{Z}^{k-1} \times \mathbb{Z}$, Theorem 3.3 says that

$$H^n(\mathbb{Z}^{k-1} \times \mathbb{Z};\mathbb{Z}) \cong \bigoplus_{i+j=n} H^i(\mathbb{Z}^{k-1};\mathbb{Z}) \otimes_{\mathbb{Z}} H^j(\mathbb{Z};\mathbb{Z}) \\ \oplus \bigoplus_{i+j=n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^i(\mathbb{Z}^{k-1};\mathbb{Z}), H^j(\mathbb{Z};\mathbb{Z})\big).$$

However, by flatness, the Tor terms vanish, and the induction hypothesis implies that

$$\begin{aligned} H^n(\mathbb{Z}^k;\mathbb{Z}) &\cong \big(H^n(\mathbb{Z}^{k-1};\mathbb{Z}) \otimes \mathbb{Z}\big) \oplus \big(H^{n-1}(\mathbb{Z}^{k-1};\mathbb{Z}) \otimes \mathbb{Z}\big) \\ &\cong \mathbb{Z}^{\binom{k-1}{n}} \oplus \mathbb{Z}^{\binom{k-1}{n-1}} \cong \mathbb{Z}^{\binom{k}{n}}. \end{aligned}$$

Here cohomology in negative degrees is zero.

Example 3.6. Let $k \geq 1$ and give $\mathbb{Q}$ the trivial $(\mathbb{Z}/2)^k$-action. Let us compute $H^\bullet((\mathbb{Z}/2)^k;\mathbb{Q})$. We proceed by induction on k. For $k = 1$, we know

$$H^n(\mathbb{Z}/2;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n = 0, \\ 0 & n > 0. \end{cases}$$

Assume the formula for k and consider $k+1$. By Theorem 3.3 and Remark 3.4

$$H^n((\mathbb{Z}/2)^k \times (\mathbb{Z}/2);\mathbb{Q}) \cong \bigoplus_{i+j=n} H^i((\mathbb{Z}/2)^k;\mathbb{Q}) \otimes_{\mathbb{Q}} H^j(\mathbb{Z}/2;\mathbb{Q}),$$

our claim is immediate.

Example 3.7. Let $\ell, k \geq 2$ and give $\mathbb{Z}$ the trivial action. We now compute $H^\bullet(\mathbb{Z}/\ell \times \mathbb{Z}/k; \mathbb{Z})$. We apply Theorem 3.3: its hypothesis holds since $H_n(\mathbb{Z}/\ell; \mathbb{Z})$ is finitely generated for every n. So we get that

$$H^n(\mathbb{Z}/\ell \times \mathbb{Z}/k; \mathbb{Z}) \cong \bigoplus_{i+j=n} H^i(\mathbb{Z}/\ell; \mathbb{Z}) \otimes H^j(\mathbb{Z}/k; \mathbb{Z}) \\ \oplus \bigoplus_{i+j=n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^i(\mathbb{Z}/\ell; \mathbb{Z}), H^j(\mathbb{Z}/k; \mathbb{Z})\big).$$

Now recall from Corollary 2.15 that

$$H^n(\mathbb{Z}/\ell; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \text{ odd}, \\ \mathbb{Z}/\ell & n \geq 2 \text{ even}. \end{cases}$$

Substitution gives the desired groups.

3.2. **Amalgamated products.** We next consider amalgamated products. As before, we begin with the corresponding topological result.

Theorem 3.8 (Mayer–Vietoris)**.** Consider a pushout of CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ {\scriptstyle j}\downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

where j is the inclusion of a subcomplex and f is cellular. Then there exists a long exact sequence

$$\cdots \to H^n(X; \mathbb{Z}) \to H^n(X_1; \mathbb{Z}) \oplus H^n(X_2; \mathbb{Z}) \to H^n(X_0; \mathbb{Z}) \xrightarrow{\partial} H^{n+1}(X; \mathbb{Z}) \to \cdots$$

in cohomology.

We explain the role of the hypotheses in this theorem. A pushout in Top, or in CW, need not give a Mayer–Vietoris sequence. A homotopy pushout does give such a sequence. Let us explain what we mean. Suppose we are given two maps $X \xleftarrow{f} W \xrightarrow{g} Y$ in Top_*. We define the *double mapping cylinder* of f and g as the quotient

$$M(f, g) = \big(X \vee (W \wedge I_+) \vee Y\big)/\!\sim,$$

where we identify

$$f(w) \sim (w, 0), \qquad g(w) \sim (w, 1).$$

Here $I = [0, 1]$ and I_+ is obtained by adjoining a disjoint basepoint. Note that we have two maps $i_X : X \to M(f, g)$ and $i_Y : Y \to M(f, g)$ and a homotopy commutative square

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow[i_X]{} & M(f, g) \end{array}$$

where the homotopy is $\psi(w, t) = [w, t]$ for all $(w, t) \in W \times I$. This square is called the *standard homotopy pushout* of f and g. With it we can define what a homotopy pushout is. Suppose we are given a homotopy commutative square

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[h]{} & Z \end{array}$$

with a homotopy $F : hf \simeq kg$. From these data we can construct a comparison map

$$\theta_F : M(f,g) \to Z, \qquad \theta_F(x) = h(x), \quad \theta_F([w,t]) = F(w,t), \quad \theta_F(y) = k(y),$$

with which we compare the given square to the standard homotopy pushout. With this we can give the following definition.

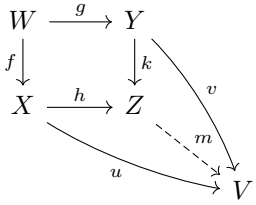
Definition 3.9. A homotopy commutative square of pointed spaces

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[h]{} & Z \end{array}$$

is said to be a *homotopy pushout* if there exists a homotopy $F : hf \simeq kg$ for which $\theta_F : M(f,g) \to Z$ is a homotopy equivalence.

Hence it is not sufficient for Z to be homotopy equivalent to $M(f,g)$: the homotopy equivalence must be given by θ_F for some $F : hf \simeq kg$.

Remark 3.10. The homotopy pushout has the universal property "up to homotopy": given maps $u : X \to V$ and $v : Y \to V$ with $uf \simeq vg$, there exists $m : Z \to V$ making $mh \simeq u$ and $mk \simeq v$.



In the homotopy category this gives a weak pushout: the induced map need not be unique.

Now what we need is the following proposition.

Proposition 3.11. Assume that the following commutative square of pointed CW-complexes is a pushout:

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & Z. \end{array}$$

If f or g is a cofibration, then the square is a homotopy pushout.

We now return to cohomology.

Example 3.12. Let X_1, X_2 be pointed connected CW-complexes, whose basepoints are vertices, and let $X_0 = *$. Our pushout is given by the one-point union $X = X_1 \vee X_2$, and so the cohomology of X can be computed by looking at the exact sequence of Theorem 3.8. In particular, for $n \geq 1$,

$$H^n(X_1 \vee X_2; \mathbb{Z}) \cong H^n(X_1; \mathbb{Z}) \oplus H^n(X_2; \mathbb{Z}),$$

and for $n = 0$ we have the exact sequence

$$0 \to H^0(X; \mathbb{Z}) \to H^0(X_1; \mathbb{Z}) \oplus H^0(X_2; \mathbb{Z}) \to H^0(X_0; \mathbb{Z}) \to 0,$$

making

$$H^0(X_1; \mathbb{Z}) \oplus H^0(X_2; \mathbb{Z}) \cong \mathbb{Z} \oplus H^0(X; \mathbb{Z}),$$

since $\mathbb{Z}$ is projective.

To obtain a group cohomology analogue, we ask when the classifying space construction takes a pushout of groups to a homotopy pushout of spaces. That is, if

$$\begin{array}{ccc} G_0 & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \cong G_1 *_{G_0} G_2 \end{array}$$

is a pushout of groups, then we want

$$\begin{array}{ccc} BG_0 & \longrightarrow & BG_1 \\ \downarrow & & \downarrow \\ BG_2 & \longrightarrow & BG \end{array}$$

to be a homotopy pushout of spaces. This need not hold in general. For example, look at

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \qquad \rightsquigarrow \qquad \begin{array}{ccc} B\mathbb{Z} \simeq S^1 & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & *. \end{array}$$

The second square is not a homotopy pushout, since its homotopy pushout is $\Sigma S^1 \simeq S^2$. However, we can prove the following lemma.

Lemma 3.13. Let

$$\begin{array}{ccc} G_0 & \hookrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{array}$$

be a pushout square of discrete groups where the two maps out of G_0 are monomorphisms. Then there exists a homotopy pushout of CW-complexes

$$\begin{array}{ccc} BG_0 & \longrightarrow & BG_1 \\ \downarrow & & \downarrow \\ BG_2 & \longrightarrow & BG. \end{array}$$

Proof. Let $X_i = BG_i$ and let X be the homotopy pushout of $BG_1 \leftarrow BG_0 \rightarrow BG_2$. We want to show that X is homotopy equivalent to BG, for $G = G_1 *_{G_0} G_2$. We verify that X is a model for BG. The Seifert–van Kampen theorem gives $\pi_1(X) \cong G$, so it remains to show that $\widetilde{X}$ is contractible. To show that $\widetilde{X}$ is contractible, we use Corollary 4.22: a CW-complex Y is contractible if and only if it is simply connected and acyclic. The universal cover $\widetilde{X}$ is simply connected, so it suffices to prove acyclicity. Denote by $p : \widetilde{X} \to X$ the covering map, and use mapping cylinders to regard the X_i as subcomplexes with $X = X_1 \cup_{X_0} X_2$. Then

$$\widetilde{X} = p^{-1}(X_1) \cup_{p^{-1}(X_0)} p^{-1}(X_2).$$

Note that the spaces $p^{-1}(X_i)$ need not be connected. Let $A^i_\alpha \subseteq p^{-1}(X_i)$ be a path component, and choose a basepoint in it. The pullback square

$$\begin{array}{ccc} p^{-1}(X_i) & \longrightarrow & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X_i & \longrightarrow & X \end{array}$$

shows that $A^i_\alpha \to X_i$ is a connected covering. The induced map $\pi_1(A^i_\alpha) \to \pi_1(X_i) = G_i$ is injective, and its composite with the injection $G_i \hookrightarrow G$ is zero, since it factors through $\pi_1(\widetilde{X}) = 0$. Hence $\pi_1(A^i_\alpha) = 0$, making A^i_α the universal cover of X_i. Thus A^i_α is contractible. Mayer–Vietoris now gives $H_n(\widetilde{X}; \mathbb{Z}) = 0$ for $n \geq 2$. Also $H_1(\widetilde{X}; \mathbb{Z}) = 0$ since $\widetilde{X}$ is simply connected. Therefore $\widetilde{X}$ is acyclic, as required. □

Hence Lemma 3.13, combined with Theorem 3.8, allows us to prove the following theorem.

Theorem 3.14 (Mayer–Vietoris)**.** Let $G_0 \hookrightarrow G_1$ and $G_0 \hookrightarrow G_2$ be monomorphisms of discrete groups, and put $G = G_1 *_{G_0} G_2$. With trivial integer coefficients, there exists a long exact sequence

$$\cdots \to H^n(G; \mathbb{Z}) \to H^n(G_1; \mathbb{Z}) \oplus H^n(G_2; \mathbb{Z}) \to H^n(G_0; \mathbb{Z}) \xrightarrow{\partial} H^{n+1}(G; \mathbb{Z}) \to \cdots.$$

Let us point out that the argument just given uses trivial coefficients, through the comparison $H^\bullet(G; \mathbb{Z}) \cong H^\bullet(BG; \mathbb{Z})$. The more general version uses local coefficients. Anyway, injectivity is necessary in Theorem 3.14. Indeed, the square

$$\begin{array}{ccc} 2\mathbb{Z} & \hookrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ 0 & \to & \mathbb{Z}/2\mathbb{Z} \end{array}$$

is a pushout of groups, but $H^\bullet(\mathbb{Z}/2; \mathbb{Z})$ is nontrivial in arbitrarily high degrees. As an application, let us compute the cohomology of the free product. For discrete groups G_1, G_2, the free product is a pushout along inclusions of the trivial group, giving

$$BG_1 \vee BG_2 \simeq B(G_1 * G_2).$$

The Mayer–Vietoris sequence therefore gives

$$0 \to H^n(G_1 * G_2; \mathbb{Z}) \to H^n(G_1; \mathbb{Z}) \oplus H^n(G_2; \mathbb{Z}) \to 0$$

for $n \geq 1$, and

$$0 \to H^0(G_1 * G_2; \mathbb{Z}) \to H^0(G_1; \mathbb{Z}) \oplus H^0(G_2; \mathbb{Z}) \to \mathbb{Z} \to 0,$$

making

$$H^n(G_1 * G_2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, \\ H^n(G_1; \mathbb{Z}) \oplus H^n(G_2; \mathbb{Z}) & n \geq 1. \end{cases}$$

Example 3.15. We can compute the cohomology of the infinite dihedral group

$$D_\infty = \langle t, s \mid s^2 = 1,\ sts = t^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the action is by inversion and the coefficient action on $\mathbb{Z}$ is trivial. This group is the symmetry group of the degenerate "n-gon", for $n \to +\infty$. How do we compute its cohomology? We note that

$$D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle, \qquad t \mapsto s_1 s_2, \quad s \mapsto s_1,$$

and so the pushout

$$\begin{array}{ccc} 0 & \to & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/2\mathbb{Z} & \to & D_\infty \end{array}$$

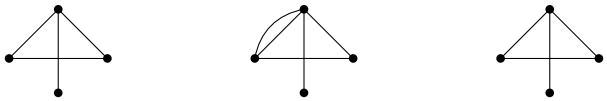
gives us the long exact sequence of Theorem 3.14. Therefore

$$\begin{aligned}\widetilde{H}^n(D_\infty;\mathbb{Z}) &\cong \widetilde{H}^n(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) \oplus \widetilde{H}^n(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) \\ &\cong \begin{cases} 0 & n=0 \text{ or } n \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z})^2 & n \geq 2 \text{ even}. \end{cases}\end{aligned}$$

Moreover, with trivial rational coefficients,

$$\widetilde{H}^\bullet(D_\infty;\mathbb{Q}) \cong \widetilde{H}^\bullet(\mathbb{Z}/2\mathbb{Z};\mathbb{Q}) \oplus \widetilde{H}^\bullet(\mathbb{Z}/2\mathbb{Z};\mathbb{Q}) = 0.$$

3.3. **Right-angled Coxeter and Artin groups.** Before turning to subgroups, we apply these methods to right-angled Coxeter and Artin groups. These groups lead to proposed counterexamples to Conjecture 1.13. Their definitions use finite simplicial graphs. A *finite simplicial graph*, or *simple graph*, is a finite graph $\Gamma = (V, E)$ that has no loops and no multiple edges. For example,



simple multiple edges a loop

To a finite simplicial graph $\Gamma = (V, E)$ we can associate a group.

Definition 3.16. Let $\Gamma = (V, E)$ be a finite simplicial graph. The *right-angled Coxeter group* associated to Γ is the group W_Γ presented as

$$W_\Gamma = \langle s \in V \mid s^2 = 1,\ [s_i, s_j] = 1 \text{ if } \{s_i, s_j\} \in E \rangle.$$

We consider some examples.

Example 3.17. Let Γ be a finite simplicial graph.

(1) If Γ has one vertex, then $W_\Gamma = \langle s \mid s^2 = 1 \rangle \cong \mathbb{Z}/2$ is finite cyclic of order two.
(2) If Γ consists of two isolated vertices, then $W_\Gamma = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong \mathbb{Z}/2 * \mathbb{Z}/2$ is a free product.
(3) If Γ consists of one edge and its two endpoints, then $W_\Gamma \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, as the presentation shows.
(4) For graphs with at least three vertices, further cases arise. First of all, the discrete graph on three vertices and the triangle produce $*_{i=1}^3 \mathbb{Z}/2$ and $\prod_{i=1}^3 \mathbb{Z}/2$, respectively. The graph with three vertices and one edge produces $(\mathbb{Z}/2 \times \mathbb{Z}/2) * \mathbb{Z}/2$, and the path with three vertices produces
$$\begin{aligned} W_\Gamma &= \langle s, t, v \mid s^2 = t^2 = v^2 = 1,\ [s, t] = [s, v] = 1 \rangle \\ &\cong (\mathbb{Z}/2)^2 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2 \cong \mathbb{Z}/2 \times D_\infty. \end{aligned}$$

These examples suggest the following general formulas:

(1) if the graph Γ is discrete, that is, $E = \varnothing$, then $W_\Gamma \cong *_{v \in V} \mathbb{Z}/2$;
(2) if the graph Γ is complete, that is, every two distinct vertices are connected by an edge, then $W_\Gamma \cong \prod_{v \in V} \mathbb{Z}/2$.

We also have the following lemma.

Lemma 3.18. Let Γ be a finite simplicial graph and let $\Lambda \subseteq \Gamma$ be a full subgraph, that is, every edge of Γ between vertices of Λ also belongs to Λ. Then $W_\Lambda \to W_\Gamma$ is a split injection.

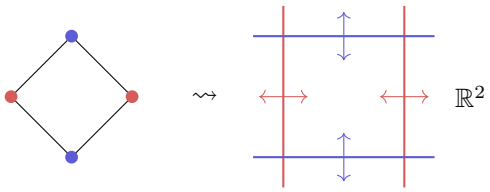
Proof. Indeed, the map $W_\Lambda \to W_\Gamma$ admits a left inverse $W_\Gamma \to W_\Lambda$ defined by fixing the generators in Λ and sending every generator outside Λ to 1. □

The amalgamated-product example generalizes as follows.

