FINITE METHODS FOR STABLE ∞-CATEGORIES

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Abstract. We survey a collection of finite methods for stable ∞-categories, aimed at studying generation and dimension.

Contents

Introduction

The goal of these notes is to understand the notion of dimension of a stable category, or rather, the possible different notions of dimension, and the consequence of having finite dimension. We want to see how finiteness control produces representability results.

Linear overview.

Prerequisites. In these notes "category" means "∞-category". If we want to emphasise that a category has discrete mapping space, we will call it a "1-category". These notes assume that the reader is reasonably familiar with the basic notions of higher category theory. This includes: limits and colimits; cofinality; adjoint functors; Kan extensions and the pointwise formulas to compute them; the Yoneda embedding and its universal property. We assume familiarity with commutative algebra of categories.

Material. There is nothing new in these pages. All the material is extracted from [Lur17], [Aut18], The bibliographic references are probably rather incomplete. My apologies to everyone who does not receive the appropriate credit.

Notation. We adopt the use of universes, which we call small, large and very large. We let Spc denote the large category of small spaces and Cat the large category of small categories. With Pr^{L} we denote the very large category of presentable categories and left adjoints. We use the homological indexing convention for t-structures.

Disclaimer. I wrote various parts of these notes in the process of learning the notion of dimension(s) of a stable category, and decided only afterwards to put them together into a fully developed document. In order to make things more readable, I added some introductory sections (like).

1. Stable categories

In this chapter we collect the basic background on stable categories that will be used throughout the text. We begin with pointed categories and the definition of stability, and we discuss the formal properties that make stable categories behave as a convenient replacement for triangulated categories. We then review the main examples coming from spectra, module categories, and quasi-coherent sheaves. We conclude with a brief discussion of t-structures, mostly to fix notation and conventions for later use.

1.1. Pointed categories.

We begin by introducing pointed categories.

Definition 1.1.1. A category is called *pointed* if it admits an initial object $\varnothing$ and a terminal object $*$ and the canonical map $\varnothing \to *$ is an equivalence. In this case we will denote by 0 the initial/terminal object and refer to it as the *zero object*.

Notation 1.1.2. Let $\mathcal{C}$ be a pointed category. Given objects $x, y \in \mathcal{C}$ there are unique morphisms $x \to 0$ and $0 \to y$ to and from the zero object. We will denote their composite by $0 : x \to y$, and call it the *zero map*.

We let $\mathrm{Cat}^{\mathrm{pt}}$ denote the non-wide non-full subcategory of Cat spanned by the pointed categories and functors preserving the zero object. Let Cat^{*} denote the non-wide non-full subcategory of Cat spanned by the categories with a terminal object and functors which preserve the terminal object and similarly we let $\mathrm{Cat}^{\varnothing}$ be the non-wide non-full the subcategory of Cat spanned by categories with an initial object and functors which preserve the initial object. Then $\mathrm{Cat}^{\mathrm{pt}}$ identifies with a full subcategory of both Cat^{*} and $\mathrm{Cat}^{\varnothing}$.

There is a universal way to turn a category with a terminal object into a pointed category.

Construction 1.1.3. Let $\mathcal{C} \in \mathrm{Cat}^*$ be a category with a terminal object $* \in \mathcal{C}$. We define its *category of pointed objects* $\mathcal{C}_*$ via the following pullback square

$$\begin{array}{ccc} \mathcal{C}_* & \longrightarrow & \mathrm{Fun}([1], \mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ * & \xrightarrow[*]{} & \mathcal{C} \end{array}$$

Objects of $\mathcal{C}_*$ are objects $x \in \mathcal{C}$ equipped with a map $* \to x$ from the terminal object. It is clear that $\mathcal{C}_*$ is a pointed category (with zero object given by the point) and that the forgetful functor $\mathcal{C}_* \to \mathcal{C}$, given by evaluation at 1, is an equivalence if and only if $\mathcal{C}$ is pointed.

Lemma 1.1.4. The full inclusion of $\mathrm{Cat}^{\mathrm{pt}} \subseteq \mathrm{Cat}^*$ into pointed categories extends to an adjunction $\mathrm{incl} : \mathrm{Cat}^{\mathrm{pt}} \rightleftarrows \mathrm{Cat}^* : (-)_*$.

