ENRICHMENTS IN FINITE FILTERED SPECTRA

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Abstract. This note studies enrichments valued in finite filtered spectra. For a stable category with a bounded t-structure, the Whitehead tower of a bounded object is a finite filtration, and this induces a canonical $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment on the subcategory of bounded objects. As a consequence, mapping spectra carry finite spectral sequences whose E_1-page is described by the homotopy objects.

Contents

1. Introduction

Filtered objects are ubiquitous in homological algebra and stable homotopy theory, as they encode extension data and naturally give rise to spectral sequences [Lur17, Section 1.2.2]. This note studies the *finite* part of this theory, with the goal of computing hom-spectra in stable categories; equivalently, it studies enrichments valued in *finite* filtered spectra.

Starting from the Day convolution symmetric monoidal structure on $\mathrm{Fil}(\mathrm{Sp})$, we show that if $\mathcal{C}$ is a presentable stable category, then $\mathrm{Fil}(\mathcal{C})$ admits a natural enrichment in filtered spectra. We then isolate the full subcategory $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$ of finite filtered spectra and prove, by a finite dévissage argument, that it is symmetric monoidal. A large class of categories enriched in finite filtered spectra is provided by the following:

Proposition 1.1 (Proposition 2.33)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a bounded t-structure. Then the Whitehead tower provides a $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment of $\mathcal{C}^{\mathrm{b}}$.

As a consequence, the hom-spectra of $\mathcal{C}$ become the target of a spectral sequence which has only finitely many nonzero columns, hence converges strongly and collapses at a finite stage. This spectral sequence is relatively easy to understand:

Proposition 1.2 (Proposition 3.2)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a bounded t-structure. Then for every $x, y \in \mathcal{C}$ there exists a functorial finite decreasing filtration $F^\bullet$ on the $\mathrm{Hom}_{h\mathcal{C}}(x, y)$ such that:

(1) The filtration is multiplicative, in that if $f \in F^p\mathrm{Hom}_{h\mathcal{C}}(x,y)$ and $g \in F^q\mathrm{Hom}_{h\mathcal{C}}(y,z)$, then $g \circ f \in F^{p+q}\mathrm{Hom}_{h\mathcal{C}}(x,z)$.
(2) There is a strongly convergent spectral sequence
$$E_1^{p,q}(x,y) \simeq \prod_{r\in\mathbb{Z}} \mathrm{Ext}^{-q}_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y)) \Rightarrow \pi_{p+q}\mathrm{hom}_{\mathcal{C}}(x,y)$$
and its induced filtration on $\pi_0\mathrm{hom}_{\mathcal{C}}(x,y) = \mathrm{Hom}_{h\mathcal{C}}(x,y)$ is precisely $F^\bullet$.
(3) For every $p \geq 0$, the quotient $F^p\mathrm{Hom}_{h\mathcal{C}}(x,y)\big/F^{p+1}\mathrm{Hom}_{h\mathcal{C}}(x,y)$ is a subquotient of
$$\prod_{r\in\mathbb{Z}} \mathrm{Ext}^{p}_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y)).$$
(4) The first piece of the filtration
$$F^1\mathrm{Hom}_{h\mathcal{C}}(x,y) = \ker(\mathrm{Hom}_{h\mathcal{C}}(x,y) \to \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_n(x), \pi_n(y)))$$
is the kernel of the homotopy maps.

This may be viewed as a generalization of [Ola21, Proposition 1], a result used by Olander to compute the Rouquier dimension of quasi-affine noetherian regular schemes.

1.1. **Organization.** The paper is organised as follows. Section 2.1 recalls basic facts on filtered objects. Section 2.2 constructs the Fil(Sp)-enrichment on filtered categories. Section 2.3 discusses the associated spectral sequence. Section 2.4 introduces finite filtered spectra and proves that bounded objects in a stable category with bounded t-structure admit a natural enrichment in $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$ via Whitehead towers. Section 3 applies this construction to the induced filtration on π_0 of mapping spectra.

1.2. **Notation and terminology.** We will refer to "∞-categories" simply as "categories". If we want to emphasise that a category has discrete mapping space, we will call it a "1-category". Given a category $\mathcal{C}$ we let $h\mathcal{C}$ denote the corresponding 1-category.

Notation 1.3. Recall that a category $\mathcal{C}$ is called *stable* if it is pointed, admits finite limits and finite colimits, and a commutative square is a pullback if and only if it is a pushout. Equivalently, $\mathcal{C}$ is pointed and every morphism admits a fibre and a cofibre, which canonically agree. We let $\mathrm{hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ denote the mapping spectrum of $\mathcal{C}$. This is a bi-exact functor. Recall that a functor is *exact* if it preserves finite limits and colimits. We let $\mathrm{Cat}^{\mathrm{st}}$ be the category of stable categories and exact functors.

Notation 1.4. Let $\mathcal{C}$ be a stable category. A t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ will be graded homologically[1]. That is, we imagine $\mathcal{C}$ as linearized in the following way $\cdots \to \bullet_{n+1} \to \bullet_n \to \bullet_{n-1} \to \ldots$. We will think of objects in $\mathcal{C}_{\geq n}$ as existing to the left of n, whereas objects in $\mathcal{C}_{\leq n}$ will exist on the right. In particular, we will call these objects *n-connective* and *n-coconnective*. The inclusions of the connective and coconnective objects admit a left and right adjoint respectively, that is $\tau_{\leq n} : \mathcal{C} \rightleftarrows \mathcal{C}_{\leq n} : i_{\leq n}$ and $i_{\geq n} : \mathcal{C}_{\geq n} \rightleftarrows \mathcal{C} : \tau_{\geq n}$. Let $\mathcal{C}^\heartsuit = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ denote the *heart of the t-structure*. We will denote by $\pi_n : \mathcal{C} \to \mathcal{C}^\heartsuit$ the functor $\tau_{\leq 0}\tau_{\geq 0}[-n]$ for any $n \in \mathbb{Z}$, and refer to π_n as the *n-th homotopy group of the t-structure*. Finally, we will denote by
$$\mathcal{C}^- = \bigcup_{n>0} \mathcal{C}_{\geq -n}, \qquad \mathcal{C}^+ = \bigcup_{n>0} \mathcal{C}_{\leq n}, \qquad \mathcal{C}^b = \mathcal{C}^- \cap \mathcal{C}^+$$
the full subcategories of $\mathcal{C}$ spanned by the *bounded below* , *bounded above* and *bounded objects.*

[1]Recall that to switch to the cohomological convention it suffices to define $\mathcal{C}^{\geq n} = \mathcal{C}_{\leq -n}$ for every $n \in \mathbb{Z}$.

2. Filtered categories

2.1. Filtered objects.

Notation 2.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. We let $\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$ denote the category of *filtered objects in* $\mathcal{C}$. We will denote filtered objects adding a subscripted dot (that is, by $F_\bullet$). In the case where the category is given by Sp, we will refer to filtered objects in Sp as *filtered spectra.*

The category of filtered objects in a stable category is clearly stable. It also admits "additional shifts".

Notation 2.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. For every $n \in \mathbb{Z}$ and every $F_\bullet \in \mathrm{Fil}(\mathcal{C})$, we denote by $F_\bullet\langle n\rangle$ the shifted filtration defined by

$$(F_\bullet\langle n\rangle)_r \simeq F_{r+n}.$$

In other words, the transition maps of $F_\bullet\langle n\rangle$ are those of $F_\bullet$ shifted by n. Since the shift by n is induced by the translation $\mathbb{Z} \to \mathbb{Z}$ by n, it is clear that $\langle n\rangle : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ is an equivalence.

Notation 2.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $x \in \mathcal{C}$ be an object. For $r \in \mathbb{Z}$, let $\langle r, x\rangle \in \mathrm{Fil}(\mathcal{C})$ be the filtration

$$\langle r,x\rangle_n \simeq \begin{cases} x & n \le r, \\ 0 & n > r \end{cases} \qquad \simeq \qquad (\cdots \to 0 \to 0 \to x \to x \to \dots)$$

where the first non zero object is placed in degree r. We will refer to $\langle r, x\rangle$ as the *r-step filtration of x*.