Lemma 3.19. Let Γ be a finite simplicial graph and suppose $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where Γ_1, Γ_2 are full subgraphs and $\Gamma_0 = \Gamma_1 \cap \Gamma_2$. Then

$$W_\Gamma \cong W_{\Gamma_1} *_{W_{\Gamma_0}} W_{\Gamma_2}.$$

The term "right-angled" has a geometric interpretation. We can view the generators as reflections across hyperplanes, with commutation relations corresponding to perpendicular hyperplanes. For example, consider the square graph and its associated group $W_\Gamma = D_\infty \times D_\infty$. The vertices of Γ are now reflections that act on $\mathbb{R}^2$:



The reflections corresponding to adjacent vertices commute. We next prove a result about the commutator subgroup.

Lemma 3.20. For a finite simplicial graph Γ, the group $[W_\Gamma, W_\Gamma]$ is torsion-free and of finite index in W_Γ.

Proof. We first show that every finite subgroup $F \subseteq W_\Gamma$ is conjugate into W_Λ for some complete subgraph $\Lambda \subseteq \Gamma$. We use the following fact: a finite subgroup of $G_1 *_{G_0} G_2$, with injective amalgamating maps, is conjugate into G_1 or G_2. We proceed by induction on the number of vertices of Γ. If Γ is complete, then $W_\Gamma \cong \prod_{v \in V} \mathbb{Z}/2$ and the claim is trivial. If Γ is not complete, then $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where the Γ_i are full subgraphs with fewer vertices than Γ. Now by the general fact there exists $g \in W_\Gamma$ such that $gFg^{-1} \subseteq W_{\Gamma_1}$, for example. By induction, there exists $h \in W_{\Gamma_1}$ such that $(hg)F(hg)^{-1} \subseteq W_{\Lambda_1}$, with $\Lambda_1 \subseteq \Gamma_1 \subseteq \Gamma$ complete. Now consider the exact sequence

$$1 \to [W_\Gamma, W_\Gamma] \to W_\Gamma \xrightarrow{\pi} \prod_{v \in V} \mathbb{Z}/2 \to 1$$

given by the abelianization map π. The quotient is finite, so $[W_\Gamma, W_\Gamma]$ has finite index. Let $F \subseteq W_\Gamma$ be a nontrivial finite subgroup. We show that $\pi(F) \neq 1$, so that $F \nsubseteq \ker(\pi) = [W_\Gamma, W_\Gamma]$, which then has no torsion. By the preceding argument, there exists $g \in W_\Gamma$ with $gFg^{-1} \subseteq W_\Lambda$ for some complete $\Lambda \subseteq \Gamma$. This happens if and only if $F \subseteq g^{-1} W_\Lambda g$, but the map

$$\pi|_{g^{-1}W_\Lambda g} : g^{-1} W_\Lambda g \to \prod_{v \in V} \mathbb{Z}/2$$

is injective since Λ is complete. Hence $\pi(F) \neq 1$. □

We can now compute rational cohomology using the methods developed above. Indeed, we can prove the following theorem.

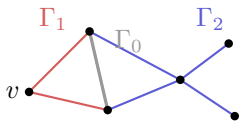
Theorem 3.21. Let Γ be a finite simplicial graph, and give $\mathbb{Q}$ the trivial W_Γ-action. Then

$$H^n(W_\Gamma; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

Proof. We proceed by induction on the number of vertices of Γ. For the empty graph $W_\Gamma = 1$, and for $|V| = 1$ we have $W_\Gamma \cong \mathbb{Z}/2$, so the statement is clearly true. So let us assume the statement for every graph Λ with $|V^\Lambda| < |V^\Gamma|$, and let us prove it for Γ. We distinguish the complete and noncomplete graphs. By Definition 3.16, if Γ is complete, then $W_\Gamma \cong \prod_{v \in V} \mathbb{Z}/2$, and so Theorem 3.3 and Remark 3.4 allow us to conclude. Therefore, we can assume that Γ is not complete. Then we can

find $v \in V$ which is not connected to all other vertices, that is, there exists $w \neq v$ with $\{v, w\} \notin E$. The idea now is to decompose $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ into full subgraphs:

- Γ_1 is spanned by v and all vertices connected to it;
- Γ_0 is spanned by all vertices connected to v, but not v;
- Γ_2 is spanned by $V \setminus \{v\}$.



By Lemma 3.19,

$$W_\Gamma \cong W_{\Gamma_1} *_{W_{\Gamma_0}} W_{\Gamma_2},$$

and $W_{\Gamma_0} \to W_{\Gamma_1}$ and $W_{\Gamma_0} \to W_{\Gamma_2}$ are monomorphisms. So by Mayer–Vietoris we get a long exact sequence

$$\cdots \to H^{n-1}(W_{\Gamma_0}; \mathbb{Q}) \to H^n(W_\Gamma; \mathbb{Q}) \to H^n(W_{\Gamma_1}; \mathbb{Q}) \oplus H^n(W_{\Gamma_2}; \mathbb{Q}) \to \cdots.$$

But now we can use induction, since $\Gamma_0, \Gamma_1, \Gamma_2$ all have fewer vertices than Γ. Indeed, $v \notin \Gamma_0, \Gamma_2$, while $w \notin \Gamma_1$. We conclude that $H^n(W_\Gamma; \mathbb{Q}) = 0$ for $n \geq 2$. For $n = 1$, the map $H^0(W_{\Gamma_1}; \mathbb{Q}) \oplus H^0(W_{\Gamma_2}; \mathbb{Q}) \to H^0(W_{\Gamma_0}; \mathbb{Q})$ is the surjection $(a, b) \mapsto a - b$, so exactness also gives $H^1(W_\Gamma; \mathbb{Q}) = 0$. □

Together with right-angled Coxeter groups, there are also right-angled Artin groups.

Definition 3.22. Let $\Gamma = (V, E)$ be a finite simplicial graph. The *right-angled Artin group* associated to Γ is

$$A_\Gamma = \langle s \in V \mid [s_i, s_j] = 1 \text{ if } \{s_i, s_j\} \in E \rangle.$$

As before, let us have a look at the examples.

Example 3.23. Let Γ be a finite simplicial graph.

(1) If Γ has one vertex, then $A_\Gamma \cong \mathbb{Z}$. Similarly, if $E = \varnothing$, then $A_\Gamma \cong *_{v \in V} \mathbb{Z}$.
(2) If Γ consists of one edge and its endpoints, then $A_\Gamma \cong \mathbb{Z} \times \mathbb{Z}$. Similarly, if Γ is complete, then $A_\Gamma \cong \prod_{v \in V} \mathbb{Z}$.

These examples parallel the Coxeter-group examples, with $\mathbb{Z}$ replacing $\mathbb{Z}/2$. In this picture we have replaced reflections, given by $\mathbb{Z}/2$, by translations, given by $\mathbb{Z}$. As before, we can show that if $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ is a union of full subgraphs with intersection Γ_0, then

$$A_\Gamma \cong A_{\Gamma_1} *_{A_{\Gamma_0}} A_{\Gamma_2}.$$

We now compute the cohomology of these groups.

Theorem 3.24. Let Γ be a finite simplicial graph and let $n \geq 0$. Give $\mathbb{Z}$ the trivial A_Γ-action. Then

$$H^n(A_\Gamma; \mathbb{Z}) \cong \bigoplus_{\substack{\Lambda \subseteq \Gamma \text{ complete} \\ |V^\Lambda| = n}} \mathbb{Z},$$

where the empty subgraph is included when $n = 0$.

Proof. We again proceed by induction on the number of vertices of Γ. For the empty graph the statement is immediate. If $|V| = 1$, then

$$H^n(A_\Gamma; \mathbb{Z}) \cong H^n(\mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 1, \\ 0 & n \geq 2, \end{cases}$$

as computed in Section 2.2. So let us assume the claim for every graph Λ with $|V^\Lambda| < |V^\Gamma|$, and let us prove it for Γ. We distinguish the complete and noncomplete graphs. If Γ is complete, then

$A_\Gamma \cong \prod_{v\in V} \mathbb{Z}$, and we can use Theorem 3.3 once more to conclude. So let us assume that Γ is not complete, so that we can find a vertex $v \in V$ which is not connected to all other vertices. In particular, we can decompose $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ using the full subgraphs

- Γ_1 is spanned by v and all vertices connected to it;
- Γ_0 is spanned by all vertices connected to v, but not v;
- Γ_2 is spanned by $V \setminus \{v\}$.

The induction hypothesis applies. Now, as before,

$$A_\Gamma \cong A_{\Gamma_1} *_{A_{\Gamma_0}} A_{\Gamma_2},$$

where all the maps out of A_{Γ_0} are monomorphisms. By Theorem 3.14 we can use the long exact sequence

$$\cdots \to H^{n-1}(A_{\Gamma_0};\mathbb{Z}) \to H^n(A_\Gamma;\mathbb{Z}) \to H^n(A_{\Gamma_1};\mathbb{Z}) \oplus H^n(A_{\Gamma_2};\mathbb{Z}) \to H^n(A_{\Gamma_0};\mathbb{Z}) \to \cdots.$$

Now if we define

$$r_n(\Gamma) = \#\{\text{complete subgraphs of } \Gamma \text{ with } n \text{ vertices}\},$$

we can count these subgraphs. Since $\Gamma_0 \subseteq \Gamma_1$ is full, $A_{\Gamma_0} \to A_{\Gamma_1}$ is a split injection, by the same retraction as in Lemma 3.18. By applying the cohomology functor, we see that $H^n(A_{\Gamma_1};\mathbb{Z}) \to H^n(A_{\Gamma_0};\mathbb{Z})$ is an epimorphism in every degree, giving us the short exact sequence

$$0 \to H^n(A_\Gamma;\mathbb{Z}) \to H^n(A_{\Gamma_1};\mathbb{Z}) \oplus H^n(A_{\Gamma_2};\mathbb{Z}) \to H^n(A_{\Gamma_0};\mathbb{Z}) \to 0.$$

In particular, we see that $H^n(A_\Gamma;\mathbb{Z})$ is free, say $\mathbb{Z}^x$. The sequence allows us to conclude that

$$x = r_n(\Gamma_1) + r_n(\Gamma_2) - r_n(\Gamma_0) = r_n(\Gamma).$$

□

3.4. **Subgroups and Shapiro's lemma.** We now study the relation between the cohomology of a group and that of its subgroups. Let $H \leq G$ be a subgroup of a discrete group, and let $i : H \hookrightarrow G$ be the inclusion homomorphism. We first examine the induced functors between their module categories. We begin by studying the restriction functor

$$\mathrm{res}^G_H = \mathrm{res}_i : {}_G\mathrm{Mod} \to {}_H\mathrm{Mod}$$

that sends $(M, a : G \times M \to M)$ to $(M, a \circ (i \times 1_M) : H \times M \to M)$, simply restricting the action of G to H. We begin with some examples.

Example 3.25. Let $H \leq G$ be a subgroup of a discrete group, and let M, N be left $\mathbb{Z}G$-modules.

(1) We claim that $\mathrm{res}^G_H(\mathbb{Z}) = \mathbb{Z}$ when $\mathbb{Z}$ has trivial action. Restricting a trivial action gives a trivial action.
(2) We claim that
$$\mathrm{res}^G_H(\mathbb{Z}G) \cong \bigoplus_{H\backslash G} \mathbb{Z}H.$$
Indeed, by restricting the action of G on $\mathbb{Z}G$ we get a free H-module with a basis given by representatives of $H\backslash G$.
(3) We claim that $\mathrm{res}^G_H(M \oplus N) \cong \mathrm{res}^G_H(M) \oplus \mathrm{res}^G_H(N)$.

These computations give the following homological properties.

Lemma 3.26. Let $H \leq G$ be a subgroup of a discrete group. The functor $\mathrm{res}^G_H : {}_G\mathrm{Mod} \to {}_H\mathrm{Mod}$ is exact and preserves projectives.

Proof. The functor res^G_H is exact, since exactness can be checked on the underlying abelian groups. It preserves projectives, since projective modules are direct summands of free modules and restriction preserves both freeness and direct summands. □

Now, together with the restriction functor, we should consider the *induction functor*

$$\begin{aligned} \mathrm{ind}_H^G : {}_H\mathrm{Mod} &\to {}_G\mathrm{Mod} \\ N &\mapsto \mathrm{ind}_H^G(N) = \mathbb{Z}G \otimes_{\mathbb{Z}H} N. \end{aligned}$$

To be more precise, the action of G is given by

$$g \cdot (g' \otimes n) = gg' \otimes n,$$

that is, on the left, whereas the tensor product uses the right $\mathbb{Z}H$-action on $\mathbb{Z}G$. Induction is related to restriction by the following adjunction.

Lemma 3.27. Let $H \leq G$ be a subgroup of a discrete group. There exists an adjunction

$$\mathrm{ind}_H^G : {}_H\mathrm{Mod} \rightleftarrows {}_G\mathrm{Mod} : \mathrm{res}_H^G .$$

Proof. We construct the bijection

$$\mathrm{Hom}_G(\mathrm{ind}_H^G(N), M) \cong \mathrm{Hom}_H(N, \mathrm{res}_H^G(M))$$

for every left $\mathbb{Z}H$-module N and left $\mathbb{Z}G$-module M. The two directions are

$$f \mapsto F_f, \quad F_f(g \otimes n) = g \cdot f(n), \qquad T \mapsto f_T, \quad f_T(n) = T(e_G \otimes n).$$

If we start with $f : N \to \mathrm{res}_H^G(M)$ and then construct F_f and f_{F_f}, we can compute

$$f_{F_f}(n) = F_f(e_G \otimes n) = e_G \cdot f(n) = f(n)$$

for every $n \in N$, proving that one composite is the identity. For the other, start with T and construct F_{f_T}. Then for every $n \in N$ and $g \in G$ we have

$$F_{f_T}(g \otimes n) = g \cdot f_T(n) = g \cdot T(e_G \otimes n) = T(g \otimes n).$$

□

We consider two examples.

Example 3.28. Let $H \leq G$ be a subgroup of a discrete group.

(1) By a straightforward computation, with trivial H-action on $\mathbb{Z}$,

$$\mathrm{ind}_H^G(\mathbb{Z}) = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} \cong \mathbb{Z}[G/H],$$

since $\mathbb{Z}G \cong \bigoplus_{G/H} \mathbb{Z}H$ as a right $\mathbb{Z}H$-module.

(2) We have

$$\mathrm{ind}_H^G(\mathbb{Z}H) = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong \mathbb{Z}G.$$

Thus induction is left adjoint to restriction. Restriction also has a right adjoint. Indeed, we can construct the *coinduction functor*

$$\begin{aligned} \mathrm{coind}_H^G : {}_H\mathrm{Mod} &\to {}_G\mathrm{Mod} \\ N &\mapsto \mathrm{coind}_H^G(N) = \mathrm{Hom}_H(\mathbb{Z}G, N) \end{aligned}$$

by using the fact that $\mathbb{Z}G$ is also a left $\mathbb{Z}H$-module. As always, the action is on the left and is given by

$$(g \cdot f)(g') = f(g'g)$$

for every $g \in G$, $g' \in \mathbb{Z}G$ and $f \in \mathrm{coind}_H^G(N)$. The adjunction is as follows.

Lemma 3.29. Let $H \leq G$ be a subgroup of a discrete group. There is an adjunction

$$\mathrm{res}_H^G : {}_G\mathrm{Mod} \rightleftarrows {}_H\mathrm{Mod} : \mathrm{coind}_H^G .$$

Proof. We construct the bijection

$$\mathrm{Hom}_H(\mathrm{res}^G_H(M), N) \cong \mathrm{Hom}_G(M, \mathrm{coind}^G_H(N))$$

for every left $\mathbb{Z}G$-module M and left $\mathbb{Z}H$-module N. The two directions are

$$f \mapsto F_f, \quad F_f(m) = f(m)(e_G), \qquad T \mapsto f_T, \quad f_T(m)(g) = T(gm).$$

Indeed, if we start with f and construct f_{F_f}, we can compute for every $m \in M$ and $g \in G$:

$$f_{F_f}(m)(g) = F_f(gm) = f(gm)(e_G) = (g \cdot f(m))(e_G) = f(m)(g),$$

as claimed. For the converse, start with T and construct F_{f_T}. Then for every $m \in M$ we have

$$F_{f_T}(m) = f_T(m)(e_G) = T(e_G m) = T(m).$$

□

We have therefore constructed adjunctions

$$\mathrm{ind}^G_H \dashv \mathrm{res}^G_H \dashv \mathrm{coind}^G_H,$$

and for subgroups of finite index the two adjoints agree.

Lemma 3.30. Let $H \leq G$ be a subgroup of finite index in a discrete group, and let N be a left $\mathbb{Z}H$-module. Then

$$\mathrm{coind}^G_H(N) \cong \mathrm{ind}^G_H(N).$$

Proof. Exercise; see [Bro82, Chapter III]. □

We can now prove the main result of this section. See [Bro82, Chapter III] for Shapiro's lemma.

Lemma 3.31 (Shapiro)**.** Let $H \leq G$ be a subgroup of a discrete group, and let N be a left $\mathbb{Z}H$-module. Then, for every $n \geq 0$,

$$H^n(H; N) \cong H^n(G; \mathrm{coind}^G_H(N)).$$

Proof. We compute cohomology with coefficients in the coinduced module. Choose a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$ and compute

$$\begin{aligned} H^n(G; \mathrm{coind}^G_H(N)) &= H^n\big(\mathrm{Hom}_G(P_\bullet, \mathrm{coind}^G_H(N))\big) \\ &\cong H^n\big(\mathrm{Hom}_H(\mathrm{res}^G_H(P_\bullet), N)\big). \end{aligned}$$

By Lemma 3.26, the restricted complex $\mathrm{res}^G_H(P_\bullet) \to \mathbb{Z} \to 0$ is a projective resolution over $\mathbb{Z}H$. So we can compute

$$H^n(G; \mathrm{coind}^G_H(N)) \cong H^n(H; N).$$

□

Lemma 3.31 will be useful when we study dimensions of groups. We first record some consequences for group cohomology.

Corollary 3.32. Let G be a discrete group.