Proof. This is an application of [Lur09, Proposition 5.2.7.4]. In particular it suffices to construct a natural transformation $\varepsilon : \mathrm{incl} \circ (-)_* \to \mathrm{id}$ of functors $\mathrm{Cat}^* \to \mathrm{Cat}^*$ such that $(\varepsilon_{\mathcal{C}})_*, \varepsilon_{\mathcal{C}_*} : (\mathcal{C}_*)_* \to \mathcal{C}_*$ coincide for every $\mathcal{C} \in \mathrm{Cat}^*$. To achieve this, it suffices to define $\varepsilon_{\mathcal{C}} : \mathcal{C}_* \to \mathcal{C}$ to be evaluation at 1 for every $\mathcal{C} \in \mathrm{Cat}^*$. Indeed, since an object of $(\mathcal{C}_*)_*$ is a morphism $\mathrm{id}_* \to (* \to x)$ in $\mathcal{C}_*$, it must be unique, and both functors send it to $(* \to x)$. □

The above adjunction can be lifted to an adjunction of 2-categories.

Remark 1.1.5. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^*$ and assume that $\mathcal{D} \in \mathrm{Cat}^{\mathrm{pt}}$ is pointed. Then post-composition with the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence $\mathrm{Fun}^*(\mathcal{D}, \mathcal{C}_*) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$. Here $\mathrm{Fun}^*(-,-)$ denotes the full subcategory of the functor category spanned by those functors that preserve the terminal object. Indeed, by the Yoneda lemma it suffices to show that $\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, \mathrm{Fun}^*(\mathcal{D}, \mathcal{C}_*)) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, \mathrm{Fun}^*(\mathcal{D}, \mathcal{C}))$ for all $\mathcal{E} \in \mathrm{Cat}$. By currying and uncurrying, this is equivalent to show that the map $\mathrm{Hom}_{\mathrm{Cat}^*}(\mathcal{D}, \mathrm{Fun}(\mathcal{E}, \mathcal{C}_*)) \to \mathrm{Hom}_{\mathrm{Cat}^*}(\mathcal{D}, \mathrm{Fun}(\mathcal{E}, \mathcal{C}))$ is an equivalence. Since $\mathrm{Fun}(\mathcal{E}, -)$ preserves pullbacks we deduce that $\mathrm{Fun}(\mathcal{E}, \mathcal{C}_*) \simeq \mathrm{Fun}(\mathcal{E}, \mathcal{C})_*$, so this is an instance of the adjunction from Lemma 1.1.4.

Remark 1.1.6. Let $\mathcal{C} \in \mathrm{Cat}^*$ be a category with a terminal object. If $\mathcal{C}$ admits pullbacks, then $\mathcal{C}_*$ admits pullbacks and the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ preserves them. Similarly, if $\mathcal{C}$ admits pushouts, then $\mathcal{C}_*$ admits pushouts and the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ preserves them.

1.2. **Stable categories.** We now turn our attention to stable categories.

Definition 1.2.1. A category is called *stable* if it is pointed, admits finite limits and finite colimits, and a commutative square is a pullback if and only if it is a pushout.

Notation 1.2.2. Let $\mathcal{C}$ be a stable category. We will say that a nullsequence $x \to y \to z$ is *exact* if the associated square

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array}$$

is an exact square, that is, it is both a pullback and pushout square.

Definition 1.2.3. A pointed category $\mathcal{A}$ is called *semiadditive* if it has finite products and coproducts and the canonical map $x \sqcup y \to x \times y$ is an equivalence for every $x, y \in \mathcal{A}$. In this case, we will refer to biproducts in $\mathcal{A}$ as *direct sums* and denote them by $x \oplus y$.

Remark 1.2.4. Let $\mathcal{A}$ be a semiadditive category. Then for every pair of objects $x, y \in \mathcal{A}$ the set $\pi_0 \mathrm{Hom}_{\mathcal{A}}(x, y)$ can be equipped with the structure of an abelian monoid. Given morphisms $f, g : x \to y$, their sum $f + g$ is defined as $x \to x \oplus x \xrightarrow{f \oplus g} y \oplus y \to y$ via the obvious diagonal and codiagonal maps. It is a rather tedious exercise to check that this operation is indeed unital, associative and commutative.

To get on $\pi_0 \operatorname{Hom}_{\mathcal{A}}(x, y)$ the structure of an abelian group it is sufficient to define, for every morphism $f : x \to y$, its negative $-f : x \to y$ by requiring the shear map

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix} : z \oplus z \to z \oplus z$$

to be an equivalence for every $z \in \mathcal{A}$.