Lemma 2.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $r \in \mathbb{Z}$. Then there exists an adjunction $\langle r, -\rangle : \mathcal{C} \rightleftarrows \mathrm{Fil}(\mathcal{C}) : \mathrm{ev}_r$ where the right adjoint evaluates at r.

Proof. Let $\iota_r : * \to \mathbb{Z}^{\mathrm{op}}$ be the inclusion of the object r. Since $\mathrm{Fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$, precomposition with ι_r defines the evaluation functor $\mathrm{ev}_r = \iota_r^* : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ given by $F_\bullet \mapsto F_r$. Since $\mathcal{C}$ admits finite colimits, the Kan extensions along the functor $* \to \mathbb{Z}^{\mathrm{op}}$ exists. In particular, the functor ι_r^* admits a left adjoint given by left Kan extension $\mathrm{Lan}(\iota_r) : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$. It remains to identify this left Kan extension with the r-step filtration.

Let $x \in \mathcal{C}$. For every $n \in \mathbb{Z}^{\mathrm{op}}$, one has

$$(\mathrm{Lan}(\iota_r)(x))_n \simeq \mathrm{colim}_{(* \times_{\mathbb{Z}^{\mathrm{op}}} (\mathbb{Z}^{\mathrm{op}})_{/n})}\, x.$$

Since $*$ has the unique object r, the indexing category is empty if there is no morphism $r \to n$ in $\mathbb{Z}^{\mathrm{op}}$, and is contractible otherwise. Now a morphism $r \to n$ in $\mathbb{Z}^{\mathrm{op}}$ exists if and only if $n \le r$ in $\mathbb{Z}$. Therefore

$$(\mathrm{Lan}(\iota_r)(x))_n \simeq \begin{cases} x & n \le r, \\ 0 & n > r, \end{cases}$$

with identity transition maps whenever both sides are nonzero. This is exactly the filtration $\langle r, x\rangle$. □

In the case of spectra, we obtain the following consequence.

Corollary 2.5. Let $r \in \mathbb{Z}$ and $X_\bullet \in \mathrm{Fil}(\mathrm{Sp})$. Then there is a natural equivalence

$$\hom_{\mathrm{Fil}(\mathrm{Sp})}(\langle r, \mathbb{S}\rangle, X_\bullet) \simeq X_r$$

of spectra.

Proof. Notice that
$$\hom_{\mathrm{Fil(Sp)}}(\langle r,\mathbb{S}\rangle, X_\bullet) \simeq \hom_{\mathrm{Fil(Sp)}}(\mathrm{Lan}(\iota_r)(\mathbb{S}), X_\bullet) \simeq \hom_{\mathrm{Sp}}(\mathbb{S}, \iota_r^*(X_\bullet)).$$
Here the first equivalence is by definition, the second one by Lemma 2.4. Since $\iota_r^*(X_\bullet) = X_r$ and $\mathbb{S}$ is the unit of Sp, this gives $\hom_{\mathrm{Sp}}(\mathbb{S}, X_r) \simeq X_r$ and hence the claim. □

Lemma 2.6. The step filtrations generate Fil(Sp) under small colimits.

Proof. Let $\mathcal{L} \subseteq \mathrm{Fil(Sp)}$ be the smallest full stable subcategory closed under small colimits and containing all step filtrations $\langle r, S\rangle$. Since $\mathrm{Fil(Sp)} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$ is a presentable stable category, it is enough to show that the right orthogonal $\mathcal{L}^\perp$ is zero. Let $X_\bullet \in \mathcal{L}^\perp$ so that $\hom_{\mathrm{Fil(Sp)}}(\langle r, S\rangle, X_\bullet) \simeq 0$ for every $r \in \mathbb{Z}$. Use Corollary 2.5 to deduce that $X_r \simeq 0$ for every $r \in \mathbb{Z}$. This implies $X_\bullet \simeq 0$ objectwise and so $\mathcal{L}^\perp = 0$. Thus $\mathcal{L} = \mathrm{Fil(Sp)}$, which proves the claim. □

Notation 2.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $F_\bullet \in \mathrm{Fil}(\mathcal{C})$ be a filtered object. For $r \in \mathbb{Z}$, we let $\sigma_{\le r} F_\bullet$ denote the filtration defined by
$$(\sigma_{\le r} F_\bullet)_n \simeq \begin{cases} 0 & n > r, \\ \mathrm{cofib}(F_{r+1} \to F_n) & n \le r. \end{cases}$$
In other words, $\sigma_{\le r} F_\bullet$ is obtained from $F_\bullet$ by killing the tail above r. Notice that $\sigma_{\le r}$ assembles into a functor $\mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$, and for $r \le s$ there is a natural transformation $\sigma_{\le r} \to \sigma_{\le s}$.

Notation 2.8. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $F_\bullet \in \mathrm{Fil}(\mathcal{C})$ be a filtered object. For $r \in \mathbb{Z}$ we let $\mathrm{gr}_r(F_\bullet)$ denote the cofibre
$$\mathrm{gr}_r(F_\bullet) = \mathrm{cofib}(F_{r+1} \to F_r)$$
and we will refer to it as the *r-graded piece of $F_\bullet$*. Notice that taking the r-graded piece assembles into a functor $\mathrm{gr}_r : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$.

Notation 2.9. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. For $r \in \mathbb{Z}$ and $x \in \mathcal{C}$, let $\delta_r(x) \in \mathrm{Fil}(\mathcal{C})$ be the filtration concentrated in degree r, that is
$$\delta_r(z)_n \simeq \begin{cases} x & n = r, \\ 0 & n \neq r, \end{cases}$$
with all transition maps zero. We will refer to $\delta_r(x)$ as the *r-delta of x*.

Lemma 2.10. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $r \in \mathbb{Z}$. Then there exists an adjunction $\mathrm{gr}_r : \mathrm{Fil}(\mathcal{C}) \rightleftarrows \mathcal{C} : \delta_r$.

Proof. Let $j_r : (\Delta^1)^{\mathrm{op}} \hookrightarrow \mathbb{Z}^{\mathrm{op}}$ be the functor sending the unique non-identity arrow of $(\Delta^1)^{\mathrm{op}}$ to $r+1 \to r$. Then restriction along j_r defines a functor $j_r^* : \mathrm{Fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, and for every $F_\bullet \in \mathrm{Fil}(\mathcal{C})$ one has $j_r^*(F_\bullet) \simeq (F_{r+1} \to F_r)$. It follows that $\mathrm{gr}_r = \mathrm{cofib} \circ j_r^*$. Now the functor $\mathrm{cofib} : \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ has right adjoint $s : \mathcal{C} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$ defined by $x \mapsto (0 \to x)$. Moreover, j_r^* has right adjoint given by right Kan extension $(j_r)_* : \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ so that $\mathrm{gr}_r = \mathrm{cofib} \circ j_r^* \dashv (j_r)_* \circ s$.

It remains to identify $(j_r)_* \circ s$ with δ_r. For $x \in \mathcal{C}$, the object $s(x)$ is the arrow $0 \to x$ on the edge $r+1 \to r$. Its right Kan extension along j_r is the filtration with value x in degree r and 0 in every other degree, namely $\delta_r(x)$. Thus $(j_r)_* \circ s \simeq \delta_r$ and hence $\mathrm{gr}_r \dashv \delta_r$. □

Lemma 2.11. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $F_\bullet \in \mathrm{Fil}(\mathcal{C})$ be a filtered object. Then for every $r \in \mathbb{Z}$ there is an exact sequence
$$\sigma_{\le r-1} F_\bullet \to \sigma_{\le r} F_\bullet \to \langle r, \mathrm{gr}_r(F_\bullet)\rangle$$
in $\mathrm{Fil}(\mathcal{C})$.