(1) If $H^n(G; M) = 0$ for every left $\mathbb{Z}G$-module M and every $n > 0$, then $G = 1$.
(2) If G contains a nontrivial element of finite order, then for every $k \geq 0$ there exist $n \geq k$ and a left $\mathbb{Z}G$-module M such that $H^n(G; M) \neq 0$.
(3) If $H \leq G$ is of finite index, then, for $n \geq 0$,

$$H^n(H; \mathbb{Z}) \cong H^n(G; \mathbb{Z}[G/H]), \qquad H^n(H; \mathbb{Z}H) \cong H^n(G; \mathbb{Z}G),$$

where $\mathbb{Z}$ has trivial action and the group rings have their left multiplication actions. In particular, if G is finite, then $H^n(G; \mathbb{Z}G) = 0$ for $n > 0$.

Proof. (1) If $G \neq 1$, we can consider the subgroup $\langle g \rangle \leq G$ generated by a nontrivial element $g \in G \setminus \{e_G\}$. By Lemma 3.31, for every module N over $\langle g \rangle$ we have

$$H^n(\langle g \rangle; N) \cong H^n(G; \operatorname{coind}_{\langle g \rangle}^G(N)) = 0$$

for $n > 0$. Our calculations for cyclic groups imply $\langle g \rangle = \{e_G\}$, and so $G = \{e_G\}$.

(2) Let $H \leq G$ be cyclic of finite order $\ell \geq 2$. Then

$$H^n(G; \operatorname{coind}_H^G(\mathbb{Z})) \cong H^n(H; \mathbb{Z}) \neq 0$$

for every positive even n. So we can choose $n \geq k$ as large as we want, taking $M = \operatorname{coind}_H^G(\mathbb{Z})$.

(3) Suppose now $H \leq G$ is of finite index, so that $\operatorname{coind}_H^G(N) \cong \operatorname{ind}_H^G(N)$ for every left $\mathbb{Z}H$-module N. In this case, Lemma 3.31 tells us

$$H^n(H; N) \cong H^n(G; \operatorname{ind}_H^G(N)).$$

If we choose $N = \mathbb{Z}$ and $N = \mathbb{Z}H$, we get the first part of the claim. To conclude, if G is finite and $n > 0$, we have

$$H^n(G; \mathbb{Z}G) \cong H^n(1; \mathbb{Z}) = 0.$$

□

We now show a dimension-shifting result.

Corollary 3.33. Let G be a discrete group and let M be a left $\mathbb{Z}G$-module. Then there exists a left $\mathbb{Z}G$-module M' such that

$$H^n(G; M) \cong H^{n-1}(G; M')$$

for every $n \geq 2$.

Proof. Let us note that short exact sequences of G-modules

$$0 \to M \to M'' \to M' \to 0$$

produce long exact sequences in cohomology

$$\cdots \to H^{n-1}(G; M') \to H^n(G; M) \to H^n(G; M'') \to \cdots .$$

Hence to show the claim it is sufficient to construct an injective map $M \hookrightarrow M''$ such that $H^n(G; M'') = 0$ for $n \geq 1$. We will then define $M' = \operatorname{coker}(M \to M'')$. To arrange this vanishing, we look for $M'' = \operatorname{coind}_H^G(N)$ for some $H \leq G$ and left $\mathbb{Z}H$-module N. If $H = 1$, then $H^n(1; N) = 0$ for every $n \geq 1$, and so $H^n(G; M'') = 0$. Take

$$\begin{aligned} M &\hookrightarrow \operatorname{coind}_1^G(\operatorname{res}_1^G M) = M'' \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \operatorname{res}_1^G M) \\ m &\mapsto (g \mapsto g \cdot m) \end{aligned}$$

which gives the required embedding. □

These algebraic constructions also have a topological interpretation.

Let $H \leq G$ be a subgroup. Topologically, we have

$$\begin{array}{ccc} EG & \longrightarrow & EG/H \simeq BH \\ \downarrow & \swarrow & \\ BG & & \end{array}$$

where $EG = \widetilde{BG}$.

3.5. **Finite index subgroups.** We continue our study of subgroups, now assuming that the index is finite. Our goal is to compute the rational cohomology of finite groups. So let us consider an inclusion $i : H \hookrightarrow G$ of discrete groups and a left $\mathbb{Z}G$-module M. As we have seen, by functoriality of group cohomology we have a map

$$i^* : H^n(G; M) \to H^n(H; \operatorname{res}^G_H M).$$

We now want to define a map that goes in the other direction. Suppose $[G : H] < \infty$. We define the transfer map

$$\operatorname{tr}_* : H^n(H; \operatorname{res}^G_H M) \to H^n(G; M)$$

by using the following construction. Suppose we have a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ by G-modules, and let us define

$$\operatorname{tr}_* : \operatorname{Hom}_H(\operatorname{res}^G_H P_\bullet, \operatorname{res}^G_H M) \to \operatorname{Hom}_G(P_\bullet, M)$$

by the sum

$$\operatorname{tr}_*(f)(p) = \sum_{gH \in G/H} g \cdot f(g^{-1}p)$$

degreewise, choosing one representative of each coset.

Lemma 3.34. Let $H \leq G$ have finite index, let M be a left $\mathbb{Z}G$-module, and let $P_\bullet \to \mathbb{Z}$ be a projective resolution over $\mathbb{Z}G$. The formula defines a map of cochain complexes tr_*.

Proof. We first check independence of the coset representatives. If $g' = gh$ for $h \in H$, then

$$gh \cdot f(h^{-1}g^{-1}p) = g \cdot f(g^{-1}p),$$

since f is H-equivariant. Left multiplication permutes the cosets, so reindexing the sum proves G-equivariance, and the G-equivariance of the differentials of $P_\bullet$ shows that the sum commutes with the cochain differential. □

Therefore we have the following definition.

Definition 3.35. Let $H \leq G$ be a subgroup of finite index in a discrete group, and let M be a left $\mathbb{Z}G$-module. The induced map

$$\operatorname{tr}_* : H^n(H; \operatorname{res}^G_H M) \to H^n(G; M)$$

is called the *transfer map*.

The transfer map satisfies the following identity.

Lemma 3.36. Let $H \leq G$ be of finite index in a discrete group, let $i : H \hookrightarrow G$ be the inclusion, and let M be a left $\mathbb{Z}G$-module. For every $n \geq 0$, the composite $\operatorname{tr}_* \circ i^*$ on $H^n(G; M)$ is multiplication by $[G : H]$.

Proof. We check this at the level of cochain complexes, that is,

$$\operatorname{Hom}_G(P_\bullet, M) \xrightarrow{i^*} \operatorname{Hom}_H(\operatorname{res}^G_H P_\bullet, \operatorname{res}^G_H M) \xrightarrow{\operatorname{tr}_*} \operatorname{Hom}_G(P_\bullet, M).$$

For $f \in \operatorname{Hom}_G(P_n, M)$ and $p \in P_n$ we have

$$(\operatorname{tr}_* \circ i^*)(f)(p) = \sum_{gH \in G/H} g \cdot (i^* f)(g^{-1}p) = \sum_{gH \in G/H} gg^{-1} \cdot f(p) = [G : H] f(p).$$

□

We can now compute the rational cohomology of finite groups.

Corollary 3.37. Let G be a finite group, and give $\mathbb{Q}$ the trivial G-action. Then $H^n(G; \mathbb{Q}) = 0$ for all $n \geq 1$.

Proof. We choose $H = \{e_G\} \leq G$. Then $[G : H] = |G|$ is finite, and so we can apply Lemma 3.36 to get that $\mathrm{tr}_* \circ i^*$ is multiplication by $|G|$:

$$H^n(G;\mathbb{Q}) \xrightarrow{i^*} H^n(H;\mathbb{Q}) \xrightarrow{\mathrm{tr}_*} H^n(G;\mathbb{Q}).$$

These are $\mathbb{Q}$-vector spaces, so multiplication by $|G|$ is injective. Since $H^n(H;\mathbb{Q}) = 0$ for $n \geq 1$, the claim follows. □

Over the integers this is not true, since multiplication by $|G|$ need not be injective on cohomology. If $|G|$ is invertible in the coefficient ring, then we can apply the same argument.

Remark 3.38. Let $H \leq G$ be a subgroup of finite index in a discrete group. In the covering model above, $BH \to BG$ is a finite covering, and the transfer also has a topological construction.

4. Dimensions of groups

We return to the two notions of dimension introduced at the start of the course. We first study geometric dimension and then cohomological dimension. We compare them using Theorem 4.23.

4.1. **Geometric dimension.** The first notion of dimension of a group uses the notion of dimension of a CW-complex.

Definition 4.1. Let G be a discrete group. The *geometric dimension* of G is

$$\mathrm{gd}(G) = \inf\{n \geq 0 \mid \text{there exists an } n\text{-dimensional CW-model for } BG\}.$$

Clearly, $\mathrm{gd}(G) \in \mathbb{N} \cup \{0, \infty\}$, and it is infinite when every model for BG is infinite-dimensional. We begin with some examples.

Example 4.2. Let us compute $\mathrm{gd}(\{1\})$. We know that $B\{1\} \simeq *$, which is zero-dimensional, proving that $\mathrm{gd}(\{1\}) = 0$. The converse is also true, since if $BG \simeq *$, then G must be trivial.

Example 4.3. Let $n \geq 1$ and let $F_n = *_{i=1}^n \mathbb{Z}$ be the free group on n generators. Let us compute its geometric dimension. By Lemma 3.13, we have $BF_n \simeq \bigvee_{i=1}^n S^1$, thanks to $B\mathbb{Z} \simeq S^1$. Thus $\mathrm{gd}(F_n) \leq 1$; since $F_n \neq 1$, equality holds. Actually, this is an if-and-only-if condition: every connected CW-complex of dimension one is homotopy equivalent to a wedge of circles, and hence is a classifying space for some free group.

Example 4.4. Let $k \geq 1$. Let us compute $\mathrm{gd}(\mathbb{Z}^k)$. By Lemma 3.2, we immediately get $B\mathbb{Z}^k \simeq (S^1)^k$, so that $\mathrm{gd}(\mathbb{Z}^k) \leq k$. But since $H^k((S^1)^k;\mathbb{Z}) \cong \mathbb{Z}$, every CW-complex homotopy equivalent to $B\mathbb{Z}^k$ must have a cell of dimension k, proving $\mathrm{gd}(\mathbb{Z}^k) = k$. The same argument gives $\mathrm{gd}(\pi_1(\Sigma_g)) = 2$ for a closed orientable surface Σ_g of genus $g \geq 1$.

As in Example 4.4, the comparison

$$H^\bullet(BG;\mathbb{Z}) \cong H^\bullet(G;\mathbb{Z})$$

allows us to bound, and even compute, the geometric dimension of G. If we know that $\mathrm{gd}(G) \leq k$, nonzero cohomology in degree k establishes equality. This comparison also determines the geometric dimension of $\mathbb{Z}/k$ for $k \geq 2$. By Corollary 2.15, $H^n(\mathbb{Z}/k;\mathbb{Z})$ does not vanish in arbitrarily high positive even degrees: no finite-dimensional CW-complex has this property, implying $\mathrm{gd}(\mathbb{Z}/k) = \infty$. For right-angled Artin groups, the computation requires a more detailed construction.

Lemma 4.5. Let Γ be a finite simplicial graph. Let $\omega(\Gamma)$ be its *clique number*, the maximum number of vertices in a complete subgraph, with $\omega(\varnothing) = 0$. Then

$$\mathrm{gd}(A_\Gamma) = \omega(\Gamma).$$

Proof. Let $A_\Gamma = \langle s \in V \mid [s_i, s_j] = 1 \text{ if } \{s_i, s_j\} \in E\rangle$ be the right-angled Artin group associated to Γ. We compute $\mathrm{gd}(A_\Gamma)$ by constructing a model for BA_Γ, called the Salvetti complex. The *Salvetti complex* associated to Γ is the CW-complex

$$S_\Gamma = \bigcup_{\substack{\Lambda \subseteq \Gamma \\ \Lambda \text{ complete}}} (S^1)^{V^\Lambda} \subseteq (S^1)^V,$$

where each torus is the coordinate subtorus with all coordinates outside V^Λ equal to the basepoint. For each complete subgraph $\Lambda \subseteq \Gamma$, take the torus $(S^1)^{V^\Lambda}$ and glue these tori along their common coordinate subtori. Thus each complete subgraph contributes a coordinate torus. We prove $S_\Gamma \simeq BA_\Gamma$ by induction on the number of vertices. For the empty graph both spaces are points. Indeed, if $|V| = 1$, then $A_\Gamma \cong \mathbb{Z}$ and $S_\Gamma \cong S^1 \simeq B\mathbb{Z}$, proving the one-vertex case. So assume the hypothesis for all graphs which have fewer vertices than Γ. If now Γ is complete, then $S_\Gamma \cong \prod_{v\in V} S^1$ and $A_\Gamma \cong \prod_{v\in V} \mathbb{Z}$, so $BA_\Gamma \simeq \prod_{v\in V} S^1$, since B preserves products, proving the claim. So we are left with the case where Γ is not complete. Choose a vertex $v \in V$ that is not adjacent to every other vertex. In particular, we can decompose $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where

- Γ_1 is the full subgraph spanned by v and all vertices connected to it;
- Γ_0 is the full subgraph spanned by all vertices connected to v, but not v;
- Γ_2 is the full subgraph spanned by $V \setminus \{v\}$.

In particular, they are full subgraphs with fewer vertices than Γ, so that the induction hypothesis applies. We have

$$A_\Gamma \cong A_{\Gamma_1} *_{A_{\Gamma_0}} A_{\Gamma_2},$$

and by construction of the Salvetti complex,

$$S_\Gamma \cong S_{\Gamma_1} \cup_{S_{\Gamma_0}} S_{\Gamma_2}.$$

To be more precise, the map realizing this identification exists since any complete subgraph $\Lambda \subseteq \Gamma$ lies in Γ_1 or Γ_2. Now, since $S_{\Gamma_i} \simeq BA_{\Gamma_i}$ by induction, the previous pushout

$$\begin{array}{ccc} S_{\Gamma_0} & \hookrightarrow & S_{\Gamma_2} \\ \downarrow & & \downarrow \\ S_{\Gamma_1} & \longrightarrow & S_\Gamma \end{array}$$

is cellular, since $S_{\Gamma_0} \to S_{\Gamma_2}$ is an inclusion of subcomplexes, and reads, up to homotopy, as

$$\begin{array}{ccc} BA_{\Gamma_0} & \longrightarrow & BA_{\Gamma_2} \\ \downarrow & & \downarrow \\ BA_{\Gamma_1} & \longrightarrow & S_\Gamma. \end{array}$$

By Lemma 3.13, this implies $S_\Gamma \simeq BA_\Gamma$. We can now compute the geometric dimension of A_Γ. The CW-dimension of S_Γ is

$$\dim S_\Gamma = \sup\{|V^\Lambda| \mid \Lambda \subseteq \Gamma \text{ complete}\},$$

the maximal number of vertices contained in a complete subgraph. Indeed, the dimension of S_Γ is given by the largest coordinate torus we can find in it. This number is the clique number $\omega(\Gamma)$. Therefore $\mathrm{gd}(A_\Gamma) \leq \omega(\Gamma)$. However, by Theorem 3.24,

$$H^n(A_\Gamma; \mathbb{Z}) \cong \bigoplus_{\substack{\Lambda \subseteq \Gamma \text{ complete} \\ |V^\Lambda| = n}} \mathbb{Z},$$

so $H^{\omega(\Gamma)}(A_\Gamma; \mathbb{Z}) \neq 0$, showing $\mathrm{gd}(A_\Gamma) = \omega(\Gamma)$. □

The constructions in Section 3 also give general bounds on geometric dimension.

Proposition 4.6. Let G, G_0, G_1, G_2 be discrete groups.

(1) If $H \leq G$ is a subgroup, then $\mathrm{gd}(H) \leq \mathrm{gd}(G)$. In particular, if G has a nontrivial element of finite order, then $\mathrm{gd}(G) = \infty$.
(2) We have $\mathrm{gd}(G_1 \times G_2) \leq \mathrm{gd}(G_1) + \mathrm{gd}(G_2)$.
(3) If $G_0 \hookrightarrow G_1$ and $G_0 \hookrightarrow G_2$ are monomorphisms, then

$$\mathrm{gd}(G_1 *_{G_0} G_2) \leq \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2), \mathrm{gd}(G_0) + 1\},$$

and $\mathrm{gd}(G_1 * G_2) = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2)\}$.

The proof uses the corresponding constructions of classifying spaces.

Proof. (1) If $H \leq G$, then the covering associated to H gives a model $BH \to BG$, and its lifted CW-structure has the same dimension as BG. Hence $\mathrm{gd}(H) \leq \mathrm{gd}(G)$. The torsion assertion follows by taking a nontrivial finite cyclic subgroup.

(2) By Lemma 3.2, a model for $B(G_1 \times G_2)$ is given by $BG_1 \times BG_2$, whose product CW-structure has dimension $\dim BG_1 + \dim BG_2$. Therefore

$$\mathrm{gd}(G_1 \times G_2) \leq \dim BG_1 + \dim BG_2,$$

and taking the infimum on both sides concludes the argument.

(3) By Lemma 3.13, the pushout of groups induces a homotopy pushout of classifying spaces:

$$\begin{array}{ccc} G_0 & \hookrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \to & G_1 *_{G_0} G_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} BG_0 & \to & BG_1 \\ \downarrow & & \downarrow \\ BG_2 & \to & B(G_1 *_{G_0} G_2). \end{array}$$

In particular, since we need to introduce the mapping cylinder, the dimension is bounded by the maximum of $\mathrm{gd}(G_1)$, $\mathrm{gd}(G_2)$ and $\mathrm{gd}(G_0)+1$. For $G_0 = 1$, we get $B(G_1 * G_2) \simeq BG_1 \vee BG_2$. Since the dimension of a wedge sum is the maximum of the dimensions of the summands, this gives the upper bound. The inclusions $G_i \hookrightarrow G_1 * G_2$ and part (1) give the reverse inequality. □

Let us note that there are groups $G = G_1 *_{G_0} G_2$ with

$$\mathrm{gd}(G) = \mathrm{gd}(G_0) + 1 > \mathrm{gd}(G_i).$$

4.2. **Cohomological dimension.** We now introduce the algebraic notion of dimension. We work first over a ring R, which need not be commutative.