Definition 1.2.5. A category is *additive* if it is semiadditive and every shear map is an equivalence. We let $\mathrm{Cat}^{\mathrm{add}}$ the non-full non-wide subcategory of Cat spanned by the additive categories and the functor preserving direct sums.

Lemma 1.2.6. Every stable category is additive.

Proof. Let $\mathcal{C}$ be a stable category. Since $\mathcal{C}$ is pointed and admits every finite product and coproduct, to check that it is semiadditive it suffices to check that the canonical map $x \sqcup y \to x \times y$ is an equivalence for every $x, y \in \mathcal{C}$. For that it suffices to note that the squares

$$\begin{array}{ccc} 0 & \longrightarrow & x \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ 0 & \longrightarrow & x \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ y & \xrightarrow[\mathrm{id}]{} & y \end{array}$$

are pullbacks, so their product is also a pullback. Since $\mathcal{C}$ is stable, this square is also a pushout, thus showing semiadditivity. For additivity, notice that the shear map on x fits into a map of exact sequences

$$\begin{array}{ccccc} x & \xrightarrow{i_2} & x \oplus x & \xrightarrow{p_1} & x \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{shear}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ x & \xrightarrow[i_2]{} & x \oplus x & \xrightarrow[p_1]{} & x \end{array}$$

This implies that the shear map is an equivalence. □

Lemma 1.2.7. Let $\mathcal{C}$ be a pointed category with finite colimits, and let $\mathcal{D}$ be a pointed category with finite limits. Then a pointed functor $f : \mathcal{C} \to \mathcal{D}$ carries pushouts to pullbacks[1] if and only if the canonical map $\eta_x : f(x) \to \Omega_{\mathcal{D}} f(\Sigma_{\mathcal{C}} x)$ is an equivalence for every $x \in \mathcal{C}$.

Proof. The implication (⇒) is by definition, so consider (⇐). Assume that the canonical map $\eta_x : f(x) \to \Omega_{\mathcal{D}} f(\Sigma_{\mathcal{C}} x)$ is an equivalence for every $x \in \mathcal{C}$. Consider a pushout

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}$$

in $\mathcal{C}$. Applying f to the commutative diagram

$$\begin{array}{ccccccc} w & \longrightarrow & x & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ z & \longrightarrow & y & \longrightarrow & \mathrm{cofib}(w \to z) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(w \to x) & \longrightarrow & \Sigma w & \longrightarrow & \Sigma x \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Sigma z & \longrightarrow & \Sigma y \end{array}$$

in which every square is a pushout, produces maps $f(w) \to f(x) \times_{f(y)} f(z) \to \Omega f(\Sigma w) \to \Omega f(\Sigma x) \times_{\Omega f(\Sigma y)} \Omega f(\Sigma z)$, and by the 2-out of-6 it follows that f carries pushouts to pullbacks. □

[1] That is, it is reduced and 1-excisive.

Lemma 1.2.8. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{rex,pt}}$ be a pointed category with finite colimits.

(1) The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful if and only if every pushout square in $\mathcal{C}$ is also a pullback square.

(2) If the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful, then every commutative square

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}$$

for which the induced map $\mathrm{cofib}(w \to x) \to \mathrm{cofib}(z \to y)$ is an equivalence, is a pullback.

(3) The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence if and only if $\mathcal{C}$ stable.

Proof. Consider (1). If pushout squares in $\mathcal{C}$ are pullback squares, then the pushout Σx of the cospan $0 \leftarrow x \to 0$ shows that x is a pullback of the span $0 \to \Sigma x \leftarrow 0$. In particular, $\mathrm{Hom}_{\mathcal{C}}(y, x) \simeq \Omega\,\mathrm{Hom}_{\mathcal{C}}(y, \Sigma x) \simeq \mathrm{Hom}_{\mathcal{C}}(\Sigma y, \Sigma x)$ so that Σ is fully-faithful. Conversely, assume that Σ is fully-faithful and consider the pushout Σ of the cospan of functors $\mathrm{const}_0 \leftarrow \mathrm{id} \to \mathrm{const}_0$. To check that the identity is the pullback of the span $\mathrm{const}_0 \to \Sigma \leftarrow \mathrm{const}_0$ apply $\mathrm{Hom}_{\mathcal{C}}(x, -)$ for every $x \in \mathcal{C}$ and notice that the resulting square is a pullback. Since $\mathrm{Hom}_{\mathcal{C}}(x, -)$ is reduced, Lemma 1.2.7 implies $\mathrm{Hom}_{\mathcal{C}}(x, -)$ that sends pushouts to pullbacks, which precisely means that pushout squares in $\mathcal{C}$ are pullback squares.