Proof. Since $\mathrm{Fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{C})$, cofibres are computed objectwise. Fix $n \in \mathbb{Z}$. If $n > r$, then $(\sigma_{\le r-1} F_\bullet)_n \simeq 0 \simeq (\sigma_{\le r} F_\bullet)_n$, so the cofibre is zero. Assume now that $n \le r$. Then $(\sigma_{\le r-1} F_\bullet)_n \simeq \mathrm{cofib}(F_r \to F_n)$ and $(\sigma_{\le r} F_\bullet)_n \simeq \mathrm{cofib}(F_{r+1} \to F_n)$. The map between these two objects is induced by the composable maps $F_{r+1} \to F_r \to F_n$. Applying the octahedral axiom produces an exact sequence

$$\mathrm{cofib}(F_r \to F_n) \to \mathrm{cofib}(F_{r+1} \to F_n) \to \mathrm{cofib}(F_{r+1} \to F_r).$$

By definition, $\mathrm{cofib}(F_{r+1} \to F_r) \simeq \mathrm{gr}_r(F_\bullet)$. Therefore, for every $n \le r$, the cofibre of the map $(\sigma_{\le r-1} F_\bullet)_n \to (\sigma_{\le r} F_\bullet)_n$ is canonically equivalent to $\mathrm{gr}_r(F_\bullet)$, while for $n > r$ it is zero. These identifications are natural in n, hence the cofibre filtration is precisely the step filtration $\langle r, \mathrm{gr}_r(F_\bullet)\rangle$. □

2.2. **A monoidal structure and enrichments.** We now observe the existence of a symmetric monoidal structure.

Remark 2.12. By [Lur17, Section 2.2.6], the category of filtered spectra $\mathrm{Fil}(\mathrm{Sp})$ is presentably symmetric monoidal for the Day convolution associated with the additive monoidal structure on $\mathbb{Z}^{\mathrm{op}}$. Explicitly, given two filtered spectra $X_\bullet, Y_\bullet \in \mathrm{Fil}(\mathrm{Sp})$, their tensor product is given by

$$(X_\bullet \otimes_{\mathrm{Day}} Y_\bullet)_n = \mathrm{colim}_{i+j \ge n} X_i \otimes_{\mathrm{Sp}} Y_j.$$

The unit object is given by the 0-step filtration $\langle 0, \mathbb{S}\rangle$.

We now construct the claimed enrichment.

Construction 2.13. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Then $\mathcal{C}$ is a module over Sp. Let $- \otimes - : \mathrm{Sp} \times \mathcal{C} \to \mathcal{C}$ be the module action. Let $\otimes$ denote the composite

$$\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp}) \times \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}, \mathrm{Sp} \times \mathcal{C}) \xrightarrow{+} \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp} \times \mathcal{C}) \xrightarrow{\mathrm{Lan}} \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$$

where the first functor is given componentwise, the second one by the monoidal structure on the integers and the third one is left Kan extension. In formulae,

$$(X_\bullet \otimes F_\bullet)_n \simeq \mathrm{colim}_{i+j \ge n} X_i \otimes F_j$$

for $X_\bullet \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$ and $F_\bullet \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$. It is easy to see that $\otimes$ preserves small colimits in each variable separately and equips $\mathrm{Fil}(\mathcal{C})$ with the structure of a $\mathrm{Fil}(\mathrm{Sp})$-module in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. In particular, for $F_\bullet$ as above, the functor $- \otimes F_\bullet$ admits a right adjoint $\hom^{\mathrm{fil}}_{\mathrm{Ind}(\mathcal{C}}(F_\bullet, -)$. Letting $F_\bullet$ vary produces a functor

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(-,-) : \mathrm{Fil}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathrm{Sp})$$

which equips $\mathrm{Fil}(\mathcal{C})$ with the structure of a $\mathrm{Fil}(\mathrm{Sp})$-enriched category.

Remark 2.14. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Taking the ind-completion $\mathrm{Ind}(\mathcal{C})$ and then applying Construction 2.13 produces $\mathrm{Fil}(\mathrm{Sp})$-enriched category $\mathrm{Fil}(\mathrm{Ind}(\mathcal{C}))$. This enrichment restricts to $\mathrm{Fil}(\mathcal{C})$ via the fully-faithful inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$.

Remark 2.15. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. To produce an enrichment of $\mathcal{C}$ in $\mathrm{Fil}(\mathrm{Sp})$ it suffices to produce a functor $\mathcal{C} \to \mathrm{Fil}(\mathcal{C})$. There are plenty of such functors, and in general, they produce non-equivalent enrichments. For example, the constant-filtration functor and the functor $x \mapsto \langle n, x\rangle$, defined for $n \in \mathbb{Z}$, are not equivalent. Notice that the various $x \mapsto \langle n, x\rangle$ provide equivalent enrichments.

Lemma 2.16. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Let $F_\bullet \in \mathrm{Fil}(\mathcal{C})$ and $r \in \mathbb{Z}$. Then there is a natural equivalence

$$\langle r, \mathbb{S}\rangle \otimes F_\bullet \simeq F_\bullet\langle -r\rangle$$

in $\mathrm{Fil}(\mathcal{C})$.

Proof. It is enough to compute the value in degree n. By the definition of Day convolution, $(\langle r, \mathbb{S}\rangle \otimes F_\bullet)_n \simeq \operatorname{colim}_{i+j\geq n}\langle r,\mathbb{S}\rangle_i \otimes F_j$. Since $\langle r,\mathbb{S}\rangle_i \simeq 0$ for $i > r$ and $\langle r,\mathbb{S}\rangle_i \simeq \mathbb{S}$ for $i \leq r$, this colimit reduces to $\operatorname{colim}_{i+j\geq n, i\leq r} F_j$. The condition that there exists $i \leq r$ such that $i + j \geq n$ is equivalent to $j \geq n - r$. Hence $(\langle r,\mathbb{S}\rangle \otimes F_\bullet)_n \simeq \operatorname{colim}_{j\geq n-r} F_j$. Now this colimit is taken in $\mathbb{Z}^{\mathrm{op}}$. The full subcategory on the objects $j \geq n-r$ has terminal object $n-r$, since for every $j \geq n-r$ there is a unique morphism $j \to n-r$ in $\mathbb{Z}^{\mathrm{op}}$. Therefore $\operatorname{colim}_{j\geq n-r} F_j \simeq F_{n-r}$. and it follows that $(\langle r,\mathbb{S}\rangle \otimes F_\bullet)_n \simeq F_{n-r} \simeq (F_\bullet\langle -r\rangle)_n$ and hence the claim. □

We shall now identify this enrichment.

Lemma 2.17. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $X_\bullet, Y_\bullet \in \mathrm{Fil}(\mathcal{C})$. Then for every $n \in \mathbb{Z}$ there is an equivalence

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet)_n \simeq \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Y_\bullet\langle n\rangle)$$

of spectra.

Proof. Consider the n-step filtration $\langle n, \mathbb{S}\rangle$ and compute

$$\begin{aligned}\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet)_n &\simeq \hom_{\mathrm{Fil}(\mathrm{Sp})}(\langle n,\mathbb{S}\rangle, \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet))\\ &\simeq \hom_{\mathrm{Fil}(\mathrm{Sp})}(\langle n,\mathbb{S}\rangle \otimes X_\bullet, Y_\bullet)\\ &\simeq \hom_{\mathrm{Fil}(\mathrm{Sp})}(X_\bullet\langle -n\rangle, Y_\bullet)\\ &\simeq \hom_{\mathrm{Fil}(\mathrm{Sp})}(X_\bullet, Y_\bullet\langle n\rangle).\end{aligned}$$

Here the first equivalence is by Corollary 2.5, the second one by adjunction, the third one by Lemma 2.16 and the last one since the shift by n is an equivalence. □

2.3. **A spectral sequence.** In [Lur17, Section 1.2.2] Lurie generalises the spectral sequence associated to a filtered chain complex in ordinary homological algebra to a spectral sequence associated to a cofiltered object in a stable category (notice our different convention of decreasing/increasing filtrations).