Definition 4.7. Let R be a unital ring and let M be an R-module, with a fixed choice of left or right modules throughout. We define the *projective dimension* of M over R by

$$\mathrm{pd}_R(M) = \inf\{n \geq 0 \mid \text{there exists a projective resolution } 0 \to P_n \to \cdots \to P_0 \to M \to 0\}.$$

Thus $\mathrm{pd}_R(M)$ is the least length of a projective resolution of M, when a finite resolution exists. The following lemma characterizes projective dimension in terms of Ext.

Lemma 4.8. Let R be a unital ring, let M be an R-module, and let $n \geq 0$. The following conditions are equivalent:

(1) $\mathrm{pd}_R(M) \leq n$;
(2) $\mathrm{Ext}^i_R(M, N) = 0$ for every R-module N and every $i \geq n+1$;
(3) $\mathrm{Ext}^{n+1}_R(M, N) = 0$ for every R-module N;

(4) if
$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$
is exact with all P_i projective, then K is projective, where for $n = 0$ this means that M itself is projective.

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial, as is (4) ⇒ (1). We only need to prove (3) ⇒ (4). To do that, we complete the given sequence to a projective resolution by choosing a free surjection $P_n \to K$ and continuing to the left. Write $K' = \ker(P_n \to K)$ and choose a projective surjection $P_{n+1} \to K'$. The composite $P_{n+1} \to K' \hookrightarrow P_n$ is the differential. The projection $q : P_{n+1} \to K'$ is a cocycle in $\mathrm{Hom}_R(P_\bullet, K')$ of degree $n+1$. By the assumption $\mathrm{Ext}_R^{n+1}(M, K') = 0$, there exists $r : P_n \to K'$ with $r \circ d_{n+1} = q$. Since q is surjective, the restriction of r to K' is the identity. Thus $P_n \cong K' \oplus K$, so K is projective. The same argument applies for $n = 0$, with $K = M$. □

Projective modules can be difficult to describe explicitly, especially over group rings. It is therefore useful to compare projective resolutions with free resolutions. To compare the two notions, we first need the Eilenberg trick.

Lemma 4.9 (Eilenberg)**.** Let R be a unital ring and let P be a projective R-module. Then there exists a free R-module F such that $P \oplus F \cong F$.

Proof. Since P is projective, there exists an R-module Q such that $P \oplus Q$ is free. Take $F = \bigoplus_{j \geq 0}(P \oplus Q)$. This module is free, and rearranging the countable direct sums of copies of P and Q gives
$$P \oplus F \cong F.$$
□

So we can prove the following comparison.

Corollary 4.10. Let R be a unital ring and let M be an R-module with $\mathrm{pd}_R(M) \geq 1$, allowing the value ∞. Then
$$\mathrm{pd}_R(M) = \inf\{n \geq 0 \mid M \text{ admits a free resolution of length } n\}.$$
If M is projective, the shortest free-resolution length is zero when M is free and one otherwise.

Proof. Call the infimum on the right f. Since every free module is projective, we have $\mathrm{pd}_R(M) \leq f$. To prove the converse when $n = \mathrm{pd}_R(M) \geq 1$ is finite, take a projective resolution
$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$
The idea is to add free summands to make its terms free without changing its homology. For example, replacing P_0 by $P_0 \oplus F_0$ changes the kernel of its surjection onto M from K_0 to $K_0 \oplus F_0$, so the next term must also acquire the summand F_0. By Lemma 4.9, we can choose one free module F such that $P_i \oplus F \cong F$ for all $0 \leq i \leq n$, and $F \oplus F \cong F$. Indeed, choose Q_i with $P_i \oplus Q_i$ free and take countably many copies of the direct sum of these free modules. Add the contractible complex $F \xrightarrow{1} F$ in each pair of degrees $(i, i-1)$, for $1 \leq i \leq n$. The resulting resolution has length n and all its terms are free. If $\mathrm{pd}_R(M) = \infty$, the first inequality already gives $f = \infty$. Finally, if M is projective, choose F with $M \oplus F \cong F$. The split exact sequence
$$0 \to F \to M \oplus F \to M \to 0$$
is a free resolution of length one, and a resolution of length zero exists exactly when M is free. □

We now apply this comparison to group cohomology.

Definition 4.11. For a discrete group G, we define its *cohomological dimension* to be

$$\operatorname{cd}(G) = \operatorname{pd}_{\mathbb{Z}G}(\mathbb{Z}),$$

where $\mathbb{Z}$ is a left $\mathbb{Z}G$-module with trivial action.

As before, the cohomological dimension of G is

$$\begin{aligned}\operatorname{cd}(G) &= \inf\{n \geq 0 \mid 0 \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0 \text{ is a projective resolution}\}\\ &= \inf\{n \geq 0 \mid H^i(G; M) = 0 \text{ for all left } \mathbb{Z}G\text{-modules } M \text{ and } i \geq n+1\}\\ &= \sup\{n \geq 0 \mid H^n(G; M) \neq 0 \text{ for some left } \mathbb{Z}G\text{-module } M\}.\end{aligned}$$

Thus cohomological dimension measures how high nonzero cohomology can occur as the coefficient module varies. In particular, this allows us to show the following theorem.

Theorem 4.12. Let G be a discrete group. Then $\operatorname{cd}(G) \leq \operatorname{gd}(G)$.

Proof. If $\operatorname{gd}(G) = \infty$, there is nothing to prove. So let us assume that $n = \operatorname{gd}(G) < \infty$. This means that there exists an n-dimensional model X for BG. Then the augmented cellular complex $(C_\bullet^{\mathrm{CW}}(\widetilde{X}; \mathbb{Z}), \varepsilon)$ is a free, hence projective, resolution of $\mathbb{Z}$ by G-modules. Since $\dim X = n$, we have $C_k^{\mathrm{CW}}(\widetilde{X}; \mathbb{Z}) = 0$ for all $k > n$, implying $\operatorname{cd}(G) \leq n = \operatorname{gd}(G)$. □

We revisit the preceding examples.

Example 4.13. Let G be a discrete group. It is clear from Corollary 3.32 and Lemma 4.8 that $G = 1$ if and only if $\operatorname{cd}(G) = 0$. Hence

$$G = 1 \iff \operatorname{cd}(G) = 0 \iff \operatorname{gd}(G) = 0.$$

Example 4.14. Let $k \geq 1$ and let F_k be the free group on k generators. Since $F_k \neq 1$, Theorem 4.12 and Example 4.3 give $\operatorname{cd}(F_k) = \operatorname{gd}(F_k) = 1$.

Example 4.15. Let $k \geq 1$. By Theorem 4.12 we know that $\operatorname{cd}(\mathbb{Z}^k) \leq \operatorname{gd}(\mathbb{Z}^k) = k$. However, Example 3.5 gives $H^k(\mathbb{Z}^k; \mathbb{Z}) \cong H^k(B\mathbb{Z}^k; \mathbb{Z}) \neq 0$, implying $\operatorname{cd}(\mathbb{Z}^k) = k$.

Example 4.16. Let Σ_g be a closed orientable surface of genus $g \geq 1$. We have $\operatorname{cd}(\pi_1(\Sigma_g)) \leq \operatorname{gd}(\pi_1(\Sigma_g)) = 2$, and since

$$H^2(\pi_1(\Sigma_g); \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z},$$

we obtain equality.

Example 4.17. Let Γ be a finite simplicial graph. The two dimensions also agree for right-angled Artin groups:

$$\operatorname{cd}(A_\Gamma) = \operatorname{gd}(A_\Gamma) = \omega(\Gamma).$$

Indeed, Theorem 4.12 and Lemma 4.5 give $\operatorname{cd}(A_\Gamma) \leq \operatorname{gd}(A_\Gamma) = \omega(\Gamma)$, while Theorem 3.24 gives

$$H^{\omega(\Gamma)}(A_\Gamma; \mathbb{Z}) \cong H^{\omega(\Gamma)}(S_\Gamma; \mathbb{Z}) \neq 0.$$

Example 4.18. For $k \geq 2$, we also have $\operatorname{cd}(\mathbb{Z}/k) = \infty$, since $H^n(\mathbb{Z}/k; \mathbb{Z})$ does not vanish in arbitrarily large degrees.

The results of Section 3 give analogous bounds on cohomological dimension.

Proposition 4.19. Let G, G_0, G_1, G_2 be discrete groups.

(1) If $H \leq G$ is a subgroup, then $\operatorname{cd}(H) \leq \operatorname{cd}(G)$. In particular, if G has a nontrivial element of finite order, then $\operatorname{cd}(G) = \infty$.
(2) We have $\operatorname{cd}(G_1 \times G_2) \leq \operatorname{cd}(G_1) + \operatorname{cd}(G_2)$.

(3) If $G_0 \hookrightarrow G_1$ and $G_0 \hookrightarrow G_2$ are monomorphisms, then

$$\operatorname{cd}(G_1 *_{G_0} G_2) \le \max\{\operatorname{cd}(G_1), \operatorname{cd}(G_2), \operatorname{cd}(G_0)+1\},$$

and $\operatorname{cd}(G_1 * G_2) = \max\{\operatorname{cd}(G_1), \operatorname{cd}(G_2)\}$.

Proof. (1) The claim follows from Lemma 3.31: given a left $\mathbb{Z}H$-module M, we have

$$H^n(H; M) \cong H^n(G; \operatorname{coind}_H^G(M)).$$

Indeed, if $\operatorname{cd}(H) = n$ is realized by M, then $n \le \operatorname{cd}(G)$ by using the characterization as a supremum. If $\operatorname{cd}(H) = \infty$, then for arbitrarily large n we can choose a module M_n with $H^n(H; M_n) \neq 0$ and apply the same argument, proving $\operatorname{cd}(G) = \infty$. The torsion assertion follows by taking a finite cyclic subgroup.

(2) Let $P_\bullet \to \mathbb{Z} \to 0$ and $Q_\bullet \to \mathbb{Z} \to 0$ be projective resolutions by G_1-modules and G_2-modules, respectively. If either cohomological dimension is infinite, the statement is trivial. So assume $\operatorname{cd}(G_1) = \operatorname{length}(P_\bullet) = n$ and $\operatorname{cd}(G_2) = \operatorname{length}(Q_\bullet) = m$ are finite. Consider now

$$\operatorname{Tot}^{\oplus}(P_\bullet \otimes_{\mathbb{Z}} Q_\bullet) \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z},$$

the total complex by sums of the double chain complex $P_\bullet \otimes_{\mathbb{Z}} Q_\bullet$, whose term in degree a is $\bigoplus_{b+c=a} P_b \otimes_{\mathbb{Z}} Q_c$. This is a projective resolution of $\mathbb{Z}$ by $G_1 \times G_2$-modules. Indeed, P_b and Q_c are direct summands of free modules over $\mathbb{Z}G_1$ and $\mathbb{Z}G_2$, respectively, so $P_b \otimes_{\mathbb{Z}} Q_c$ is a direct summand of a free $\mathbb{Z}[G_1 \times G_2]$-module. After forgetting the group actions, both resolutions consist of free abelian groups and their augmented exact sequences split, since each successive kernel is free abelian. Thus each is chain homotopy equivalent over $\mathbb{Z}$ to $\mathbb{Z}[0]$, and their tensor product has homology $\mathbb{Z}$ in degree zero and zero in positive degrees. Since its last possibly nonzero term is $P_n \otimes_{\mathbb{Z}} Q_m$ in degree $n+m$, we conclude that

$$\operatorname{cd}(G_1 \times G_2) \le n + m = \operatorname{cd}(G_1) + \operatorname{cd}(G_2).$$

(3) Assume all the cohomological dimensions $\operatorname{cd}(G_i)$ are finite; otherwise the asserted upper bound is clear. We can apply the algebraic Mayer–Vietoris sequence with an arbitrary left $\mathbb{Z}G$-module M, where $G = G_1 *_{G_0} G_2$, to get

$$\begin{aligned} \cdots \to H^{i-1}(G_0; \operatorname{res}^G_{G_0} M) &\to H^i(G; M) \\ &\to H^i(G_1; \operatorname{res}^G_{G_1} M) \oplus H^i(G_2; \operatorname{res}^G_{G_2} M) \to \cdots . \end{aligned}$$

Put $k = \max\{\operatorname{cd}(G_1), \operatorname{cd}(G_2), \operatorname{cd}(G_0)+1\}$. For $i > k$, both terms $H^i(G_j; \operatorname{res}^G_{G_j} M)$, for $j = 1, 2$, vanish. Also $H^{i-1}(G_0; \operatorname{res}^G_{G_0} M) = 0$, since $i - 1 > \operatorname{cd}(G_0)$. Therefore $H^i(G; M) = 0$ for every M and every $i > k$, meaning $\operatorname{cd}(G) \le k$. For the free product, the same sequence gives the upper bound in positive dimensions, and part (1) gives the lower bound; if both groups have dimension zero they are both trivial.

□

4.3. **Eilenberg–Ganea theorem.** By Theorem 4.12, cohomological dimension is bounded above by geometric dimension, and in all our examples the two dimensions are equal. We now want to prove the converse inequality by constructing a model for BG of the appropriate dimension. Recall that $\pi_n(X) = [S^n, X]_*$ is the set of based homotopy classes of based maps, with concatenation as the operation. The construction uses the following theorem. See [Hat02, Theorem 4.32], with the degree-one case given by abelianization.

Theorem 4.20 (Hurewicz)**.** Let X be a topological space and let $n \ge 1$. If X is $(n-1)$-connected, the map

$$h_n : \pi_n(X) \to H_n(X; \mathbb{Z}), \qquad [f] \mapsto H_n(f)([S^n]),$$

where $[S^n]$ generates $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$, is the abelianization map for $n = 1$ and an isomorphism for $n \geq 2$. In particular, it induces $\pi_1(X)^{\mathrm{ab}} \cong H_1(X;\mathbb{Z})$ when $n = 1$.

The other theorem we will need is the following. See [Hat02, Theorem 4.5].

Theorem 4.21 (Whitehead)**.** Let $f : X \to Y$ be a map of CW-complexes. If f is a weak homotopy equivalence, then f is a homotopy equivalence.

In particular, a weakly contractible CW-complex is contractible. As a corollary, we get the following.

Corollary 4.22. Let X be a simply connected CW-complex. If X is acyclic, that is, $\widetilde{H}_*(X;\mathbb{Z}) = 0$, then X is contractible.

Proof. Since X is simply connected, $\pi_1(X) = 0$. Inductively, if X is $(n-1)$-connected for some $n \geq 2$, then Theorem 4.20 gives $\pi_n(X) \cong H_n(X;\mathbb{Z}) = 0$. Thus X is a weakly contractible CW-complex, and Theorem 4.21 implies that X is contractible. □

We can now prove the main theorem of the course.

Theorem 4.23 (Eilenberg–Ganea)**.** Let G be a discrete group. Then

$$\mathrm{gd}(G) \leq \max\{\mathrm{cd}(G), 3\}.$$

In particular, if $\mathrm{cd}(G) \geq 3$, then $\mathrm{gd}(G) = \mathrm{cd}(G)$.

Proof. Let $n = \max\{\mathrm{cd}(G), 3\}$. We will construct an n-dimensional model Y for BG by inductively constructing its skeleta. Since we also need to control the universal cover, we write $X^{(k)} = \widetilde{Y^{(k)}}$ for $k \geq 2$. We need to construct a filtration

$$\begin{array}{ccccccc} X^{(2)} & \hookrightarrow & X^{(3)} & \hookrightarrow & \cdots & \hookrightarrow & X^{(n)} \simeq * \\ \downarrow & & & & & & \downarrow \\ Y^{(2)} & \hookrightarrow & Y^{(3)} & \hookrightarrow & \cdots & \hookrightarrow & Y^{(n)} \simeq BG \end{array}$$

when n is finite, where the bottom space has fundamental group G and the top space is contractible. Actually, by Corollary 4.22, it is enough to construct X as a simply connected, acyclic CW-complex. We take $Y^{(0)}$ to be a point and begin by constructing the 1- and 2-skeleta. Choose a presentation $G = \langle S \mid R \rangle$ and set $Y^{(1)} = \bigvee_S S^1$. Then construct

$$\begin{array}{ccc} \coprod_R S^1 & \longrightarrow & \bigvee_S S^1 = Y^{(1)} \\ \downarrow & \text{(po)} & \downarrow \\ \coprod_R D^2 & \longrightarrow & Y^{(2)} \end{array}$$

by attaching discs along the loops representing the relators. Van Kampen's theorem gives $\pi_1(Y^{(2)}) \cong G$. According to our notation, let $X^{(2)} = \widetilde{Y^{(2)}}$ be its universal cover. Since it is simply connected, $H_1(X^{(2)};\mathbb{Z}) = 0$.