Consider (2) and consider the map of cofibre sequences

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(t, w) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(t, x) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(t, \mathrm{cofib}(w \to x)) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \mathrm{Hom}_{\mathcal{C}}(t, z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(t, y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(t, \mathrm{cofib}(z \to y)) \end{array}$$

Since Σ is fully-faithful, by (1) the above can be regarded as a map of fibre sequences. Thus $w \simeq x \times_y z$.

For (3), if $\mathcal{C}$ is stable, then the square defining Ω and Σ are exact, implying that Σ is invertible. Conversely, if Σ is an equivalence then (1) implies that pushout squares in $\mathcal{C}$ are pullback squares so that it suffices to show that $\mathcal{C}$ admits finite limits, and that pullback squares in $\mathcal{C}$ are pushout squares. Given a span $x \to y \leftarrow z$ define $w = \mathrm{cofib}(\Sigma^{-1} x \to \Sigma^{-1}\mathrm{cofib}(z \to y))$. Then this object fits canonically into a square

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}$$

and the induced map on horizontal cofibres is an equivalence. Thus (2) implies that $w \simeq x \times_y z$ so that $\mathcal{C}$ has finite limits. To conclude that pullback squares in $\mathcal{C}$ are pushout squares notice that Ω is an equivalence (since Σ is), so that an application of (3) to $\mathcal{C}^{\mathrm{op}}$ concludes. □

Corollary 1.2.9. Let $\mathcal{C}$ be a stable category. Then a commutative square

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}$$

in $\mathcal{C}$ is a pushout if and only if the induced map $\mathrm{cofib}(w \to x) \to \mathrm{cofib}(y \to z)$ is an equivalence.

Proof. The implication ($\Rightarrow$) holds in every category, whereas the implication ($\Leftarrow$) follows from Lemma 1.2.8 and stability. □

Corollary 1.2.10. Let $f : \mathcal{C} \to \mathcal{D}$ be a pointed functor between stable categories. The following are equivalent:

(1) The functor f preserves finite colimits.

(2) The functor f preserves suspensions.

(3) The functor f preserves finite limits.

(4) The functor f preserves desuspensions.

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) are by definition. The converse implications follow from Lemma 1.2.8. Finally, (2) is equivalent to (4) since Σ and Ω are mutually inverse functors. □

Definition 1.2.11. We will say that a functor between stable categories is *exact* if it is both left and right exact.

Definition 1.2.12. We will denote by $\mathrm{Cat}^{\mathrm{st}}$ the non-wide non-full subcategory of Cat spanned by the stable categories and the exact functors between them.

We conclude this section by discussing some examples.

Example 1.2.13. Let Ab be the category of abelian groups and denote by $\mathrm{Ab}^{\mathbb{Z}}$ the category of graded abelian groups. A *generalized cohomology theory* is a pair (E^*, ∂) consisting of a functor $E^* : \mathrm{Spc}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ and a natural isomorphism $\partial : E^* \to E^{*+1} \circ \Sigma$ such that:

(1) For every small collection of pointed spaces $(X_i)_{i\in I}$, the inclusions $X_i \hookrightarrow \vee_{i\in I} X_i$ induce an isomorphism of graded abelian groups $E^*(\vee_{i\in I} X_i) \to \prod_{i\in I} E^*(X_i)$.

(2) The functor E^* sends cofibre sequences to exact sequences of graded abelian groups.

The natural isomorphism ∂ is called the *suspension isomorphism*, and conditions (1)-(2) are also called the *wedge axiom* and the *exactness axiom*. The famous Brown representability theorem [Bro62, Theorem 1] implies that every generalized cohomology theory can be represented by a spectrum E, that is by a sequence of pointed spaces $(E_n)_{n\in N}$ with equivalences $E_n \to \Omega E_{n+1}$ for every $n \in \mathbb{N}$. Conversely, every spectrum E determines a cohomology theory by declaring $E^n(-) = \pi_0 \operatorname{Hom}_{\mathrm{Spc}_*}(-, E_n)$. Spectra, together with the relevant notion of morphism of spectra, can be organized into a stable category Sp, defined as the colimit