Remark 2.18. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equipped with a t-structure. Given a filtered object $X_\bullet \in \mathrm{Fil}(\mathcal{C})$, Lurie constructs a spectral sequence $E_1^{p,q} = \pi_{p+q}\mathrm{gr}_p(X_\bullet)$. If $\mathcal{C}$ admits sequential colimits and $\mathcal{C}_{\leq 0}$ is closed under such colimits and if $X_n \simeq 0$ for $n >> 0$, then [Lur17, Proposition 1.2.2.14] states that this spectral sequence converges

$$E_1^{p,q} = \pi_{p+q}\mathrm{gr}_p(X_\bullet) \Rightarrow \pi_{p+q}\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} X_\bullet$$

of the colimit in $\mathcal{C}^\heartsuit$.

Example 2.19. The hypotheses of [Lur17, Proposition 1.2.2.14] are certainly satisfied for Sp with its accessible natural t-structure.

We apply this philosophy to the Fil(Sp)-enrichment.

Corollary 2.20. Let $\mathcal{D}$ be a Fil(Sp)-enriched category and let $x, y \in \mathcal{D}$. If $\hom^{\mathrm{fil}}_{\mathcal{D}}(x,y)_n \simeq 0$ for $n >> 0$, then there exists a convergent spectral sequence

$$E_1^{p,q}(x,y) \coloneqq \pi_{p+q}\mathrm{gr}_p\hom^{\mathrm{fil}}_{\mathcal{D}}(x,y) \Rightarrow \pi_{p+q}(\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}}\hom^{\mathrm{fil}}_{\mathcal{D}}(x,y)_\bullet)$$

in spectra.

We are left to construct Fil(Sp)-enriched categories for which the filtered objects vanish in high degrees and for which the target of the spectral sequence can be easily described. Regarding this last problem, Lemma 2.17 suggests that the most well-behaved filtrations are the *finite* ones, that is, the ones that become constant for small indexes and zero for high indexes.

Definition 2.21. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $X \in \mathrm{Fil}(\mathcal{C})$ be a filtered object.

(1) We will say that $X_\bullet$ is *bounded above* if $X_n \simeq 0$ for every $n >> 0$.
(2) We will say that $X_\bullet$ is *bounded below* if $X_{n+1} \to X_n$ is an equivalence for every $n << 0$.
(3) We will say that $X_\bullet$ is *finite* if bounded above and bounded below.

We will denote by $\mathrm{Fil}^+(\mathcal{C})$, $\mathrm{Fil}^-(\mathcal{C})$, $\mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ the full subcategories of $\mathrm{Fil}(\mathcal{C})$ spanned by the bounded above, bounded below and finite filtered objects.

Remark 2.22. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. It is clear that the subcategories of bounded above, bounded below and finite filtered objects are stable full subcategories of $\mathrm{Fil}(\mathcal{C})$.

Notice that the bounded above condition controls the convergence of the spectral sequence of Remark 2.18 (it is an assumption!), whereas the bounded below condition controls the target of the spectral sequence (since it makes the colimit constant). Therefore:

Lemma 2.23. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equip it with a t-structure. Assume that $\mathcal{C}$ admits sequential colimits and $\mathcal{C}_{\leq 0}$ is closed under such colimits. If $X_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ is a finite filtered object, then the spectral sequence

$$E_1^{p,q} = \pi_{p+q}\mathrm{gr}_p(X_\bullet) \Rightarrow \pi_{p+q}\,\mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} X_\bullet$$

converges strongly. To be precise, the spectral sequence collapses at a finite stage.

Proof. If $X_\bullet$ is finite, then there exist integers $a \leq b$ such that $X_p \simeq 0$ for $p > b$ and the transition maps $X_{p+1} \to X_p$ are equivalences for $p < a$. It follows that $\mathrm{gr}_p(X_\bullet) \simeq 0$ for $p \notin [a,b]$, so the E_1-page has only finitely many nonzero columns. Moreover,

$$\mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} X_\bullet \simeq X_a \simeq X_{a-1} \simeq \cdots,$$

hence the filtration is eventually constant in low degrees, and the spectral sequence converges strongly to $\pi_*(\mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} X_\bullet)$. □

2.4. **Enrichment in finite filtered spectra.** As before, the goal is to find categories which are enriched in finite filtered spectra. We start with the following finite devissage result.

Lemma 2.24. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then every $F_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ may be obtained by finitely many extensions from the step filtrations $\langle r, \mathrm{gr}_r(F_\bullet)\rangle$.

Proof. Choose integers $a \leq b$ such that $F_n \simeq 0$ for $n > b$ and $F_{n+1} \to F_n$ is an equivalence for $n < a$. For every $r \in \{a-1, \ldots, b\}$ consider the filtered object $\sigma_{\leq r} F_\bullet \in \mathrm{Fil}(\mathcal{C})$. Then $\sigma_{\leq a-1} F_\bullet \simeq 0$, since for $n \leq a-1$ the map $F_a \to F_n$ is an equivalence, and $\sigma_{\leq b} F_\bullet \simeq F_\bullet$, because $F_{b+1} \simeq 0$. Since Lemma 2.11 implies that for every $r \in \{a, \ldots, b\}$ there is an exact sequence

$$\sigma_{\leq r-1} F_\bullet \to \sigma_{\leq r} F_\bullet \to \langle r, \mathrm{gr}_r(F_\bullet)\rangle,$$

starting from $\sigma_{\leq a-1} F_\bullet \simeq 0$ and iterating these extensions for $r = a, \ldots, b$, one obtains $\sigma_{\leq b} F_\bullet \simeq F_\bullet$ from the finitely many step filtrations $\langle r, \mathrm{gr}_r(F_\bullet)\rangle$. □

The analogue of Lemma 2.24 for bounded-above or bounded-below filtrations is not finite anymore. Rather, such a filtration is recovered as a sequential colimit of finite truncations, each of which is obtained by finitely many extensions from the corresponding step filtrations.

Corollary 2.25. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $F_\bullet \in \mathrm{Fil}(\mathcal{C})$. For $a \leq b$, define a finite truncation $F_\bullet[a,b] \in \mathrm{Fil}(\mathcal{C})$ by

$$F_n[a,b] \simeq \begin{cases} 0 & n > b, \\ F_n & a \leq n \leq b, \\ F_a & n < a, \end{cases}$$

with the evident transition maps. Then $F_\bullet[a,b] \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ and

$$\mathrm{gr}_r(F_\bullet[a,b]) \simeq \begin{cases} \mathrm{gr}_r(F_\bullet) & a \le r \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover:

(1) If $F_\bullet \in \mathrm{Fil}^+(\mathcal{C})$ and $F_n \simeq 0$ for $n > b$, then $F_\bullet \simeq \mathrm{colim}_{a\to-\infty} F_\bullet[a,b]$.
(2) If $F_\bullet \in \mathrm{Fil}^-(\mathcal{C})$ and $F_{n+1} \to F_n$ is an equivalence for $n < a$, then $F_\bullet \simeq \mathrm{colim}_{b\to+\infty} F_\bullet[a,b]$.

In particular, every object of $\mathrm{Fil}^+(\mathcal{C})$ or $\mathrm{Fil}^-(\mathcal{C})$ is a sequential colimit of finite filtrations, each of which is obtained by finitely many extensions from the corresponding step filtrations $\langle r, \mathrm{gr}_r(F_\bullet)\rangle$.