We now proceed inductively. The base case $Y^{(2)}, X^{(2)}$ satisfies $\widetilde{H}_i(X^{(2)};\mathbb{Z}) = 0$ for $i \leq 1$. Assume that, for some $k \geq 3$, we have constructed $Y^{(k-1)}$ such that its universal cover satisfies

$$\widetilde{H}_i(X^{(k-1)};\mathbb{Z}) = 0 \qquad \text{for } i \leq k-2.$$

By the inductive form of Theorem 4.20, the map

$$\pi_{k-1}(X^{(k-1)}) \to H_{k-1}(X^{(k-1)};\mathbb{Z})$$

is an isomorphism. Since the fundamental group depends only on the 2-skeleton, we have $\pi_1(Y^{(k-1)}) \cong G$, so G acts on $X^{(k-1)}$ by deck transformations. In particular, it acts on homology, making $H_{k-1}(X^{(k-1)};\mathbb{Z})$ a G-module. Choose generators $\{z_\alpha\}_{\alpha\in A}$ of this G-module, that is, a surjection

$$\bigoplus_{\alpha\in A} \mathbb{Z}G \to H_{k-1}(X^{(k-1)};\mathbb{Z}), \qquad 1_\alpha \mapsto z_\alpha.$$

By Theorem 4.20, we can choose based maps $f_\alpha : S^{k-1} \to X^{(k-1)}$ such that $H_{k-1}(f_\alpha)([S^{k-1}]) = z_\alpha$. We can now proceed with the construction of $Y^{(k)}$ using the pushout

$$\begin{array}{ccc} \coprod_{\alpha\in A} S^{k-1} & \xrightarrow{\coprod f_\alpha} & X^{(k-1)} \longrightarrow Y^{(k-1)} \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in A} D^k & \longrightarrow & Y^{(k)}. \end{array}$$

In particular, the universal cover $X^{(k)}$ of $Y^{(k)}$ is obtained from $X^{(k-1)}$ by the equivariant pushout

$$\begin{array}{ccc} \coprod_{\alpha\in A} G\times S^{k-1} & \xrightarrow{(g,x)\mapsto g f_\alpha(x)} & X^{(k-1)} \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in A} G\times D^k & \longrightarrow & X^{(k)}. \end{array}$$

We have to check that $X^{(k)}$ has the right homology, using the long exact sequence of the pair $(X^{(k)}, X^{(k-1)})$. By construction,

$$H_i(X^{(k)}, X^{(k-1)};\mathbb{Z}) \cong \bigoplus_{\alpha\in A} H_i(G\times D^k, G\times S^{k-1};\mathbb{Z}) \cong \begin{cases} \bigoplus_{\alpha\in A}\mathbb{Z}G, & i=k, \\ 0, & i\neq k. \end{cases}$$

It follows that $H_i(X^{(k-1)};\mathbb{Z}) \to H_i(X^{(k)};\mathbb{Z})$ is an isomorphism for $i \le k-2$. The remaining part of the exact sequence is

$$\begin{aligned} 0 \to H_k(X^{(k)};\mathbb{Z}) \to \bigoplus_{\alpha\in A}\mathbb{Z}G \xrightarrow{\partial} H_{k-1}(X^{(k-1)};\mathbb{Z}) \\ \to H_{k-1}(X^{(k)};\mathbb{Z}) \to 0. \end{aligned}$$

We need to show that $H_{k-1}(X^{(k)};\mathbb{Z}) = 0$, or equivalently that ∂ is surjective. Under the above identification, ∂ is precisely the map sending 1_α to z_α, which is surjective by construction. This completes the induction step. If ∂ were an isomorphism, we would also have $H_k(X^{(k)};\mathbb{Z}) = 0$. By Corollary 4.22, $X^{(k)}$ would then be contractible and $Y^{(k)}$ would be a model for BG.

If $n = \infty$, we carry out this construction for all k and set $Y = \operatorname{colim}_k Y^{(k)} = \bigcup_k Y^{(k)}$. By construction, $Y \simeq BG$. If instead $n < \infty$, consider $X^{(n-1)}$ and its augmented cellular chain complex

$$C_{n-1}(X^{(n-1)}) \to C_{n-2}(X^{(n-1)}) \to \cdots \to C_0(X^{(n-1)}) \to \mathbb{Z} \to 0.$$

This is a partial free, and hence projective, resolution of $\mathbb{Z}$ by G-modules, exact below degree $n-1$. Since $\operatorname{cd}(G) \le n$, Lemma 4.8 shows that

$$H_{n-1}(X^{(n-1)};\mathbb{Z}) = \ker\bigl(C_{n-1}(X^{(n-1)}) \to C_{n-2}(X^{(n-1)})\bigr)$$

is projective. By Lemma 4.9, there is a free G-module F such that $H_{n-1}(X^{(n-1)};\mathbb{Z}) \oplus F \cong F$ is free. Choose a basis B of F over $\mathbb{Z}G$ and replace $Y^{(n-1)}$ by

$$Y^{(n-1)} \vee \bigvee_{b\in B} S^{n-1}.$$

This adds F to the top homology of its universal cover, so we may assume that $H_{n-1}(X^{(n-1)};\mathbb{Z})$ is free. Choose a basis of this free module and repeat the cell-attachment construction for $k = n$, attaching one n-cell downstairs for each basis element. The map ∂ is now an isomorphism, so the resulting universal cover is acyclic and simply connected, hence contractible. Thus $Y^{(n)}$ is a model for BG of dimension at most n. □

We illustrate the construction in the proof with several examples.

Example 4.24. Take $G = \mathbb{Z}^2 = \langle a, b \mid [a,b] = 1\rangle$. The construction begins with the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{[a,b]} & S^1 \vee S^1 \\ \downarrow & \text{(po)} & \downarrow \\ D^2 & \longrightarrow & Y^{(2)} = T^2. \end{array}$$

Its universal cover is $X^{(2)} = \mathbb{R}^2$, which is already contractible. Theorem 4.23 only gives an upper bound of 3 here, but the construction is already enough to conclude that $\mathrm{gd}(\mathbb{Z}^2) = 2$.

Example 4.25. Take $G = \mathbb{Z}^3 = \langle a, b, c \mid [a,b] = [b,c] = [a,c] = 1\rangle$. As before, we construct

$$\begin{array}{ccc} \coprod_1^3 S^1 & \xrightarrow{[a,b],\,[b,c],\,[a,c]} & S^1 \vee S^1 \vee S^1 \\ \downarrow & & \downarrow \\ \coprod_1^3 D^2 & \longrightarrow & Y^{(2)} = (T^3)^{(2)}. \end{array}$$

Its universal cover is the 2-skeleton of the usual cubical structure on $\mathbb{R}^3$, an infinite lattice of cubes whose faces are included but whose interiors are not. This space is simply connected, but $H_2(X^{(2)};\mathbb{Z}) \neq 0$. Attaching a 3-cell to $Y^{(2)}$ along the boundary of a cube fills all the cubes upstairs, giving $X^{(3)} \cong \mathbb{R}^3 \simeq *$.

Example 4.26. Let $G = \mathbb{Z}/2 = \langle a \mid a^2 = 1\rangle$. The method of the proof constructs $S^\infty \to \mathbb{RP}^\infty$.

We summarize the comparison between the two dimensions. For a discrete group G, we always have $\mathrm{cd}(G) \leq \mathrm{gd}(G)$, and equality holds if $\mathrm{cd}(G) \geq 3$. The cases still to discuss are $\mathrm{cd}(G) = 1$ and $\mathrm{cd}(G) = 2$. For the first case, we recall Theorem 1.12.

Theorem 4.27 (Stallings–Swan)**.** Let G be a discrete group with $\mathrm{cd}(G) = 1$. Then G is a free group.

In particular, if $\mathrm{cd}(G) = 1$, then $\mathrm{gd}(G) = 1$. The assertion that the remaining case $\mathrm{cd}(G) = 2$ and $\mathrm{gd}(G) = 3$ never occurs is the Eilenberg–Ganea conjecture, stated in Conjecture 1.13. This problem is closely related to the following conjecture.

Conjecture 4.28 (Whitehead, 1941)**.** Let Y be a connected 2-dimensional CW-complex with contractible universal cover. Then every connected subcomplex $Z \subseteq Y$ has contractible universal cover.

Indeed, we have the following theorem.

Theorem 4.29 (Bestvina–Brady, 1997)**.** At least one of Conjecture 1.13 and Conjecture 4.28 is false.

This completes our discussion of the relation between geometric and cohomological dimension.

4.4. **Cohomological dimension of finite index subgroups.** We next describe a proposed class of counterexamples to Conjecture 1.13. To discuss this class, we need to introduce virtual cohomological dimension. A group is said to have a property *virtually* if some subgroup of finite index has that property.

Definition 4.30. Let G be a discrete group. We say that G is *virtually torsion-free* if it has a torsion-free subgroup of finite index.

For example, torsion-free groups are virtually torsion-free, and the same holds for finite groups, since the trivial subgroup has finite index. Further examples are the right-angled Coxeter groups associated to finite simplicial graphs $\Gamma = (V, E)$. Indeed, for

$$W_\Gamma = \langle s \in V \mid s^2 = 1,\ [s_i, s_j] = 1 \text{ if } (s_i, s_j) \in E \rangle,$$

we have a short exact sequence

$$1 \to [W_\Gamma, W_\Gamma] \to W_\Gamma \to \prod_V \mathbb{Z}/2 \to 1,$$

whose kernel is torsion-free and has finite index by Lemma 3.20. For a suitable Γ, the group $[W_\Gamma, W_\Gamma]$ is a possible counterexample to Conjecture 1.13. Our goal is to show that its cohomological dimension is 2. We now introduce virtual cohomological dimension, which has a combinatorial formula for right-angled Coxeter groups.

Definition 4.31. Let G be a virtually torsion-free discrete group. Its *virtual cohomological dimension* is

$$\operatorname{vcd}(G) = \operatorname{cd}(H),$$

where $H \leq G$ is a torsion-free subgroup of finite index.

We must show that this quantity does not depend on the choice of H. This will follow from Theorem 4.35, for which we need some introductory results.

Lemma 4.32. Let G be a discrete group with $\operatorname{cd}(G) < \infty$. Then

$$\operatorname{cd}(G) = \max\{n \mid H^n(G; F) \neq 0 \text{ for some free left } \mathbb{Z}G\text{-module } F\}.$$

Proof. The inequality $\geq$ is clear, since we are dealing with a smaller class of modules. For the other inequality, let $n = \operatorname{cd}(G) < \infty$ and choose a G-module M with $H^n(G; M) \neq 0$. Take a surjection $F \to M$ with F free and complete it to an exact sequence $0 \to K \to F \to M \to 0$. The long exact sequence gives

$$H^n(G; F) \to H^n(G; M) \to H^{n+1}(G; K) = 0.$$

Hence $H^n(G; F) \neq 0$. □

Remark 4.33. If a discrete group G admits a finite model for BG, then $\operatorname{gd}(G) < \infty$ and consequently $\operatorname{cd}(G) < \infty$. In this case one can show that

$$\operatorname{cd}(G) = \max\{n \mid H^n(G; \mathbb{Z}G) \neq 0\}.$$

The finiteness assumption cannot simply be removed. For $G = \mathbb{Z}/2$, we know that $\operatorname{cd}(G) = \operatorname{gd}(G) = \infty$, whereas Lemma 3.31 gives

$$H^n(G; \mathbb{Z}G) \cong H^n(\{1\}; \mathbb{Z}) = 0 \qquad (n \geq 1),$$

so the maximum on the right is 0.

Recall from Proposition 4.19 that $\operatorname{cd}(H) \leq \operatorname{cd}(G)$ for every subgroup $H \leq G$. We now give a converse under additional hypotheses.

Proposition 4.34. Let G be a discrete group with $\operatorname{cd}(G) < \infty$ and let $H \leq G$ have finite index. Then $\operatorname{cd}(H) = \operatorname{cd}(G)$.

If $\operatorname{cd}(G) = \infty$, the claim need not hold: take $\{1\} \leq \mathbb{Z}/2$.

Proof. We must show that $\mathrm{cd}(H) \geq \mathrm{cd}(G) = n$. By Lemma 4.32, there is a free G-module F with $H^n(G;F) \neq 0$. Write $F = \bigoplus_I \mathbb{Z}G$ and set $F' = \bigoplus_I \mathbb{Z}H$. Then

$$F \cong \mathrm{ind}_H^G(F') \cong \mathrm{coind}_H^G(F'),$$

where the last isomorphism follows from $[G:H] < \infty$ by Lemma 3.30. Applying Lemma 3.31, we get

$$0 \neq H^n(G;F) \cong H^n(G;\mathrm{coind}_H^G(F')) \cong H^n(H;F'),$$

and hence $\mathrm{cd}(H) \geq n$. □

We can now prove Theorem 4.35. See [Dav08, Theorem F.3.4].

Theorem 4.35 (Serre)**.** Let G be a torsion-free discrete group and let $H \leq G$ be a subgroup of finite index. Then $\mathrm{cd}(H) = \mathrm{cd}(G)$.

Proof. If $\mathrm{cd}(H) = \infty$, then $\mathrm{cd}(G) = \infty$ and there is nothing to prove. Assume $\mathrm{cd}(H) < \infty$. By Theorem 4.23, $\mathrm{gd}(H) < \infty$. We will show that $\mathrm{gd}(G) < \infty$, so that $\mathrm{cd}(G) < \infty$ and Proposition 4.34 applies. Start with a finite-dimensional model Y for BH, and let H act on its universal cover $\widetilde{Y}$. Define the coinduced space

$$X = \mathrm{Map}_H(G, \widetilde{Y}) = \{f : G \to \widetilde{Y} \mid f(hg) = hf(g) \text{ for } h \in H,\ g \in G\},$$

with G-action $(g \cdot f)(g') = f(g'g)$. Write $m = [G:H]$ and choose representatives $g_1, \ldots, g_m$ for $H\backslash G$. Evaluation gives a bijection

$$X \to \prod_{i=1}^m \widetilde{Y}, \qquad f \mapsto (f(g_i))_{i=1}^m.$$

Give X the corresponding product CW-structure and weak topology. Changing representatives only changes coordinates by deck transformations, and the G-action permutes the product cells. Clearly $\dim X = m \dim \widetilde{Y} < \infty$ and $X \simeq *$. We need to check that the action of G on X is free, so that X/G is a model for BG. Evaluation at the identity,

$$X \to \widetilde{Y}, \qquad f \mapsto f(e_G),$$

is H-equivariant, since $(h \cdot f)(e_G) = f(h) = hf(e_G)$. Since H acts freely on $\widetilde{Y}$, it acts freely on X. For $x \in X$, let $G_x = \{g \in G \mid gx = x\}$ be its stabilizer. Then $H \cap G_x = \{1\}$ and

$$[G_x : H \cap G_x] \leq [G:H] < \infty.$$

Thus G_x is finite, and since G is torsion-free, $G_x = \{1\}$. The product cells also have trivial setwise stabilizers: their stabilizers intersect the normal core of H trivially, since that core acts on each factor by deck transformations, and hence are finite. Consequently X is a free G-CW-complex and X/G gives the required finite-dimensional model. □

We can now prove that virtual cohomological dimension is independent of this choice. If $H, K \leq G$ are torsion-free subgroups of finite index, then $H \cap K$ has finite index in both and is torsion-free. By Theorem 4.35,

$$\mathrm{cd}(H) = \mathrm{cd}(H \cap K) = \mathrm{cd}(K).$$

Before computing $\mathrm{vcd}(W_\Gamma)$, we consider two examples.

Example 4.36. If G is torsion-free, then $\mathrm{vcd}(G) = \mathrm{cd}(G)$, since we can take $H = G$.

Example 4.37. For a virtually torsion-free group G, we have $\mathrm{vcd}(G) = 0$ if and only if G is finite. If G is finite, take $H = \{1\}$. Conversely, if $\mathrm{vcd}(G) = 0$ and H is torsion-free of finite index, then $\mathrm{cd}(H) = 0$, so $H = \{1\}$ and G is finite.

Let us compute $\mathrm{vcd}(W_\Gamma)$.

Definition 4.38. Let L be a simplicial complex. We say that L is *flag* if every finite set of vertices that are pairwise joined by edges spans a simplex.

For example, Δ^2 is flag, $\partial\Delta^2$ is not flag, and the horn Λ^2_0 is flag.

Remark 4.39. The barycentric subdivision of any simplicial complex is flag. For example, subdividing the boundary of a triangle gives a six-cycle, which is flag.

There is a bijection between finite simplicial graphs and finite flag complexes: a graph Γ is sent to the maximal simplicial complex with 1-skeleton Γ, and a flag complex L is sent to its 1-skeleton $L^{(1)}$. For a finite flag complex L, we write $W_L = W_{L^{(1)}}$ and call it the right-angled Coxeter group associated to L. The following theorem allows us to compute its virtual cohomological dimension. The formula is [Dav08, Corollary 8.5.5], specialized to right-angled Coxeter groups.

Theorem 4.40. Let L be a finite flag complex. Then

$$\operatorname{vcd}(W_L) = \max\{n \geq 0 \mid \widetilde{H}^{n-1}(L \setminus \sigma; \mathbb{Z}) \neq 0 \text{ for some simplex } \sigma \subseteq L\}.$$

Here simplices are closed, the empty simplex is included, and $\widetilde{H}^{-1}(\varnothing; \mathbb{Z}) = \mathbb{Z}$.

We illustrate the formula with two examples.

Example 4.41. If L consists of two isolated vertices, then $W_L = D_\infty$ and $\operatorname{vcd}(D_\infty) = 1$. If L is the cone on a square, then $W_L \cong \mathbb{Z}/2 \times D_\infty \times D_\infty$ and $\operatorname{vcd}(W_L) = 2$.

For a finite flag complex L, put

$$d(L) = \max\{n \geq 0 \mid \widetilde{H}^{n-1}(L \setminus \sigma; \mathbb{Z}) \neq 0 \text{ for some simplex } \sigma \subseteq L\},$$

with the conventions of Theorem 4.40. We want to show the following.

Proposition 4.42. Let L be a finite flag complex.

(1) The complex L is a simplex, possibly empty, if and only if $d(L) = 0$.
(2) If $d(L) \leq 1$, then every connected component of L is contractible.
(3) If L is acyclic, 2-dimensional and not simply connected, then $d(L) = 2$.
(4) The 2-skeleton of a CW-structure on the Poincaré homology sphere with one 3-cell is acyclic.