$$\mathrm{Sp} = \operatorname{colim}(\mathrm{Spc}_* \xrightarrow{\Sigma} \mathrm{Spc}_* \xrightarrow{\Sigma} \dots)$$

in Pr^{L}, or dually, as the limit

$$\mathrm{Sp} \simeq \lim(\dots \xrightarrow{\Omega} \mathrm{Spc}_* \xrightarrow{\Omega} \mathrm{Spc}_*)$$

in Pr^{R}, the category of presentable categories and right adjoints. This category is related to spaces through an adjunction $\Sigma^\infty_+ : \mathrm{Spc} \rightleftarrows \mathrm{Sp} : \Omega^\infty$.

Example 1.2.14. Let $\mathcal{A}$ be an abelian category. Consider the 1-category $\mathrm{Ch}(\mathcal{A})$ of unbounded chain complexes in $\mathcal{A}$, with chain maps as morphisms. Recall that a map of chain complexes is a *quasi-isomorphism* if it induces isomorphism on homology groups. Let W be the class of quasi-isomorphisms. The *derived category of A* is the localization $\mathrm{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[W^{-1}]$. Since taking the homotopy category furnishes a left adjoint $h : \mathrm{Cat} \to \mathrm{Cat}_1$ to the inclusion of 1-categories into categories, the homotopy category $h\mathrm{D}(\mathcal{A})$ identifies with the ordinary derived category of $\mathcal{A}$. Regarding stability, [Lur17, Proposition 1.3.5.9] shows that if $\mathcal{A}$ is *Grothendieck abelian* (that is, it is presentable and the collection of monomorphisms is closed under small filtered colimits), then $\mathrm{D}(\mathcal{A})$ is a stable category. The zero object is given by the zero object of $\mathrm{Ch}(\mathcal{A})$ and the suspension functor Σ is represented by the shift in $\mathrm{Ch}(\mathcal{A})$, that is $C \mapsto \Sigma C$ defined by $(\Sigma C)_n = C_{n-1}$. The exact sequences of $\mathrm{D}(\mathcal{A})$ are given by the image trough the localization of short exact sequences in $\mathrm{Ch}(\mathcal{A})$.

Notation 1.2.15. Given a commuative ring A, we let $D(A)$ denote the stable category of A-modules, constructed via the previous example. Notice that $D(A) \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{L,\omega}_{\mathrm{st}})$ is a rigidly-compactly generated stable category, that is, it is compactly generated by the dualizable object, when equipped with the closed monoidal structure given by the derived tensor product.

Example 1.2.16. The previous example may be generalized to $\mathbb{E}_\infty$-ring as follows. In subsection 2.3 we will construct a closed symmetric monoidal structure on Sp, the *smash* product, which is compatible with colimits and for which the sphere spectrum $\mathbb{S} \simeq \Sigma^\infty_+(*)$ may be identified with the unit.

Commutative algebra obejcts therein CAlg(Sp) are called $\mathbb{E}_\infty$*-rings*. Given an $\mathbb{E}_\infty$-ring $A \in \mathrm{CAlg}(\mathrm{Sp})$, the category of A-modules $\mathrm{Mod}_A \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ is a rigidly-compaclty generated stable category, whose compact/dualizable object are the perfect ones Perf_A. Furthermore, every classical ring A identifies an $\mathbb{E}_\infty$-ring HA, the *Eilenberg-MacLane* spectrum of A, and $\mathrm{Mod}_{HA} \simeq D(\mathrm{Mod}^\heartsuit_A)$ identifies with the stable derived category of A-modules. In the following, we will suppress the funcotr H, and denote by $\mathrm{Mod}_A = D(\mathrm{Mod}^\heartsuit_A)$ the stable derived category of A-modules.

Example 1.2.17. Let X be a quasi-compact quasi-separated scheme. Then the derived stable category of *quasi-coherent sheaves on* X is defined as the limit

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec}(A)\to X} \mathrm{Mod}_A.$$

If X has affine diagonal (in particular, if it is separated), then $\mathrm{QCoh}(X)$ is the usual derived category of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology constructed in [Aut18, Chapter 08CU]. In any case, the above definition mathches the intuiton that "a complex of quasi-coherent sheaves on X should be a famliy of complexes of modules on each affine piece, plus the gluing data". It follows formally that $\mathrm{QCoh}(X) \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ is a rigidly-compactly generated stable category.