Proof. The first assertions are immediate from the definition: $F_\bullet[a,b]$ is zero in degrees $> b$ and constant in degrees $< a$, hence finite. Its graded pieces agree with those of $F_\bullet$ on the interval $[a,b]$ and vanish outside that interval. By Lemma 2.24, each $F_\bullet[a,b]$ is obtained by finitely many extensions from the step filtrations $\langle r, \mathrm{gr}_r(F_\bullet[a,b])\rangle$, hence from $\langle r, \mathrm{gr}_r(F_\bullet)\rangle$ for $a \le r \le b$.

For (1), assume that $F_\bullet \in \mathrm{Fil}^+(\mathcal{C})$ and choose b such that $F_n \simeq 0$ for $n > b$. There are evident maps $F_\bullet[a,b] \to F_\bullet[a-1,b]$ induced by the transition map $F_a \to F_{a-1}$. For every fixed n, once $a \le n$ one has $F_n[a,b] \simeq F_n$. Thus the natural map $\mathrm{colim}_{a\to-\infty} F_n[a,b] \to F_n$ is an equivalence for every n and hence the claim.

For (2), assume instead that $F_\bullet \in \mathrm{Fil}^-(\mathcal{C})$ and choose a such that $F_{n+1} \to F_n$ is an equivalence for $n < a$. There are evident maps $F_\bullet[a,b] \to F_\bullet[a,b+1]$. For every fixed n, once $b \ge n$ one has: if $n \ge a$, then $F_n[a,b] \simeq F_n$; if $n < a$, then $F_n[a,b] \simeq F_a \to F_n$ is an equivalence by assumption. Hence the natural map $\mathrm{colim}_{b\to+\infty} F_n[a,b] \to F_n$ is an equivalence for every n, and therefore the claim. □

We can now discuss the monoidal structure.

Proposition 2.26. The Day convolution restricts to symmetric monoidal structures on $\mathrm{Fil}^+(\mathrm{Sp})$, $\mathrm{Fil}^-(\mathrm{Sp})$ and $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$.

Proof. Compute first the tensor product of step filtrations. For every $a, b \in \mathbb{Z}$ and every spectra $E, F \in \mathrm{Sp}$, one has

$$\langle a, E\rangle \otimes_{\mathrm{Day}} \langle b, F\rangle \simeq \langle a+b, E \otimes_{\mathrm{Sp}} F\rangle.$$

Indeed, if $n \le a+b$, then the indexing category $\{(i,j) \mid i+j \ge n,\ i \le a,\ j \le b\}$ has a final object, for instance $(a, n-a)$, and therefore $(\langle a,E\rangle \otimes_{\mathrm{Day}} \langle b,F\rangle)_n \simeq E \otimes_{\mathrm{Sp}} F$. If instead $n > a+b$, the indexing category is empty, so the n-th term is zero.

For the actual claim, notice first that the unit is the step filtration $\langle 0, \mathbb{S}\rangle$, hence it belongs to $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$ and therefore also to $\mathrm{Fil}^+(\mathrm{Sp})$ and $\mathrm{Fil}^-(\mathrm{Sp})$. To show closure under tensor product in the three cases it is better to work separately.

(1) Consider the finite case. If $X_\bullet, Y_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$, choose integers $a \le b$ and $c \le d$ such that $X_n \simeq 0$ for $n > b$, $Y_n \simeq 0$ for $n > d$, and $X_{n+1} \to X_n$, $Y_{n+1} \to Y_n$ are equivalences for $n < a$, $n < c$, respectively. By Lemma 2.24, both $X_\bullet$ and $Y_\bullet$ are obtained by finitely many extensions from the step filtrations $\langle i, \mathrm{gr}_i(X_\bullet)\rangle$ and $\langle j, \mathrm{gr}_j(Y_\bullet)\rangle$ with $a \le i \le b$ and $c \le j \le d$. Since Day convolution preserves colimits separately in each variable, it is exact separately in each variable. Therefore $X_\bullet \otimes_{\mathrm{Day}} Y_\bullet$ is obtained by finitely many extensions from the step filtrations

$$\langle i+j, \mathrm{gr}_i(X_\bullet) \otimes_{\mathrm{Sp}} \mathrm{gr}_j(Y_\bullet)\rangle$$

via the above computation. In particular, all graded pieces are concentrated in the finite interval $[a+c, b+d]$, so $X_\bullet \otimes_{\mathrm{Day}} Y_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$.

(2) Consider the bounded above case. If $X_\bullet, Y_\bullet \in \mathrm{Fil}^+(\mathrm{Sp})$, choose integers b, d such that $X_n \simeq 0$ for $n > b$ and $Y_n \simeq 0$ for $n > d$. Then for $n > b+d$ there are no pairs (i,j) with $i+j \ge n$, $i \le b$, and $j \le d$. Hence $(X_\bullet \otimes_{\mathrm{Day}} Y_\bullet)_n \simeq 0$ for $n > b+d$, and therefore $X_\bullet \otimes_{\mathrm{Day}} Y_\bullet \in \mathrm{Fil}^+(\mathrm{Sp})$.

(3) Consider the bounded below case. Let $X_\bullet, Y_\bullet \in \mathrm{Fil}^-(\mathrm{Sp})$. Choose integers a, c such that $X_{n+1} \to X_n$ is an equivalence for $n < a$ and $Y_{n+1} \to Y_n$ is an equivalence for $n < c$. By Corollary 2.25, for every $b \geq a$ and $d \geq c$ the finite truncations $X_\bullet[a,b]$ and $Y_\bullet[c,d]$ lie in $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$, and

$$X_\bullet \simeq \mathrm{colim}_{b\to+\infty} X_\bullet[a,b], \qquad Y_\bullet \simeq \mathrm{colim}_{d\to+\infty} Y_\bullet[c,d].$$

Since Day convolution preserves colimits separately in each variable, one obtains

$$X_\bullet \otimes_{\mathrm{Day}} Y_\bullet \simeq \mathrm{colim}_{b,d}(X_\bullet[a,b] \otimes_{\mathrm{Day}} Y_\bullet[c,d]).$$

Each term on the right belongs to $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$ by the finite case, and its graded pieces are concentrated in degrees $[a+c, b+d]$. In particular, each $X_\bullet[a,b] \otimes_{\mathrm{Day}} Y_\bullet[c,d]$ is constant in degrees $< a+c$. Since this lower bound is independent of b and d, the colimit is again constant in degrees $< a+c$. Therefore $X_\bullet \otimes_{\mathrm{Day}} Y_\bullet \in \mathrm{Fil}^-(\mathrm{Sp})$.

Putting everything together shows the claim. □

Notice, however, that $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$ is not a presentable category, since it lacks small coproducts. In particular, the enrichment strategy of Construction 2.13 and Remark 2.14 via tensoring and restriction does not apply for the general small stable category. However, it still makes sense to enrich over $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$, and t-structures are useful in this sense.

Notation 2.27. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equipped with a t-structure. Given an object $x \in \mathcal{C}$, (co)truncating the functorial exact sequence $\tau_{\geq n}x \to x \to \tau_{\leq n-1}x$ produces the following two filtered objects:

(1) The *Whitehead tower* of x, that is

$$W(x) = (\cdots \to \tau_{\geq n+1}x \to \tau_{\geq n}x \to \tau_{\geq n-1}x \to \dots).$$

(2) The *Postnikov tower* of x, that is

$$P(x) = (\cdots \to \tau_{\leq n+1}x \to \tau_{\leq n}x \to \tau_{\leq n-1}x \to \dots).$$

Notice furthermore that they extend to functors $P, W : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$.

Remark 2.28. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equipped with a t-structure and let $x \in \mathcal{C}$ be an object. Then the Postnikov and Whitehead towers are related by taking an opposite, in that the Postnikov tower of $x \in \mathcal{C}$ is the Whitehead tower of $x^{\mathrm{op}} \in \mathcal{C}^{\mathrm{op}}$ with respect to the opposite t-structure.