We recall the star and link notation used in the notes. For a vertex v of a simplicial complex L, the *closed star* $\operatorname{st}_L(v)$ consists of the simplices containing v and all their faces. The *link* $\operatorname{lk}_L(v)$ consists of simplices τ not containing v such that $\tau \cup \{v\}$ is a simplex of L. If v^c denotes the full subcomplex on the other vertices, then

$$L = \operatorname{st}_L(v) \cup_{\operatorname{lk}_L(v)} v^c, \qquad \operatorname{st}_L(v) = v * \operatorname{lk}_L(v).$$

Proof. (1) Suppose that L is a simplex. Then W_L is a finite product of copies of $\mathbb{Z}/2$, so $d(L) = \operatorname{vcd}(W_L) = 0$ by Theorem 4.40. Conversely, if $d(L) = 0$, then $\operatorname{vcd}(W_L) = 0$ and hence W_L is finite. Any two nonadjacent vertices would generate a subgroup $\mathbb{Z}/2 * \mathbb{Z}/2$ by Lemma 3.18, contradicting finiteness. Thus the graph is complete, and since L is flag, it is a simplex.

(2) We proceed by induction on the number of vertices. The assertion is vacuous for the empty complex, and if $d(L) = 0$ the conclusion follows from part (1). If L is disconnected, each component satisfies the same vanishing of positive-degree cohomology after removal of a simplex, so induction applies to its components. Suppose therefore that L is connected and $d(L) = 1$. There is a simplex σ such that $\widetilde{H}^0(L \setminus \sigma; \mathbb{Z}) \neq 0$. The complement $L \setminus \sigma$ deformation retracts onto the full subcomplex obtained by deleting the vertices of σ. Partition the components of that subcomplex into two nonempty collections, and let L_1, L_2 be the full subcomplexes

obtained by adding the vertices of σ to each collection. Then $L = L_1 \cup_\sigma L_2$, and both L_i have fewer vertices than L. For a simplex $\tau \subseteq L_1$, we have

$$L \setminus \tau = (L_1 \setminus \tau) \cup_{\sigma \setminus (\tau \cap \sigma)} (L_2 \setminus (\tau \cap L_2)).$$

The intersection is empty or contractible. Applying Mayer–Vietoris, or applying it to the corresponding full subcomplexes after deleting vertices, gives

$$0 = H^j(L \setminus \tau; \mathbb{Z}) \cong H^j(L_1 \setminus \tau; \mathbb{Z}) \oplus H^j(L_2 \setminus (\tau \cap L_2); \mathbb{Z}) \quad (j \geq 1).$$

Hence $d(L_1) \leq 1$, and similarly $d(L_2) \leq 1$. By induction, every component of each L_i is contractible. Gluing the components meeting the nonempty simplex σ along that simplex gives a contractible component, while the other components are unchanged. This completes the induction.

(3) Since L is acyclic, it is connected. By part (2), its nontrivial fundamental group implies $d(L) \geq 2$. For a simplex σ, let D be the full subcomplex obtained by deleting its vertices, so that $L \setminus \sigma \simeq D$. The cohomology sequence of (L, D) contains

$$0 = H^2(L; \mathbb{Z}) \to H^2(D; \mathbb{Z}) \to H^3(L, D; \mathbb{Z}) = 0,$$

where the last equality follows from the dimension of the cellular pair. Thus $H^2(L \setminus \sigma; \mathbb{Z}) = 0$, and all higher cohomology vanishes as well, proving $d(L) \leq 2$.

(4) Recall that the Poincaré homology sphere M is a closed 3-manifold with

$$H_*(M; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}),$$

but with nontrivial fundamental group of order 120. It can be constructed by identifying opposite faces of a dodecahedron with the appropriate twists. Choose the CW-structure with one 3-cell, and let L be a flag triangulation of its 2-skeleton. For a point p in the open 3-cell, $M \setminus \{p\} \simeq L$. The homology sequence of $(M, M \setminus \{p\})$ gives $H_1(L; \mathbb{Z}) = 0$ and

$$H_3(M; \mathbb{Z}) \cong \mathbb{Z} \to H_3(M, M \setminus \{p\}; \mathbb{Z}) \cong \mathbb{Z} \to H_2(L; \mathbb{Z}) \to H_2(M; \mathbb{Z}) = 0.$$

The first map sends the orientation class to the local orientation class and is an isomorphism, so $H_2(L; \mathbb{Z}) = 0$. The remaining reduced homology groups vanish by connectedness and dimension.

□

Let L be this flag triangulation of the 2-skeleton of M, obtained by removing the interior of its 3-cell. Since $\pi_1(L) \cong \pi_1(M) \neq 1$, Proposition 4.42 and Theorem 4.40 give

$$\operatorname{vcd}(W_L) = d(L) = 2, \qquad \operatorname{cd}([W_L, W_L]) = \operatorname{vcd}(W_L) = 2.$$

The non-embedding claim used in the notes has the following precise form. This is [BLN01, Proposition 5].

Proposition 4.43 (Brady–Leary–Nucinkis)**.** Let L be a finite acyclic 2-dimensional CW-complex with finite, nontrivial fundamental group. Then L cannot be a subcomplex of a contractible 2-dimensional CW-complex.

Proof. We use the Gerstenhaber–Rothaus theorem: a finite system of equations over a finite group, with nonsingular exponent-sum matrix, has a solution in a group containing it [GR62]. Suppose $L \subseteq X$ with X contractible and 2-dimensional. A nullhomotopy in X of a nontrivial element of $\pi_1(L)$ lies in a finite subcomplex $K \supseteq L$. Since X has no 3-cells, $H_2(K; \mathbb{Z}) = 0$ and

$$H_1(K; \mathbb{Z}) \cong H_2(X, K; \mathbb{Z})$$

is free abelian. Attach 2-cells representing a basis to obtain an acyclic complex K'. After collapsing a forest relative to L, the pair (K', L) has equally many 1- and 2-cells, and its relative boundary matrix is invertible over $\mathbb{Z}$. Their attaching words give equations over $\pi_1(L)$ with that exponent-sum

matrix. Gerstenhaber–Rothaus implies that $\pi_1(L) \to \pi_1(K')$ is injective, contradicting the chosen nullhomotopy [BLN01, Proposition 5]. □

To conclude, the notes propose that

$$\mathrm{gd}([W_L, W_L]) = 3.$$

This remains a conjectural conclusion here.

4.5. **Upper bounds for geometric and cohomological dimension via actions.** The goal of this section is to compute upper bounds for the geometric and cohomological dimensions of a group G. We study actions on the universal cover EG of the classifying space BG. Its main features can be studied using the language of G-CW-complexes.

Definition 4.44. Let G be a discrete group. A *G-CW-complex* is a G-space X with a G-invariant filtration

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq \bigcup_{n\geq 0} X^{(n)} = X$$

such that X has the weak topology and $X^{(n)}$ is constructed by a pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & X^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \longrightarrow & X^{(n)}, \end{array}$$

where $H_i \leq G$, all maps are G-equivariant, and G acts trivially on the sphere and disc factors. The complex is *free* if every H_i is trivial.

We begin with examples.

Example 4.45. The action $n \cdot x = x + n$ makes $\mathbb{R}$ a $\mathbb{Z}$-CW-complex. Set $\mathbb{R}^{(0)} = \mathbb{Z} \times D^0 \cong \mathbb{Z}$ and form the pushout

$$\begin{array}{ccc} \mathbb{Z} \times S^0 & \xrightarrow{\varphi} & \mathbb{R}^{(0)} \\ \downarrow & & \downarrow \\ \mathbb{Z} \times D^1 & \longrightarrow & \mathbb{R}^{(1)} = \mathbb{R}, \end{array} \qquad \varphi(n,-1) = n, \quad \varphi(n,1) = n+1.$$

Thus we attach an interval between n and $n+1$ for every integer n, recovering the translation action. This is the lifted CW-structure along $\mathbb{R} \to S^1$.

Example 4.46. For $n \geq 0$, the antipodal action makes S^n a free $\mathbb{Z}/2$-CW-complex. Starting with $S^0 \cong \mathbb{Z}/2$, we inductively construct S^n from its equatorial S^{n-1} by

$$\begin{array}{ccc} \mathbb{Z}/2 \times S^{n-1} & \xrightarrow{(\xi,x)\mapsto(-1)^\varepsilon x} & S^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 \times D^n & \longrightarrow & S^n. \end{array}$$

Here we are lifting the CW-structure along $S^n \to \mathbb{RP}^n$.

These examples illustrate a general fact: the universal cover $\widetilde{Y}$ of a connected CW-complex Y is a free $\pi_1(Y)$-CW-complex. If Y is aspherical, then $\widetilde{Y}$ is contractible after forgetting the action. This motivates the following definition.

Definition 4.47. Let G be a discrete group. A *model for EG* is a free G-CW-complex whose underlying space is contractible.

The map $EG \to *$ is equivariant, but for $G \neq 1$ it is not an equivariant homotopy equivalence, since there is no equivariant map $* \to EG$. If Y is a model for BG, then its universal cover is a model for EG. Conversely, if X is a model for EG, then X/G is a model for BG, since $X \to X/G$ is a universal cover and $\pi_1(X/G) \cong G$. In particular, EG is unique up to G-homotopy equivalence, and

$$\mathrm{gd}(G) = \inf\{n \in \mathbb{N} \mid \text{there is an } n\text{-dimensional model for } EG\}.$$

Thus we can study geometric dimension using models for EG. Write G-CW for the category of G-CW-complexes and equivariant maps. For a subgroup $H \leq G$, there is an adjunction

$$\mathrm{ind}_H^G : H\text{-CW} \rightleftarrows G\text{-CW} : \mathrm{res}_H^G,$$

where res_H^G restricts the action and

$$\mathrm{ind}_H^G(Z) = G \times_H Z = (G \times Z)/H, \qquad h \cdot (g, z) = (gh^{-1}, hz).$$

In particular, $[gh, z] = [g, hz]$.

Example 4.48. Let $2\mathbb{Z}$ act on $\mathbb{R}$ by translations. Then

$$\mathrm{ind}_{2\mathbb{Z}}^{\mathbb{Z}}(\mathbb{R}) = (\mathbb{Z} \times \mathbb{R})/2\mathbb{Z}, \qquad 2n \cdot (m, x) = (m - 2n, x + 2n).$$

Using representatives $0, 1$ for $\mathbb{Z}/2\mathbb{Z}$ identifies the induced space with two copies of $\mathbb{R}$. The generator $1 \in \mathbb{Z}$ sends $[0, x]$ to $[1, x]$ and $[1, x]$ to $[0, x + 2]$.

The following construction replaces a G-CW-complex by a free one. See Lück [Luc00]; the related rebuilding construction is discussed in [Geo08, Section 6.1].

Theorem 4.49 (Lück)**.** Let G be a discrete group and let X be a G-CW-complex. For each cell orbit representative σ, let G_σ be its stabilizer and choose a model for EG_σ. There is a free G-CW-complex $\widehat{X}$ with a G-invariant filtration $\{\widehat{X}_n\}_{n \geq 0}$ and a G-map $f : \widehat{X} \to X$ such that:

(1) If I_n is a set of representatives for the n-cell orbits of X, there are G-pushouts

$$\begin{array}{ccc} \coprod_{\sigma \in I_n} (G \times_{G_\sigma} EG_\sigma) \times S^{n-1} & \longrightarrow & \widehat{X}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in I_n} (G \times_{G_\sigma} EG_\sigma) \times D^n & \longrightarrow & \widehat{X}_n. \end{array}$$

(2) We have $f(\widehat{X}_n) \subseteq X^{(n)}$.
(3) The map f is a homotopy equivalence after forgetting the action.

The filtration of $\widehat{X}$ need not be its skeletal filtration.

Proof. We sketch the construction. Suppose $\widehat{X}_{n-1}$ and $f_{n-1} : \widehat{X}_{n-1} \to X^{(n-1)}$ have been constructed, with f_{n-1} an ordinary homotopy equivalence. The attaching maps for $X^{(n)}$, composed with the projections, give a map

$$\coprod_{\sigma \in I_n} (G \times_{G_\sigma} EG_\sigma) \times S^{n-1} \to X^{(n-1)}.$$

Since the domain is a free G-CW-complex, the equivariant cellular lifting argument gives a lift to $\widehat{X}_{n-1}$ up to G-homotopy. Use these lifts in the pushout defining $\widehat{X}_n$. The homotopy extension property gives a comparison map $f_n : \widehat{X}_n \to X^{(n)}$ extending f_{n-1}. The projections $G \times_{G_\sigma} EG_\sigma \to G/G_\sigma$ are ordinary homotopy equivalences, so the gluing lemma makes f_n an ordinary homotopy equivalence as well. Passing to the union completes the construction. □

Note that $EG \times X$, with its diagonal action, is a free G-CW-complex and is G-homotopy equivalent to $\widehat{X}$. The construction gives the following dimension bounds.

Corollary 4.50. Let G be a discrete group and let X be a nonempty G-CW-complex.

(1) If X is contractible, then
$$\operatorname{gd}(G) \le \sup_{\sigma} \{\operatorname{gd}(G_\sigma) + \dim \sigma\}.$$

(2) If X is acyclic over $\mathbb{Z}$, then
$$\operatorname{cd}(G) \le \sup_{\sigma} \{\operatorname{cd}(G_\sigma) + \dim \sigma\}.$$

The suprema run over the cells of X.

Proof. For (1), choose each EG_σ of dimension $\operatorname{gd}(G_\sigma)$ and apply Theorem 4.49. The resulting $\widehat{X} \simeq X \simeq *$ is free, hence is a model for EG, with the stated dimension bound. For (2), the augmented cellular chain complex of X is exact and
$$C_p(X;\mathbb{Z}) \cong \bigoplus_{\sigma \in I_p} \mathbb{Z}[G/G_\sigma] \cong \bigoplus_{\sigma \in I_p} \operatorname{ind}_{G_\sigma}^{G} \mathbb{Z}.$$
Inducing projective resolutions shows that
$$\operatorname{pd}_{\mathbb{Z}G} C_p(X;\mathbb{Z}) \le \sup_{\sigma \in I_p} \operatorname{cd}(G_\sigma).$$
Dimension shifting along the augmented exact complex therefore gives
$$\operatorname{pd}_{\mathbb{Z}G} \mathbb{Z} \le \sup_{p} \{p + \operatorname{pd}_{\mathbb{Z}G} C_p(X;\mathbb{Z})\}.$$
If this supremum is infinite the assertion is automatic; otherwise the complex has finite length and the dimension-shifting argument applies successively to its kernels. □

We apply these bounds to group extensions.

Corollary 4.51. Let $1 \to N \to G \to Q \to 1$ be an extension of discrete groups. Then
$$\operatorname{gd}(G) \le \operatorname{gd}(N) + \operatorname{gd}(Q), \qquad \operatorname{cd}(G) \le \operatorname{cd}(N) + \operatorname{cd}(Q).$$

Proof. For the geometric bound, take a model for EQ of dimension $\operatorname{gd}(Q)$ and view it as a G-space through $G \to Q$. Every stabilizer is N, so Corollary 4.50 applies. For the cohomological bound, the extension spectral sequence [Bro82, Chapter VII] for an arbitrary G-module M is
$$E_2^{p,q} = H^p(Q; H^q(N; M)) \implies H^{p+q}(G; M).$$
Its terms vanish for $p > \operatorname{cd}(Q)$ or $q > \operatorname{cd}(N)$, proving the bound. □

Example 4.52. Consider the integer Heisenberg group
$$H_{3,\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}.$$
We want to compute its geometric dimension. The elements
$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
generate $H_{3,\mathbb{Z}}$, since
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = y^c z^b x^a.$$

In particular,

$$H_{3,\mathbb{Z}} = \langle x, y, z \mid [x, y] = z,\ [x, z] = [y, z] = 1 \rangle,$$

and there is a central extension

$$1 \to \mathbb{Z} \cong \langle z \rangle \to H_{3,\mathbb{Z}} \to \mathbb{Z}^2 \to 1.$$

By Corollary 4.51, $\operatorname{gd}(H_{3,\mathbb{Z}}) \leq 3$. Conversely, we show $\operatorname{cd}(H_{3,\mathbb{Z}}) \geq 3$. The real Heisenberg group $H_{3,\mathbb{R}}$ is a Lie group homeomorphic to $\mathbb{R}^3$, hence contractible. The subgroup $H_{3,\mathbb{Z}}$ acts freely and properly discontinuously, giving a universal cover

$$H_{3,\mathbb{R}} \to H = H_{3,\mathbb{R}}/H_{3,\mathbb{Z}}.$$

Thus H is a 3-manifold and a model for $BH_{3,\mathbb{Z}}$. Every Lie group is orientable, and these translations preserve orientation, so H is orientable. The image of the coordinate cube $[0, 1]^3$ is all of H, making H compact. Poincaré duality gives $H^3(H; \mathbb{Z}) \cong \mathbb{Z} \neq 0$. Therefore $\operatorname{cd}(H_{3,\mathbb{Z}}) = \operatorname{gd}(H_{3,\mathbb{Z}}) = 3$.

The following reformulation of Corollary 4.50 will be useful later.

Corollary 4.53. Let G be a discrete group, let $n \in \mathbb{N}$, and let X be a contractible G-CW-complex of dimension at most n. If $\operatorname{gd}(G_\sigma) \leq n - \dim \sigma$ for every cell σ of X, then $\operatorname{gd}(G) \leq n$.

5. Finiteness properties of groups

References for this chapter are Brown [Bro82, Chapter VIII] and Geoghegan [Geo08, Chapters 7–8]. We will write $\operatorname{Free}(k) = *_{i=1}^{k} \mathbb{Z}$ for the free group on k generators, to distinguish it from the finiteness properties below. We now study finiteness properties of groups and CW-complexes. In Section 4, we studied the dimension of a CW-complex by looking at the highest-dimensional cells that we attach, and then defined the geometric dimension of a group as the least dimension of a model for its classifying space. We also defined a similar notion in the algebraic setting. We now count the cells in each dimension. We will also study the algebraic counterpart of this counting problem.

5.1. The geometric finiteness properties.

Let us start with the geometric case. These properties depend on the number of cells in each skeleton. The first definition concerns the finiteness of the n-skeleton.

Definition 5.1. Let G be a discrete group and let $n \in \mathbb{N}$. We say that G is of *type* F_n if there is a model for BG with finite n-skeleton, that is, with finitely many cells in dimensions at most n.

This definition captures familiar properties of groups.