Example 1.2.18. Let X be a quasi-compact quasi-separated scheme. The category $\mathrm{QCoh}(X)$ contains two small idempotent-complete stable subcategories that we wil study in the following chapters. Namely:

(1) The full subcategory of *perfect complexes* $\mathrm{Perf}(X)$ is defined to be the full subcategory of compact objects of $\mathrm{QCoh}(X)$, or equivalently, the one spanned by the dualizable ones.
(2) The full subcategory of *bounded coherent complexes* $\mathrm{Coh}(X)$, whose construction is fairly techinical (see [**?**, text]). If X is noetherian, $\mathrm{Coh}(X)$ is identified with $D^b_{\mathrm{coh}}(X)$, the full subcategory spanned by the bounded chain complexes with coherent cohomology.

In general, there is an inclusion $\mathrm{Perf}(X) \subseteq \mathrm{Coh}(X)$. Furthermore, these subcategories are also *idempotent-complete*, a property that we will study in Section 2

1.3. t-structures.

Definition 1.3.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. A t-structure on $\mathcal{C}$ is a pair of full subcategory $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that:

(1) The pair is *orthogonal*, that is, for every $x \in \mathcal{C}_{\geq 0}$ and every $y \in \mathcal{C}_{\leq 0}$ it is $\hom_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq *$.
(2) The pair is *stable under shifts*, that is, $\Sigma\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$ and $\Sigma^{-1}\mathcal{C}_{\leq 0} \subseteq \mathcal{C}_{\leq 0}$.
(3) The pair admits *truncations*, that is, for every $x \in \mathcal{C}$ there exists an exact sequence $x_{\geq 1} \to x \to x_{\leq 0}$ with $x_{\geq 1} \in \Sigma\mathcal{C}_{\geq 0}$ and $x_{\leq 0} \in \mathcal{C}_{\leq 0}$.

Notation 1.3.2. For $n \in \mathbb{Z}$ we set $\mathcal{C}_{\geq n} \coloneqq \Sigma^n \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq n} \coloneqq \Sigma^n \mathcal{C}_{\leq 0}$. We denote by $i_{\geq n} : \mathcal{C}_{\geq n} \hookrightarrow \mathcal{C}$ and $i_{\leq n} : \mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ the inclusions. Observe that we use a *homological indexing convention.*

Lemma 1.3.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then there are adjunctions $\tau_{\leq n} : \mathcal{C} \rightleftarrows \mathcal{C}_{\leq n} : i_{\leq n}$ and $i_{\geq n} : \mathcal{C}_{\geq n} \rightleftarrows \mathcal{C} : \tau_{\geq n}$ for every $n \in \mathbb{Z}$.

Proof. The case $n = 0$ and $n = 1$ is treated; the general case follows by shifting. To prove the adjunction $\tau_{\leq 0} \dashv i_{\leq 0}$, consider an object $x \in \mathcal{C}$ and choose an exact sequence $x_{\geq 1} \to x \to x_{\leq 0}$, with $x_{\geq 1} \in \mathcal{C}_{\geq 1}$, $x_{\leq 0} \in \mathcal{C}_{\leq 0}$ as provided by the t-structure axioms: For $y \in \mathcal{C}_{\leq 0}$, applying $\hom_{\mathcal{C}}(-, y)$ yields a fiber sequence

$$\hom_{\mathcal{C}}(x_{\leq 0}, y) \to \hom_{\mathcal{C}}(x, y) \to \hom_{\mathcal{C}}(x_{\geq 1}, y).$$

Since $x_{\geq 1} \in \mathcal{C}_{\geq 1} = \Sigma\mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, orthogonality implies $\hom_{\mathcal{C}}(x_{\geq 1}, y) \simeq *$. Hence the map $\hom_{\mathcal{C}}(x_{\leq 0}, y) \to \hom_{\mathcal{C}}(x, y)$ is an equivalence. This identifies $x \mapsto x_{\leq 0}$ as a left adjoint to the inclusion $i_{\leq 0} : \mathcal{C}_{\leq 0} \hookrightarrow \mathcal{C}$; it is denoted by $\tau_{\leq 0}$.