Remark 2.29. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equipped with a t-structure and let $x \in \mathcal{C}$ be an object. Then:

(1) If $x \in \mathcal{C}^+$, then $x \in \mathcal{C}_{\leq r}$ for some $r \in \mathbb{Z}$. In particular, $\tau_{\geq r+n}x \simeq 0$ for all $n \in \mathbb{N}$. Thus the Whitehead and the Postnikov tower become constant on the left, in that

$$W(x) = (\cdots \to 0 \to \tau_{\geq r}x \to \tau_{\geq r-1}x \to \dots), \qquad P(x) = (\cdots \to x \to x \to \tau_{\leq r-1}x \to \dots).$$

Thus $W(x) \in \mathrm{Fil}^+(\mathcal{C})$ is bounded above.

(2) If $x \in \mathcal{C}^-$, then $x \in \mathcal{C}_{\geq r}$ for some $r \in \mathbb{Z}$. In particular, $\tau_{\leq r-n}x \simeq 0$ for all $n \in \mathbb{N}$. Thus the Whitehead and the Postnikov tower become constant on the right, in that

$$W(x) = (\cdots \to \tau_{\geq r+1}x \to x \to x \to \dots), \qquad P(x) = (\cdots \to \tau_{\leq r+1}x \to \tau_{\leq r}x \to 0 \to \dots).$$

Thus $W(x) \in \mathrm{Fil}^-(\mathcal{C})$ is bounded below.

In particular, if $x \in \mathcal{C}^{\mathrm{b}}$, then the Whitehead tower $W(x) \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ is finite in $\mathcal{C}$, and dually, the Postnikov tower $P(x) \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C}^{\mathrm{op}})$ is finite in $\mathcal{C}^{\mathrm{op}}$.

Lemma 2.30. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category with a t-structure. Let $X_\bullet \in \mathrm{Fil}(\mathcal{C})$ and $y \in \mathcal{C}$. Then:

(1) If $X_\bullet \in \mathrm{Fil}^-(\mathcal{C})$ and $y \in \mathcal{C}^+$, then $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \in \mathrm{Fil}^+(\mathrm{Sp})$.
(2) If $X_\bullet \in \mathrm{Fil}^+(\mathcal{C})$ and $y \in \mathcal{C}^-$, then $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \in \mathrm{Fil}^-(\mathrm{Sp})$.

In particular, if $X_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ and $y \in \mathcal{C}^b$, then $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \in \mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$.

Proof. Consider first (1). Since $X_\bullet \in \mathrm{Fil}^-(\mathcal{C})$, there exists $a \in \mathbb{Z}$ such that $X_{n+1} \to X_n$ is an equivalence for every $n < a$. Since $y \in \mathcal{C}^+$, there exists $d \in \mathbb{Z}$ such that $y \in \mathcal{C}_{\le d}$. Use Corollary 2.25 to get that for every $b \ge a$ the truncation $X_\bullet[a,b] \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$ is finite and $X_\bullet \simeq \mathrm{colim}_{b\to+\infty} X_\bullet[a,b]$. Apply now $\hom^{\mathrm{fil}}_{\mathcal{C}}(-, W(y))$ to get

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \simeq \lim_{b\to+\infty} \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y)).$$

Notice now that each filtered spectrum $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y))$ is bounded above, with a bound independent of b. Indeed, by Lemma 2.24, the finite filtration $X_\bullet[a,b]$ is obtained by finitely many extensions from the step filtrations $\langle r, \mathrm{gr}_r(X_\bullet)\rangle$ with $a \le r \le b$. Since $\hom^{\mathrm{fil}}_{\mathcal{C}}(-, W(y))$ is exact in the first variable, it is enough to prove the claim for a step filtration $X_\bullet = \langle r, x\rangle$ with $r \ge a$. Now Lemma 2.17 and Lemma 2.4 give

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(\langle r,x\rangle, W(y))_n \simeq \hom_{\mathrm{Fil}(\mathcal{C})}(\langle r,x\rangle, W(y)\langle n\rangle) \simeq \hom_{\mathcal{C}}(x, W(y)_{r+n}) \simeq \hom_{\mathcal{C}}(x, \tau_{\ge r+n} y)$$

for every $n \in \mathbb{Z}$. If $n > d-a$, then $r+n > d$ since $r \ge a$, hence $\tau_{\ge r+n} y \simeq 0$. Therefore $\hom^{\mathrm{fil}}_{\mathcal{C}}(\langle r,x\rangle, W(y))_n \simeq 0$ for all $n > d-a$. This bound is uniform in r and b, so each $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y))$ belongs to $\mathrm{Fil}^+(\mathrm{Sp})$ with the same upper bound. Taking the limit, one gets $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \in \mathrm{Fil}^+(\mathrm{Sp})$.

Consider now (2). Since $X_\bullet \in \mathrm{Fil}^+(\mathcal{C})$, there exists $b \in \mathbb{Z}$ such that $X_n \simeq 0$ for every $n > b$. Since $y \in \mathcal{C}^-$, there exists $c \in \mathbb{Z}$ such that $y \in \mathcal{C}_{\ge c}$. By Corollary 2.25, for every $a \le b$ the truncation $X_\bullet[a,b] \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$, and $X_\bullet \simeq \mathrm{colim}_{a\to-\infty} X_\bullet[a,b]$. Hence

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \simeq \lim_{a\to-\infty} \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y)).$$

Now each filtered spectrum $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y))$ is bounded below, with a bound independent of a. As above, it is enough to consider a step filtration $X_\bullet = \langle r, x\rangle$ with $r \le b$. For every $n \in \mathbb{Z}$ one has

$$\hom^{\mathrm{fil}}_{\mathcal{C}}(\langle r,x\rangle, W(y))_n \simeq \hom_{\mathcal{C}}(x, \tau_{\ge r+n} y).$$

If $n < c-b$, then $r+n < c$ since $r \le b$. Therefore both $\tau_{\ge r+n} y$ and $\tau_{\ge r+n+1} y$ are equivalent to y, and the structure map $\hom_{\mathcal{C}}(x, \tau_{\ge r+n+1} y) \to \hom_{\mathcal{C}}(x, \tau_{\ge r+n} y)$ is an equivalence. Hence $\hom^{\mathrm{fil}}_{\mathcal{C}}(\langle r,x\rangle, W(y))$ is constant in degrees $< c-b$. This bound is uniform in r and a, so each $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet[a,b], W(y))$ belongs to $\mathrm{Fil}^-(\mathrm{Sp})$ with the same lower bound. Passing to the limit yields $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, W(y)) \in \mathrm{Fil}^-(\mathrm{Sp})$. The "in particular" part is a consequence of (1)-(2). ☐

Proposition 2.31. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category with a t-structure. Then the bifunctor $\hom^W_{\mathcal{C}}(x,y) := \hom^{\mathrm{fil}}_{\mathcal{C}}(W(x), W(y))$ restricts to bifunctors

$$(\mathcal{C}^-)^{\mathrm{op}} \times \mathcal{C}^+ \to \mathrm{Fil}^+(\mathrm{Sp}), \qquad (\mathcal{C}^+)^{\mathrm{op}} \times \mathcal{C}^- \to \mathrm{Fil}^-(\mathrm{Sp}).$$

In particular, its restriction to $(\mathcal{C}^{\mathrm{b}})^{\mathrm{op}} \times \mathcal{C}^{\mathrm{b}}$ lands in $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$, providing a $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enriched structure on $\mathcal{C}^{\mathrm{b}}$ via the Whitehead tower.

Proof. By Construction 2.13, the category $\mathrm{Fil}(\mathcal{C})$ is enriched in $\mathrm{Fil}(\mathrm{Sp})$. Since the Whitehead tower defines a functor $W : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$, the composition $(x,y) \mapsto \hom^{\mathrm{fil}}_{\mathcal{C}}(W(x), W(y))$ produces a bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Fil}(\mathrm{Sp})$. The claim then follows by Remark 2.29 and Lemma 2.30. ☐

It is possible to start form a small stable category with a bounded t-structure and obtain the analogous result.