(1) Every group is of type F_0, since we can choose a model for BG with only one 0-cell.
(2) A group G is of type F_1 if and only if it is finitely generated. Indeed, if a model for BG has finite 1-skeleton, collapsing a maximal tree in that skeleton gives finitely many loops whose images generate $\pi_1(BG) \cong G$. Conversely, starting with finitely many generators, the construction of BG gives a finite 1-skeleton.
(3) Similarly, G is of type F_2 if and only if it is finitely presented.

We could go on: type F_3 imposes finitely many relations between relations. The next notion captures all the types F_n.

Definition 5.2. A discrete group G is of *type* F_∞ if there is a model for BG of finite type, that is, with finitely many cells in each dimension.

Finite groups are examples of groups of type F_∞. Indeed, their standard model $\Delta^\infty(G)/G \simeq BG$ has finitely many cells in each dimension. Finite domination gives another finiteness condition.

Definition 5.3. A discrete group G is of *type* FD if it has a finitely dominated model Y for BG. Here *finitely dominated* means that there are a finite CW-complex K and maps $Y \xrightarrow{i} K \xrightarrow{r} Y$ with $ri \simeq \mathrm{id}_Y$.

The following notion is the strongest one we consider.

Definition 5.4. A discrete group G is of *type* F if there is a model for BG with finitely many cells.

Thus type F means that one model is both finite-dimensional and of finite type. Right-angled Artin groups associated to finite graphs are of type F, as their Salvetti complexes show. We now compare these finiteness conditions.

Lemma 5.5. For every discrete group and every $n \geq 1$, there is a chain of implications

$$F \implies FD \implies F_\infty \implies F_n \implies F_{n-1} \implies \cdots \implies F_1 \implies F_0.$$

The only nontrivial implication is $FD \Rightarrow F_\infty$, which is included in Proposition 5.6. We can also reverse some of these implications.

Proposition 5.6. Let G be a discrete group.

(1) If G is of type F_n for every $n \in \mathbb{N}$, then G is of type F_∞.
(2) The group G is of type FD if and only if it is of type F_∞ and $\mathrm{gd}(G) < \infty$.

Proof. See [Geo08, Section 7.2]. □

Are there any other implications? Clearly F_0 does not imply F_1, and F_1 does not imply F_2.

Example 5.7. Choose embeddings of the free group of countably infinite rank into two copies of Free(2) and form

$$G_1 = \mathrm{Free}(2) *_{\mathrm{Free}(\infty)} \mathrm{Free}(2).$$

This group is finitely generated, hence of type F_1. The defining square is a pushout

$$\begin{array}{ccc} \mathrm{Free}(\infty) & \hookrightarrow & \mathrm{Free}(2) \\ \downarrow & & \downarrow \\ \mathrm{Free}(2) & \longrightarrow & G_1. \end{array}$$

By Theorem 3.14, there is an exact sequence

$$\begin{aligned} 0 \to H^0(G_1;\mathbb{Z}) \to H^0(\mathrm{Free}(2);\mathbb{Z})^{\oplus 2} \to H^0(\mathrm{Free}(\infty);\mathbb{Z}) \\ \to H^1(G_1;\mathbb{Z}) \to H^1(\mathrm{Free}(2);\mathbb{Z})^{\oplus 2} \\ \to H^1(\mathrm{Free}(\infty);\mathbb{Z}) \to H^2(G_1;\mathbb{Z}) \to 0. \end{aligned}$$

Since $B\,\mathrm{Free}(k) \simeq \bigvee_k S^1$, the degree-zero part splits and

$$H^1(\mathrm{Free}(2);\mathbb{Z})^{\oplus 2} \cong \mathbb{Z}^4, \qquad H^1(\mathrm{Free}(\infty);\mathbb{Z}) \cong \prod_{\mathbb{N}} \mathbb{Z}.$$

Thus $H^2(G_1;\mathbb{Z})$ is the quotient of $\prod_{\mathbb{N}} \mathbb{Z}$ by the image of $\mathbb{Z}^4$. It cannot be finitely generated: otherwise the middle group would be an extension of two finitely generated abelian groups and hence finitely generated, whereas it is uncountable. If BG_1 had finite 2-skeleton, its cellular cochain groups through degree 2 would be finitely generated, forcing $H^2(G_1;\mathbb{Z})$ to be finitely generated. Therefore G_1 is not of type F_2.

For every $n \geq 2$, we can ask for a group of type F_n that is not of type F_{n+1}. Example 5.35 gives such groups. The other converse is posed in the notes as an open problem.

Conjecture 5.8. There exists a discrete group of type FD that is not of type F.

We conclude this section with some consequences of the definitions.

Lemma 5.9. Let $n \geq 1$.

(1) If a discrete group G is of type F_n and $H \leq G$ has finite index, then H is of type F_n.
(2) If discrete groups G_1, G_2 are of type F_n, then $G_1 \times G_2$ is of type F_n.
(3) If $G_0 \hookrightarrow G_1, G_2$ are monomorphisms of discrete groups, the groups G_1, G_2 are of type F_n, and G_0 is of type F_{n-1}, then $G_1 *_{G_0} G_2$ is of type F_n.

Proof. (1) Choose a model for BG with finite n-skeleton and consider its universal cover EG. The quotient EG/H is a model for BH. The covering $EG/H \to EG/G$ has finitely many sheets, so the lifted n-skeleton is finite.
(2) By Lemma 3.2, $B(G_1 \times G_2) \simeq BG_1 \times BG_2$. The product of models with finite n-skeleta again has finite n-skeleton.
(3) By Lemma 3.13, a model for $B(G_1 *_{G_0} G_2)$ is given by the homotopy pushout of $BG_1 \leftarrow BG_0 \to BG_2$. Use the double mapping cylinder, attaching $BG_0 \times [0,1]$ to $BG_1 \amalg BG_2$ along its two ends. Its new cells of dimension at most n come from cells of BG_0 of dimension at most $n-1$. Thus its n-skeleton is finite.

□

5.2. **Cohomological finiteness properties.** We now discuss the algebraic counterpart of the geometric definitions above, in the general setting of a ring R, not necessarily commutative. A resolution

$$\cdots \to P_n \to \cdots \to P_0 \to M \to 0$$

of a left R-module M is of *finite type* if every P_i is finitely generated, and is *finite* if it is of finite type and finite length. These conditions keep track of the number of generators in each degree, and of the number of degrees.

Definition 5.10. Let R be a ring, let M be a left R-module, and let $n \in \mathbb{N}$.

(1) The module M is of *type* FP_n if it admits a projective resolution $P_\bullet \to M \to 0$ in which P_i is finitely generated for $0 \leq i \leq n$.
(2) It is of *type* FP_∞ if it admits a projective resolution of finite type.
(3) It is of *type* FP if it admits a finite projective resolution.
(4) It is of *type* FL if it admits a finite free resolution.

Thus we can think of a module of type FP_n as having a finite n-skeleton, and a module of type FP_∞ as having a finite skeleton in every degree. The condition FP adds finite dimension. Theorem 5.26 relates these notions to the geometric ones. We first discuss the implications between these conditions.

Lemma 5.11. For a left module over any ring, and for every $n \geq 1$, there are implications

$$FL \implies FP \implies FP_\infty \implies FP_n \implies \cdots \implies FP_1 \implies FP_0.$$

All these implications follow immediately from the definitions. Some converses hold, as in the geometric case. Let us start with the implication from FP_n for every n to FP_∞.

Lemma 5.12 (Schanuel)**.** Let R be a ring, let M be a left R-module, and let $n \geq 1$. Consider exact sequences

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0, \qquad 0 \to K'_n \to P'_{n-1} \to \cdots \to P'_0 \to M \to 0,$$

where all P_i, P'_i are projective. Then

$$K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus P'_{n-3} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus P_{n-3} \oplus \cdots,$$

where each sum alternates down to degree 0. In particular, if every P_i, P'_i is finitely generated, then K_n is finitely generated if and only if K'_n is.

Proof. We proceed by induction on n. For $n = 1$, form the pullback $S = P_0 \times_M P'_0$. It fits into the diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K'_1 & = & K'_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & S & \longrightarrow & P'_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since P_0 and P'_0 are projective, the two sequences ending in them split. Hence $K'_1 \oplus P_0 \cong S \cong K_1 \oplus P'_0$. For the inductive step, truncate the resolutions at

$$K = \operatorname{im}(P_{n-1} \to P_{n-2}), \qquad K' = \operatorname{im}(P'_{n-1} \to P'_{n-2}).$$

Put $Q = P'_{n-2} \oplus P_{n-3} \oplus \cdots$ and $Q' = P_{n-2} \oplus P'_{n-3} \oplus \cdots$, alternating down to degree 0. Induction gives $K \oplus Q \cong K' \oplus Q'$. The sequences

$$0 \to K_n \to P_{n-1} \oplus Q \to K \oplus Q \to 0, \qquad 0 \to K'_n \to P'_{n-1} \oplus Q' \to K' \oplus Q' \to 0$$

are exact, and the case $n = 1$ gives the required isomorphism. □

Lemma 5.12 gives the following characterization.

Proposition 5.13. Let R be a ring, let M be a left R-module, and let $n \in \mathbb{N}$. The following are equivalent.

(1) The module M is of type FP_n.
(2) The module M is finitely generated and, for every $0 \leq k < n$ and every exact sequence

$$0 \to K \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$$

with all P_i finitely generated and projective, the module K is finitely generated.
(3) There is a free resolution $L_\bullet \to M \to 0$ with L_i finitely generated for $0 \leq i \leq n$.

Proof. For (1) ⇒ (2), choose a projective resolution $P'_\bullet \to M \to 0$ with P'_i finitely generated for $i \leq n$. The surjection $P'_0 \to M$ shows that M is finitely generated. For a partial resolution as in (2), let $K' = \ker(P'_k \to P'_{k-1})$, with $P'_{-1} = M$. The module K' is the image of P'_{k+1} and is finitely generated, since $k + 1 \leq n$. By Lemma 5.12, K is finitely generated as well. For (2) ⇒ (3), choose a finite free module L_0 surjecting onto M. When $n \geq 1$, condition (2) makes its kernel finitely generated, so choose a finite free module L_1 surjecting onto that kernel. Iterate this procedure through degree n, and continue with arbitrary free modules afterwards. The implication (3) ⇒ (1) is clear. □

This gives a characterization of type FP_∞.

Corollary 5.14. Let R be a ring and let M be a left R-module. The following are equivalent.

(1) The module M is of type FP_∞.
(2) The module M is of type FP_n for every $n \in \mathbb{N}$.
(3) The module M admits a free resolution of finite type.

Proof. Clearly (3) ⇒ (1) ⇒ (2). For (2) ⇒ (3), construct a free resolution inductively. The property FP_0 gives a finite free module L_0 surjecting onto M. The property FP_1 makes its kernel finitely generated, and the higher properties allow us to continue with finitely generated free modules in every degree, by Proposition 5.13. □

Finite projective dimension gives a criterion for type FP.

Proposition 5.15. Let R be a ring and let M be a left R-module. Then M is of type FP if and only if M is of type FP_∞ and $\mathrm{pd}_R(M) < \infty$.

Proof. The forward implication is immediate from a finite projective resolution. Conversely, put $n = \mathrm{pd}_R(M) < \infty$. If $n = 0$, then M is finitely generated and projective, and we are done. Otherwise choose a partial resolution

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with each P_i finitely generated and projective. By Proposition 5.13, K is finitely generated, and by Lemma 4.8, it is projective. This gives a finite projective resolution. □

We now discuss the obstruction to making a finite projective resolution free.

Definition 5.16. Let R be a ring. A finitely generated projective left R-module P is *stably free* if $P \oplus Q$ is free for some finitely generated free left R-module Q.

Lemma 5.17. Let R be a ring and let M be a left R-module of type FP. There is a resolution

$$0 \to P_n \to L_{n-1} \to \cdots \to L_0 \to M \to 0$$

in which all modules are finitely generated, P_n is projective, and each L_i is free. For any such resolution, M is of type FL if and only if P_n is stably free.

Proof. The existence follows from Proposition 5.13 and Lemma 4.8, truncating a free resolution in a degree at least $\mathrm{pd}_R(M)$. If $P_n \oplus Q$ is free with Q finitely generated and free, then

$$0 \to Q \to P_n \oplus Q \to L_{n-1} \to \cdots \to L_0 \to M \to 0$$

is a finite free resolution. Conversely, suppose that M has a finite free resolution

$$0 \to L'_m \to L'_{m-1} \to \cdots \to L'_0 \to M \to 0.$$

Pad both resolutions with zero modules to a common length and apply Lemma 5.12 with zero final kernels. Since every term except possibly P_n is finitely generated and free, the resulting alternating-sum isomorphism exhibits P_n as stably free. □

Thus the chain $FL \Rightarrow FP \Rightarrow FP_\infty \Rightarrow \cdots \Rightarrow FP_0$ has the converses described by Lemma 5.17, Proposition 5.15, and Corollary 5.14. Let us return to $R = \mathbb{Z}G$ and $M = \mathbb{Z}$.

Definition 5.18. Let G be a discrete group and let $n \in \mathbb{N}$. We say that G is of *type* FP_n, *type* FP_∞, *type* FP, or *type* FL if the trivial left $\mathbb{Z}G$-module $\mathbb{Z}$ has the corresponding property.

Let us see some examples.

Example 5.19. Every discrete group is of type FP_0, since the augmentation $\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$ begins a resolution.

Example 5.20. A discrete group G is of type FP_1 if and only if it is finitely generated. Indeed, by Lemma 2.14, the augmentation ideal

$$IG = \ker \varepsilon = \langle g - 1 \mid g \in G \rangle_{\mathbb{Z}G}$$

is finitely generated if and only if G is, and Proposition 5.13 applies.

Example 5.21. Every finite group G is of type FP_∞, since its standard resolution has finitely generated terms. If $G \neq 1$, it is not of type FP, since $\mathrm{cd}(G) = \infty$.

Example 5.22. The group $\mathrm{Free}(2) *_{\mathrm{Free}(\infty)} \mathrm{Free}(2)$ is of type FP_1 but not of type FP_2, by the argument of Example 5.7. A resolution finitely generated through degree 2 would make $H^2(G;\mathbb{Z})$ finitely generated, contradicting that example.

Example 5.35 shows that for every $n \geq 2$ there is a group of type FP_n that is not of type FP_{n+1}. As before, the notes pose the following question.

Conjecture 5.23. There exists a discrete group of type FP that is not of type FL.

We next consider passage to subgroups of finite index.

Lemma 5.24. Let G be a discrete group, let $H \leq G$ have finite index, and let $n \in \mathbb{N}$. Then G is of type FP_n if and only if H is of type FP_n.

Proof. Suppose first that G is of type FP_n and choose a projective resolution $P_\bullet \to \mathbb{Z} \to 0$ whose terms through degree n are finitely generated. Restriction res^G_H is exact and preserves projectives by Lemma 3.26. Since $[G : H] < \infty$, it also preserves finite generation, as

$$\mathrm{res}^G_H(\mathbb{Z}G) \cong \bigoplus_{H\backslash G} \mathbb{Z}H$$

is a finite free $\mathbb{Z}H$-module. The restricted resolution proves that H is of type FP_n.

For the converse, we prove by induction on n a stronger statement: if a left $\mathbb{Z}G$-module M is of type FP_n after restriction to H, then it is of type FP_n over $\mathbb{Z}G$. For $n = 0$, a finite generating set over $\mathbb{Z}H$ also generates over $\mathbb{Z}G$. For $n \geq 1$, choose an exact sequence

$$0 \to K \to F \to M \to 0$$

with F finitely generated and free over $\mathbb{Z}G$. After restriction, F is still finitely generated and free. Applying Proposition 5.13 to this partial resolution, and then to its successive extensions, shows that K is of type FP_{n-1} over $\mathbb{Z}H$. By induction, K is of type FP_{n-1} over $\mathbb{Z}G$. Append $F \to M$ to a projective resolution of K finitely generated through degree $n-1$; the resulting resolution proves that M is of type FP_n over $\mathbb{Z}G$. Taking $M = \mathbb{Z}$ proves the converse. □

Corollary 5.25. Let G be a torsion-free discrete group and let $H \leq G$ have finite index. Then G is of type FP if and only if H is of type FP.

Proof. Use Lemma 5.24 in every degree, Corollary 5.14, and the equality $\mathrm{cd}(H) = \mathrm{cd}(G)$ from Theorem 4.35. Then apply Proposition 5.15 to the trivial modules. □

5.3. **Geometric versus cohomological finiteness properties.** Let us collect the implications we have shown. In the geometric setting we have $F \Rightarrow FD \Rightarrow F_\infty \Rightarrow F_n \Rightarrow \cdots \Rightarrow F_0$, and in the algebraic setting we have $FL \Rightarrow FP \Rightarrow FP_\infty \Rightarrow FP_n \Rightarrow \cdots \Rightarrow FP_0$. The converses are those described in Proposition 5.6, Lemma 5.17, Proposition 5.15, and Corollary 5.14. The geometric and algebraic implications fit into the following diagram:

$$\begin{array}{ccccccccccccc}
F & \Longrightarrow & FD & \Longrightarrow & F_\infty & \Longrightarrow & F_n & \Longrightarrow \cdots \Longrightarrow & F_2 & \Longrightarrow & F_1 & \Longrightarrow & F_0 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Updownarrow & & \Updownarrow \\
FL & \Longrightarrow & FP & \Longrightarrow & FP_\infty & \Longrightarrow & FP_n & \Longrightarrow \cdots \Longrightarrow & FP_2 & \Longrightarrow & FP_1 & \Longrightarrow & FP_0.
\end{array}$$

Indeed, for a model of BG with i-cells indexed by I_i, we have

$$C_i(BG;\mathbb{Z}) \cong \bigoplus_{I_i} \mathbb{Z}, \qquad C_i(\widetilde{BG};\mathbb{Z}) \cong \bigoplus_{I_i} \mathbb{Z}G.$$

The augmented cellular chain complex of the universal cover is a free resolution, giving $F \Rightarrow FL$, $F_\infty \Rightarrow FP_\infty$, and $F_n \Rightarrow FP_n$. Moreover, FD implies both F_∞ and finite geometric dimension, hence FP_∞ and finite cohomological dimension, which imply FP. The equivalences $F_0 \Leftrightarrow FP_0$ and $F_1 \Leftrightarrow FP_1$ follow from the examples above: every group has the first two properties, and the latter two both mean finite generation. Finite presentation gives further implications. See also [Bro82, Chapter VIII] and [Geo08, Chapters 7–8].