Remark 2.32. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure. Then [AGH18, Proposition 2.13] states that $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \subseteq \mathrm{Ind}(\mathcal{C})$ determines the connective part of an accessible t-structure on $\mathrm{Ind}(\mathcal{C})$ which is compatible with filtered colimits and such that the inclusion functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is t-exact. Moreover, if the t-structure on $\mathcal{C}$ is bounded below, then the t-structure on $\mathrm{Ind}(\mathcal{C})$ is right complete (and hence right separated).

Then:

Proposition 2.33. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a bounded t-structure. Then the Whitehead tower provides a $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment of $\mathcal{C}^{\mathrm{b}}$.

Proof. By Remark 2.32 the ind-completion $\mathrm{Ind}(\mathcal{C})$ carries a t-structure. Apply Proposition 2.31 to deduce that $\mathrm{Ind}(\mathcal{C})^{\mathrm{b}}$ inherits a $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment. Restrict it along the inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\mathrm{b}}$. ☐

Definition 2.34. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category with a t-structure. We denote by $(\mathcal{C}^{\mathrm{b}}, W)$ the Whitehead $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment on $\mathcal{C}^{\mathrm{b}}$ of Proposition 2.31 for $W := \hom^{W}_{\mathcal{C}}$.

3. The π_0-case

We can use the theory developed above to deduce interesting consequences on the homotopy category of a stable category with a bounded t-structure. We begin with a preliminary result.

Lemma 3.1. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and let $X_\bullet, Y_\bullet \in \mathrm{Fil}^{\mathrm{fin}}(\mathcal{C})$. Then for every $p \in \mathbb{Z}$ there is a natural equivalence

$$\mathrm{gr}_p \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet) \simeq \prod_{r \in \mathbb{Z}} \hom_{\mathcal{C}}(\mathrm{gr}_r X_\bullet, \mathrm{gr}_{r+p} Y_\bullet).$$

Since $X_\bullet$ and $Y_\bullet$ are finite, the product is finite.

Proof. Consider the filtered hom-object $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet) \in \mathrm{Fil}(\mathrm{Sp})$. The n-th component, for $n \in \mathbb{Z}$, of this filtered spectrum is computed via Lemma 2.17 as $\hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet)_n \simeq \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Y_\bullet\langle n\rangle)$. Therefore the gradient piece may be computed as

$$\mathrm{gr}_p \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet)_\bullet \simeq \mathrm{cofib}(\hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Y_\bullet\langle p+1\rangle) \to \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Y_\bullet\langle p\rangle)).$$

Since $\hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, -)$ is exact in the second variable, this gives

$$\mathrm{gr}_p \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet)_\bullet \simeq \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Q^p_\bullet(Y_\bullet)),$$

where $Q^p_\bullet(Y_\bullet) := \mathrm{cofib}(Y_\bullet\langle p+1\rangle \to Y_\bullet\langle p\rangle)$ is the cofibre of the shift $Y_\bullet\langle p+1\rangle \to Y_\bullet\langle p\rangle$. The goal is to compute it, and it suffices to do it pointwise. But for every $n \in \mathbb{Z}$,

$$Q^p_n(Y_\bullet) \simeq \mathrm{cofib}\big(Y_{n+p+1} \to Y_{n+p}\big) = \mathrm{gr}_{n+p}(Y_\bullet).$$

Moreover, the transition maps of $Q^p_\bullet(Y_\bullet)$ are zero. Indeed, the map $Q^p_{n+1}(Y_\bullet) \to Q^p_n(Y_\bullet)$ is induced by the commutative square

$$\begin{array}{ccc} Y_{n+p+2} & \longrightarrow & Y_{n+p+1} \\ \downarrow & & \downarrow \\ Y_{n+p+1} & \longrightarrow & Y_{n+p} \end{array}$$

and the induced map $\mathrm{cofib}(Y_{n+p+2} \to Y_{n+p+1}) \to \mathrm{cofib}(Y_{n+p+1} \to Y_{n+p})$ is zero because it comes from two composable morphisms. Using the delta of Notation 2.9, it follows that $Q^p_\bullet(Y_\bullet) \simeq \bigoplus_{r \in \mathbb{Z}} \delta_r(\mathrm{gr}_{r+p} Y_\bullet)$, where the sum is finite because $Y_\bullet$ is finite. Putting everything together and using Lemma 2.10 yield

$$\begin{aligned} \mathrm{gr}_p \hom^{\mathrm{fil}}_{\mathcal{C}}(X_\bullet, Y_\bullet) &\simeq \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, Q^p_\bullet(Y_\bullet)) \\ &\simeq \prod_{r \in \mathbb{Z}} \hom_{\mathrm{Fil}(\mathcal{C})}(X_\bullet, \delta_r(\mathrm{gr}_{r+p} Y_\bullet)) \\ &\simeq \prod_{r \in \mathbb{Z}} \hom_{\mathcal{C}}(\mathrm{gr}_r X_\bullet, \mathrm{gr}_{r+p} Y_\bullet), \end{aligned}$$

as claimed. □

We can now prove a generalization of [Ola21, Proposition 1].

Proposition 3.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a bounded t-structure, and let $x, y \in \mathcal{C}$. For every $p \in \mathbb{Z}$, define $F^p\mathrm{Hom}_{h\mathcal{C}}(x,y) := \mathrm{im}(\pi_0 W_p(x,y) \to \pi_0 \operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} W_\bullet(x,y))$. Then:

(1) There is a canonical identification $\pi_0 \operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} M_\bullet(x,y) \simeq \mathrm{Hom}_{h\mathcal{C}}(x,y)$. In particular, $F^\bullet$ is a finite decreasing filtration on $\mathrm{Hom}_{h\mathcal{C}}(x,y)$.
(2) The filtration is functorial in (x,y) and multiplicative, in that if $f \in F^p\mathrm{Hom}_{h\mathcal{C}}(x,y)$ and $g \in F^q\mathrm{Hom}_{h\mathcal{C}}(y,z)$, then $g \circ f \in F^{p+q}\mathrm{Hom}_{h\mathcal{C}}(x,z)$.
(3) There is a strongly convergent spectral sequence $E_1^{p,q}(x,y) := \pi_{p+q}\mathrm{gr}_p W_\bullet(x,y) \Rightarrow \pi_{p+q}\mathrm{hom}_{\mathcal{C}}(x,y)$ and its induced filtration on $\pi_0\mathrm{hom}_{\mathcal{C}}(x,y) = \mathrm{Hom}_{h\mathcal{C}}(x,y)$ is precisely $F^\bullet$.
(4) One has a natural identification
$$E_1^{p,q}(x,y) \simeq \prod_{r\in\mathbb{Z}} \pi_q \mathrm{hom}_{\mathcal{C}}(\pi_r(x), \pi_{r+p}(y)) = \prod_{r\in\mathbb{Z}} \mathrm{Ext}^{-q}_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y)).$$
(5) For every $p \geq 0$, the quotient $F^p\mathrm{Hom}_{h\mathcal{C}}(x,y)/F^{p+1}\mathrm{Hom}_{h\mathcal{C}}(x,y)$ is a subquotient of
$$\prod_{r\in\mathbb{Z}} \mathrm{Ext}^{p}_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y)).$$
(6) One has
$$F^1\mathrm{Hom}_{h\mathcal{C}}(x,y) = \ker(\mathrm{Hom}_{h\mathcal{C}}(x,y) \to \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_n(x), \pi_n(y))).$$

Proof. By Proposition 2.33, the Whitehead tower provides a $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$-enrichment on $\mathcal{C}$, denoted by $W_\bullet(x,y) = \mathrm{hom}^{\mathrm{fil}}_{\mathcal{C}}(W(x), W(y))$. Consider first (1). Since $W(x)$ is finite, choose integers $a \leq b$ such that $\mathrm{gr}_r W(x) \simeq 0$ for $r \notin [a,b]$. Since $y \in \mathcal{C}$ is bounded, pick $c \in \mathbb{Z}$ such that $y \in \mathcal{C}_{\geq c}$. If $p < c - b$, then for every $r \leq b$ one has $r + p < c$, hence $\tau_{\geq r+p} y \simeq y$. By Lemma 2.24, the finite filtration $W(x)$ is obtained by finitely many extensions from the step filtrations $\langle r, \mathrm{gr}_r W(x)\rangle$ for $a \leq r \leq b$. Since $\mathrm{hom}^{\mathrm{fil}}_{\mathcal{C}}(-, W(y))$ is exact in the first variable, it is enough to check the claim for a step filtration. For every r and every p one has, by the step computation in the proof of Lemma 2.30,
$$\mathrm{hom}^{\mathrm{fil}}_{\mathcal{C}}(\langle r, u\rangle, W(y))_p \simeq \mathrm{hom}_{\mathcal{C}}(u, \tau_{\geq r+p} y).$$
If $p < c - b$, this becomes
$$\mathrm{hom}^{\mathrm{fil}}_{\mathcal{C}}(\langle r, u\rangle, W(y))_p \simeq \mathrm{hom}_{\mathcal{C}}(u, y) \simeq \mathrm{hom}_{\mathrm{Fil}(\mathcal{C})}(\langle r, u\rangle, \mathrm{const}(y)).$$
By finite dévissage, it follows that
$$W_p(x,y) \simeq \mathrm{hom}_{\mathrm{Fil}(\mathcal{C})}(W(x), \mathrm{const}(y))$$
for $p \ll 0$. Since $\mathrm{const} : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ is right adjoint to $\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}}$, one gets
$$\mathrm{hom}_{\mathrm{Fil}(\mathcal{C})}(W(x), \mathrm{const}(y)) \simeq \mathrm{hom}_{\mathcal{C}}(\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} W(x), y).$$
As $\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} W(x) \simeq x$, this gives $W_p(x,y) \simeq \mathrm{hom}_{\mathcal{C}}(x,y)$ for $p \ll 0$. Since $W_\bullet(x,y)$ is finite, its colimit is computed by any sufficiently negative stage. Therefore $\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} W_\bullet(x,y) \simeq \mathrm{hom}_{\mathcal{C}}(x,y)$ and passing to π_0 proves (1).

Consider (2). Since $\mathcal{C}$ is enriched in $\mathrm{Fil}^{\mathrm{fin}}(\mathrm{Sp})$, composition is a morphism $W_\bullet(y,z) \otimes_{\mathrm{Day}} W_\bullet(x,y) \to W_\bullet(x,z)$ in $\mathrm{Fil}(\mathrm{Sp})$. Now for fixed $p, q \in \mathbb{Z}$, the pair (q,p) contributes to degree $p+q$ in the Day convolution, hence there is a canonical map $W_q(y,z) \otimes W_p(x,y) \to W_{p+q}(x,z)$. Therefore, if f and g are represented by classes in $\pi_0 W_p(x,y)$ and $\pi_0 W_q(y,z)$, then $g \circ f$ is represented by a class in $\pi_0 W_{p+q}(x,z)$. This proves
$$F^q\mathrm{Hom}_{h\mathcal{C}}(y,z) \circ F^p\mathrm{Hom}_{h\mathcal{C}}(x,y) \subseteq F^{p+q}\mathrm{Hom}_{h\mathcal{C}}(x,z),$$

that is, that the filtration is multiplicative. For (3), notice that, since $W_\bullet(x,y)$ is a finite filtered spectrum, Corollary 2.20 and Lemma 2.23 yield a strongly convergent spectral sequence $E_1^{p,q}(x,y) = \pi_{p+q}\mathrm{gr}_p M_\bullet(x,y) \Rightarrow \pi_{p+q}\operatorname{colim}_{\mathbb{Z}^{op}} W_\bullet(x,y) \simeq \pi_{p+q}\hom_{\mathcal{C}}(x,y)$. By construction, its induced filtration on $\pi_0\hom_{\mathcal{C}}(x,y)$ is exactly $F^\bullet$.

For (4), apply Lemma 3.1 to $W(x)$ and $W(y)$. Indeed, since $\mathrm{gr}_r W(x) \simeq \Sigma^r\pi_r(x)$ and $\mathrm{gr}_{r+p}W(y) \simeq \Sigma^{r+p}\pi_{r+p}(y)$, one gets

$$\mathrm{gr}_p W_\bullet(x,y) \simeq \prod_{r\in\mathbb{Z}} \hom_{\mathcal{C}}(\Sigma^r\pi_r(x), \Sigma^{r+p}\pi_{r+p}(y)) \simeq \prod_{r\in\mathbb{Z}} \hom_{\mathcal{C}}(\Sigma^{-p}\pi_r(x), \pi_{r+p}(y)).$$

Apply now π_{p+q} to get

$$\begin{aligned} E_1^{p,q}(x,y) &= \pi_{p+q}\mathrm{gr}_p W_\bullet(x,y) \\ &\simeq \prod_{r\in\mathbb{Z}} \pi_{p+q}\hom_{\mathcal{C}}(\Sigma^{-p}\pi_r(x), \pi_{r+p}(y)) \\ &\simeq \prod_{r\in\mathbb{Z}} \pi_0\hom_{\mathcal{C}}(\Sigma^{-p}\pi_r(x), \Sigma^{-p-q}\pi_{r+p}(y)) \\ &\simeq \prod_{r\in\mathbb{Z}} \pi_0\hom_{\mathcal{C}}(\Sigma^{q}\pi_r(x), \pi_{r+p}(y)) = \prod_{r\in\mathbb{Z}} \mathrm{Ext}^{-q}_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y)) \end{aligned}$$

and hence the claim. For (5), one has $F^p\mathrm{Hom}_{h\mathcal{C}}(x,y)/F^{p+1}\mathrm{Hom}_{h\mathcal{C}}(x,y) \simeq E_\infty^{p,-p}(x,y)$ and $E_\infty^{p,-p}(x,y)$ is a subquotient of $E_1^{p,-p}(x,y)$. Using (4) yields $E_1^{p,-p}(x,y) \simeq \mathrm{Ext}^p_{\mathcal{C}^\heartsuit}(\pi_r(x), \pi_{r+p}(y))$.

Finally, consider (6). For $p=0$, the previous computation gives

$$E_1^{0,0}(x,y) \simeq \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_n(x), \pi_n(y)).$$

On the other hand, if $p<0$ and $A, B \in \mathcal{C}^\heartsuit$, then $\mathrm{Ext}^p_{\mathcal{C}^\heartsuit}(A,B) = \pi_0\hom_{\mathcal{C}}(A, \Sigma^p B) = 0$ by orthogonality of the t-structure. Hence $E_1^{p,-p}(x,y) = 0$ for $p<0$. It follows that the edge morphism $\mathrm{Hom}_{h\mathcal{C}}(x,y) \to E_1^{0,0}(x,y)$ in total degree 0 has kernel equal to $F^1\mathrm{Hom}_{h\mathcal{C}}(x,y)$. It remains to identify this edge morphism. By construction, it sends a morphism $\varphi : x \to y$ to the induced morphisms on the graded pieces of the Whitehead towers $\mathrm{gr}_r W(x) \to \mathrm{gr}_r W(y)$. Under the identifications $\mathrm{gr}_r W(x) \simeq \Sigma^r\pi_r(x)$ and $\mathrm{gr}_r W(y) \simeq \Sigma^r\pi_r(y)$ this is exactly the family

$$(\pi_n(\varphi))_{n\in\mathbb{Z}} \in \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_n(x), \pi_n(y))$$

and therefore

$$F^1\mathrm{Hom}_{h\mathcal{C}}(x,y) \cong \ker(\mathrm{Hom}_{h\mathcal{C}}(x,y) \to \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_n(x), \pi_n(y))).$$

□

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