Theorem 5.26. Let G be a finitely presented discrete group, that is, a group of type F_2.

(1) For every $n \in \mathbb{N}$, type FP_n implies type F_n.
(2) Type FP_∞ implies type F_∞.
(3) Type FP implies type FD.
(4) Type FL implies type F.

Proof. (1) For $n \leq 2$, there is nothing to prove. For $n \geq 3$, revisit the proof of Theorem 4.23 to construct a model Y for BG with finite n-skeleton. Write $X^{(k)} = \widetilde{Y^{(k)}}$ for $k \geq 2$. Start with a finite presentation $G = \langle S \mid R \rangle$ and its finite presentation complex $Y^{(2)}$. Then $\pi_1(Y^{(2)}) \cong G$ and $H_1(X^{(2)}; \mathbb{Z}) = 0$. Inductively, suppose $Y^{(k-1)}$ has been constructed with finite $(k-1)$-skeleton and

$$\widetilde{H}_i(X^{(k-1)}; \mathbb{Z}) = 0 \qquad (i \leq k-2).$$

The group G acts on the universal cover and on its homology. For $3 \leq k \leq n$, the exact sequence

$$0 \to H_{k-1}(X^{(k-1)}; \mathbb{Z}) \to C_{k-1}(X^{(k-1)}) \to \cdots \to C_0(X^{(k-1)}) \to \mathbb{Z} \to 0$$

has finitely generated free cellular terms. Since G is of type FP_n, Proposition 5.13 shows that its leftmost module is finitely generated. Choose finitely many G-module generators z_α for it. By Theorem 4.20, represent them by maps $f_\alpha : S^{k-1} \to X^{(k-1)}$ and attach k-cells downstairs along their projections. The same relative-homology calculation as in Theorem 4.23 gives $\widetilde{H}_i(X^{(k)}; \mathbb{Z}) = 0$ for $i < k$. Continue through degree n with finitely many cells, and thereafter allow arbitrary sets of generators. The union $X = \bigcup_k X^{(k)}$ is simply connected and acyclic, hence contractible. Thus $Y = \bigcup_k Y^{(k)}$ is a model for BG with finite n-skeleton.

(2) If G is of type FP_∞, then it is of type FP_n for every n. Part (1) makes it of type F_n for every n, hence of type F_∞ by Proposition 5.6.

(3) If G is of type FP, then it is of type FP_∞ and has finite cohomological dimension. Part (2) gives type F_∞, and Theorem 4.23 gives finite geometric dimension. Therefore G is of type FD by Proposition 5.6.

(4) Suppose G is of type FL and put $n = \max\{3, \mathrm{cd}(G)\} < \infty$. Carry out part (1) through degree $n-1$. The module $P = H_{n-1}(X^{(n-1)}; \mathbb{Z})$ is finitely generated and projective, and Lemma 5.17 makes it stably free. Choose a finitely generated free module Q with $P \oplus Q$ free, and wedge onto $Y^{(n-1)}$ one $(n-1)$-sphere for each basis element of Q. The top homology of the universal cover is now the finitely generated free module $P \oplus Q$. Attach one n-cell downstairs for each basis element of this module, using Hurewicz representatives as before. The relative boundary map is an isomorphism, so the resulting universal cover is acyclic and simply connected, hence contractible. Only finitely many cells have been added, giving a finite model for BG.

□

Does FP_2 imply F_2? If it did, all the corresponding geometric and algebraic conditions would agree. Example 5.36 shows that FP_2 does not imply F_2.

5.4. **Finiteness properties by actions.** To introduce examples showing that FP_n does not imply FP_{n+1}, that F_n does not imply F_{n+1}, and that FP_2 does not imply F_2, we need some machinery involving G-CW-complexes.

Definition 5.27. Let G be a discrete group and let X be a G-CW-complex. We say that X is *cocompact* if its quotient $G\backslash X$ is compact.

Clearly, G is of type F_n if and only if it admits a model for EG with cocompact n-skeleton.

Lemma 5.28. Let G be a discrete group, let $n \geq 1$, and let X be a free G-CW-complex. If X is $(n-1)$-connected and $X^{(n)}$ is cocompact, then G is of type F_n.

Proof. Construct a model for EG by attaching free G-orbits of cells of dimensions at least $n+1$ to X, successively killing its homotopy groups. The resulting complex is weakly contractible, hence contractible by Theorem 4.21. Its n-skeleton is unchanged and is cocompact, so its quotient is a model for BG with finite n-skeleton. □

We now relax the freeness assumption and then the cocompactness condition.

Theorem 5.29. Let G be a discrete group, let $n \geq 1$, and let X be a $(n-1)$-connected G-CW-complex with cocompact n-skeleton. If G_σ is of type $F_{n-\dim\sigma}$ for every cell σ with $\dim\sigma \leq n$, then G is of type F_n.

Proof. For each cell orbit representative σ of dimension at most n, choose a model for EG_σ with cocompact $(n-\dim\sigma)$-skeleton. For higher-dimensional cells, choose arbitrary models. Apply Theorem 4.49 to obtain a free G-CW-complex $\widehat{X} \simeq X$. We claim that $\widehat{X}$ has cocompact n-skeleton. The stage

$$\widehat{X}_0 = \coprod_{\sigma \in I_0} G \times_{G_\sigma} EG_\sigma$$

has cocompact n-skeleton, since I_0 is finite and the stabilizers are of type F_n. For $1 \leq i \leq n$, the next stage is formed by the pushout

$$\begin{array}{ccc} \coprod_{\sigma\in I_i}(G\times_{G_\sigma} EG_\sigma)\times S^{i-1} & \longrightarrow & \widehat{X}_{i-1} \\ \downarrow & & \downarrow \\ \coprod_{\sigma\in I_i}(G\times_{G_\sigma} EG_\sigma)\times D^{i} & \longrightarrow & \widehat{X}_{i}. \end{array}$$

The new cells are the relative cells of $(EG_\sigma \times D^i, EG_\sigma \times S^{i-1})$. A cell of dimension p in EG_σ contributes a new cell of dimension $p+i$. Hence only the cocompact $(n-i)$-skeleton of EG_σ contributes through dimension n, and there are finitely many relevant cell orbits σ. The later stages, with $i > n$, add no cells of dimension at most n. Thus $\widehat{X}^{(n)}$ is cocompact, and Lemma 5.28 applies. □

We obtain the following consequence for extensions.

Corollary 5.30. Let $1 \to N \to G \to Q \to 1$ be an extension of discrete groups and let $n \geq 1$. If N and Q are of type F_n, then G is of type F_n. If both are of type F, then G is of type F.

Proof. View a model for EQ as a G-space via $G \to Q$, so every stabilizer is N. For type F_n, choose EQ with cocompact n-skeleton and apply Theorem 5.29. For type F, choose cocompact models for both EN and EQ; the replacement in Theorem 4.49 then has finitely many cell orbits in total. □

We now relax the cocompactness condition in Lemma 5.28. This is the free-action form of Brown's criterion [Bro87]; see also [Geo08, Section 7.4].

Theorem 5.31 (Brown)**.** Let G be a discrete group, let $n \geq 1$, and let X be a free $(n-1)$-connected G-CW-complex. Let

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X = \bigcup_{i \geq 0} X_i$$

be a filtration by G-subcomplexes such that $X_i^{(n)}$ is cocompact for every i. Then G is of type F_n if and only if the filtration is *essentially* $(n-1)$*-connected*: for every i there is $j \geq i$ such that, for every $0 \leq k \leq n-1$, every map $S^k \to X_i$ extends to a map $D^{k+1} \to X_j$. For $k \geq 1$, this means that the inclusion induces the zero map on π_k at every basepoint of X_i; for $k = 0$, it means that all components of X_i lie in one component of X_j.

Proof. Suppose first that G is of type F_n and choose a model Z for EG with cocompact n-skeleton. Complete X to a model X' for EG by attaching free G-orbits of cells of dimensions at least $n+1$. By uniqueness of EG up to equivariant homotopy equivalence, choose cellular G-maps

$$f : X' \rightleftarrows Z : g$$

and a cellular G-homotopy $H : \mathrm{id}_{X'} \simeq gf$. Since $Z^{(n)}$ is cocompact, there is j_0 such that $g(Z^{(n)}) \subseteq X_{j_0}^{(n)}$. For each i, cocompactness of $X_i^{(n-1)}$ gives $j \geq \max\{i, j_0\}$ such that

$$H\big(X_i^{(n-1)} \times [0,1]\big) \subseteq X_j^{(n)}.$$

Thus the triangle

$$\begin{array}{ccc} X_i^{(n-1)} & \hookrightarrow & X_j^{(n)} \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Z^{(n)} & \end{array}$$

commutes up to equivariant homotopy. The space $Z^{(n)}$ is $(n-1)$-connected, so the inclusion $X_i^{(n-1)} \to X_j^{(n)}$ kills homotopy in degrees at most $n-1$, with the stated interpretation in degree 0. Every map $S^k \to X_i$ with $k \leq n-1$ is homotopic in X_i to a cellular map into $X_i^{(n-1)}$. It is therefore nullhomotopic in X_j, proving essential connectivity.

For the converse, we use the following standard fact: G is of type F_n if and only if a model Y for BG is homotopy dominated by a CW-complex with finite n-skeleton. Thus it suffices to construct a CW-complex Z with finite n-skeleton and maps

$$Y \xrightarrow{f} Z \xrightarrow{g} Y, \qquad gf \simeq \mathrm{id}_Y\,.$$

Discarding empty initial stages if necessary, choose a basepoint in X_0 and let $C_i \subseteq X_i$ be its component. Essential 0-connectivity gives $i_1 \geq 0$ such that $X_0 \subseteq C_{i_1}$. Since X_0 is G-invariant, every translate of the basepoint lies in C_{i_1}, so C_{i_1} and all later C_i are G-invariant. The covering $C_{i_1} \to G\backslash C_{i_1}$ gives an exact sequence

$$1 \to \pi_1(C_{i_1}) \to \pi_1(G\backslash C_{i_1}) \to G \to 1.$$

The quotient has finite 1-skeleton, so its fundamental group is finitely generated, and hence so is G. This proves the case $n = 1$.

Suppose now that $n \geq 2$. By essential 1-connectivity, choose $i_2 \geq i_1$ such that $\pi_1(C_{i_1}) \to \pi_1(C_{i_2})$ is zero. The inclusions give a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(C_{i_1}) & \longrightarrow & \pi_1(G\backslash C_{i_1}) & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle 0}\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(C_{i_2}) & \longrightarrow & \pi_1(G\backslash C_{i_2}) & \underset{\beta_2}{\longrightarrow} & G & \longrightarrow & 1. \end{array}$$

Consequently, the middle vertical map factors through a homomorphism $\alpha_2 : G \to \pi_1(G\backslash C_{i_2})$ with $\beta_2\alpha_2 = \mathrm{id}_G$. Let Y be a CW-model for BG and choose a cellular map

$$f_2 : Y^{(2)} \to G\backslash C_{i_2}$$

inducing α_2 on fundamental groups. For $2 \le k \le n-1$, essential k-connectivity gives $i_{k+1} \ge i_k$ such that $\pi_k(C_{i_k}) \to \pi_k(C_{i_{k+1}})$ is zero. Coverings induce isomorphisms on these higher homotopy groups, so $\pi_k(G\backslash C_{i_k}) \to \pi_k(G\backslash C_{i_{k+1}})$ is also zero. It follows that f_k extends after inclusion across the $(k+1)$-cells of Y:

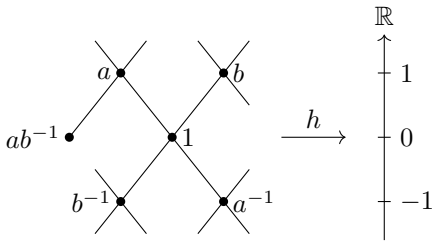
$$\begin{array}{ccc} Y^{(k)} & \xrightarrow{f_k} & G\backslash C_{i_k} \\ \downarrow & & \downarrow \\ Y^{(k+1)} & \xrightarrow[f_{k+1}]{} & G\backslash C_{i_{k+1}}. \end{array}$$

Repeating this process gives $f_n : Y^{(n)} \to G\backslash C_{i_n}$. Write $\beta_n : \pi_1(G\backslash C_{i_n}) \to G$ for the covering homomorphism; then $\beta_n \pi_1(f_n) = \mathrm{id}_G$. Construct Z from $G\backslash C_{i_n}$ by following the remaining cell attachments of Y: for each cell of dimension at least $n+1$, attach a cell of the same dimension along the image of its attaching map under the extension already constructed. Extending over these new cells gives a map $f : Y \to Z$. Only cells of dimensions at least $n+1$ have been added, so Z has finite n-skeleton and $\pi_1(Z) \cong \pi_1(G\backslash C_{i_n})$. Since Y is a model for BG, Proposition 2.31 gives a map $g : Z \to Y$ inducing β_n. Now $\pi_1(gf) = \mathrm{id}_G$, so the same universal property gives $gf \simeq \mathrm{id}_Y$. The domination criterion proves that G is of type F_n. □

Example 5.32. Let

$$\varphi : \mathrm{Free}(2) = \langle a, b \rangle \to \mathbb{Z}, \qquad a, b \mapsto 1,$$

and put $G = \ker \varphi$. We show that G is not finitely generated. Let X be the 4-regular Cayley tree of Free(2) and define a *height function* $h : X \to \mathbb{R}$ by $h(w) = \varphi(w)$ at each vertex, extending linearly over the edges. Thus the height of an arbitrary word is its total exponent sum in a and b.



The subcomplexes $X_i = h^{-1}([-i, i])$, for integers $i \ge 0$, form a G-invariant filtration of X. Each quotient $G\backslash X_i$ is finite: its vertices are the integers from $-i$ to i, with two edges between consecutive integers. However, the filtration is not essentially 0-connected. For any $j \ge 0$, choose an integer $m > j$. The vertices 1 and $a^m b^{-m}$ both lie in X_0, but their unique connecting path in the tree passes through a^m, of height m. They therefore lie in different components of X_j. By Theorem 5.31, G is not of type F_1, that is, it is not finitely generated.

5.5. **Groups with exotic finiteness properties.** We now construct two kinds of examples:

(1) for each $n \ge 1$, a group G_n of type F_n but not of type FP_{n+1};
(2) a group G of type FP but not of type F_2.

We obtain them from right-angled Artin groups associated to finite flag complexes. For a nonempty finite flag complex L, write A_L for the right-angled Artin group associated to its 1-skeleton. Let $\varphi : A_L \to \mathbb{Z}$ be the homomorphism sending every vertex generator to 1.

Definition 5.33. Let L be a nonempty finite flag complex. The *Bestvina–Brady group* associated to L is

$$H_L = \ker\big(\varphi : A_L \to \mathbb{Z}\big), \qquad \varphi(s) = 1$$

for every vertex generator s.

By definition, we have a short exact sequence

$$1 \to H_L \to A_L \xrightarrow{\varphi} \mathbb{Z} \to 1.$$

The result we need is the main theorem of Bestvina and Brady [BB97].

Theorem 5.34 (Bestvina–Brady, 1997)**.** Let L be a nonempty finite flag complex and let $n \geq 1$.

(1) The group H_L is of type FP_n if and only if L is $(n-1)$-acyclic over $\mathbb{Z}$, that is, $\widetilde{H}_i(L;\mathbb{Z}) = 0$ for $0 \leq i \leq n-1$.
(2) The group H_L is of type F_2 if and only if L is simply connected.
(3) The group H_L is of type F_n if and only if L is $(n-1)$-connected.

Proof. We give only the idea of the proof, as in the notes. Parts (1) and (2) use Morse theory. Part (3) follows from them: for $n = 1$, use $F_1 \Leftrightarrow FP_1$; for $n \geq 2$, combine Theorem 5.26 with Theorem 4.20 to pass between $(n-1)$-acyclicity and $(n-1)$-connectivity in the simply connected case. □

Theorem 5.34 gives the first family.

Example 5.35. Let $n \geq 1$ and give S^n the flag triangulation

$$L = (S^0)^{*(n+1)},$$

the join of $n+1$ copies of the two-point complex S^0. The sphere is $(n-1)$-connected but not n-acyclic, so H_L is of type F_n but not of type FP_{n+1}. In this case,

$$A_L = \prod_{i=1}^{n+1} \mathrm{Free}(2), \qquad H_L = \ker\left(\prod_{i=1}^{n+1} \mathrm{Free}(2) \xrightarrow{\varphi} \mathbb{Z}\right),$$

where φ sends all the indicated free generators to 1.

For the second example, we use an acyclic space that is not simply connected.

Example 5.36. Let L be a finite flag triangulation of the 2-dimensional spine of the Poincaré homology sphere considered in Section 4.4. The complex L is acyclic but not simply connected. By Theorem 5.34, H_L is of type FP_∞ but not of type F_2. Since $H_L \leq A_L$ and the Salvetti complex gives a finite-dimensional model for BA_L, the group H_L has finite cohomological dimension. Thus Proposition 5.15 shows that H_L is of type FP. Moreover, $\mathrm{cd}(H_L) = 2$ by [BB97, Examples 6.3(3)].

The following theorem is proved in [BB97, Theorem 8.7].

Theorem 5.37 (Bestvina–Brady)**.** Let L be a finite flag triangulation of the 2-dimensional spine of the Poincaré homology sphere. Then $\mathrm{gd}(H_L) = 3$ or the Whitehead asphericity conjecture, Conjecture 4.28, is false.

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