For the adjunction $i_{\geq 1} \dashv \tau_{\geq 1}$, take $z \in \mathcal{C}_{\geq 1}$, applying $\hom_{\mathcal{C}}(z,-)$ to the same cofiber sequence gives a fiber sequence

$$\hom_{\mathcal{C}}(z, x_{\geq 1}) \to \hom_{\mathcal{C}}(z,x) \to \hom_{\mathcal{C}}(z, x_{\leq 0}).$$

Since $z \in \mathcal{C}_{\geq 1}$ and $x_{\leq 0} \in \mathcal{C}_{\leq 0}$, orthogonality implies $\hom_{\mathcal{C}}(z, x_{\leq 0}) \simeq *$. Therefore $\hom_{\mathcal{C}}(z, x_{\geq 1}) \to \hom_{\mathcal{C}}(z,x)$ is an equivalence. This identifies $x \mapsto x_{\geq 1}$ as a right adjoint to the inclusion $i_{\geq 1} : \mathcal{C}_{\geq 1} \hookrightarrow \mathcal{C}$; it is denoted by $\tau_{\geq 1}$. □

Definition 1.3.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. The *heart* of the t-structure is defined as the intersection $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$. For each $n \in \mathbb{Z}$, we let $\pi_0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the functor $\tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0}$, and we let $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the composition $\pi_0 \circ \Sigma^{-n}$.

Proposition 1.3.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then the heart $\mathcal{C}^{\heartsuit}$ is an abelian 1-category.

Proof. For $a, b \in \mathcal{C}^{\heartsuit}$ and $m \in \mathbb{Z}$ one has $\pi_m \hom_{\mathcal{C}}(a,b) \simeq \pi_0 \hom_{\mathcal{C}}(\Sigma^m a, b)$. If $m > 0$, then $\Sigma^m a \in \mathcal{C}_{\geq 1}$ and $b \in \mathcal{C}_{\leq 0}$, hence $\hom_{\mathcal{C}}(\Sigma^m a, b) \simeq *$ by orthogonality; similarly, if $m < 0$ then $\Sigma^m a \in \mathcal{C}_{\leq -1}$ and $b \in \mathcal{C}_{\geq 0}$, hence $\hom_{\mathcal{C}}(\Sigma^m a, b) \simeq *$. Therefore $\pi_m \hom_{\mathcal{C}}(a,b) = 0$ for $m \neq 0$, so $\mathcal{C}^{\heartsuit}$ is 0-truncated (its mapping spaces are discrete), and the sets $\mathrm{Hom}_{\mathcal{C}^{\heartsuit}}(a,b) := \pi_0 \hom_{\mathcal{C}}(a,b)$ are abelian groups.

Since $\mathcal{C}$ is stable, finite biproducts exist, and $\mathcal{C}^{\heartsuit}$ is closed under them; hence $\mathcal{C}^{\heartsuit}$ is additive. Let $f : a \to b$ be a morphism in $\mathcal{C}^{\heartsuit}$. Consider the fiber and cofiber sequences $\mathrm{fib}(f) \to a \to b$ and $a \to b \to \mathrm{cofib}(f)$ in $\mathcal{C}$. Since $a, b \in \mathcal{C}_{\geq 0}$, one has $\mathrm{fib}(f) \in \mathcal{C}_{\geq 0}$, and therefore $\tau_{\leq 0}\mathrm{fib}(f) \in \mathcal{C}^{\heartsuit}$. Define then $\ker(f) := \tau_{\leq 0}\mathrm{fib}(f) \in \mathcal{C}^{\heartsuit}$. Dually, since $a, b \in \mathcal{C}_{\leq 0}$, one has $\mathrm{cofib}(f) \in \mathcal{C}_{\leq 0}$, and therefore $\tau_{\geq 0}\mathrm{cofib}(f) \in \mathcal{C}^{\heartsuit}$. Define $\mathrm{coker}(f) := \tau_{\geq 0}\mathrm{cofib}(f) \in \mathcal{C}^{\heartsuit}$. The universal properties of $\ker(f)$ and $\mathrm{coker}(f)$ follow by applying $\hom_{\mathcal{C}}(x,-)$ and $\hom_{\mathcal{C}}(-,x)$ for $x \in \mathcal{C}^{\heartsuit}$ and using orthogonality to identify maps to/from $\mathrm{fib}(f)$ and $\mathrm{cofib}(f)$ with maps to/from their truncations. Finally, consider the canonical morphism $\mathrm{coker}(\ker(f) \to a) \to \ker(b \to \mathrm{coker}(f))$ in $\mathcal{C}^{\heartsuit}$. Its fiber and cofiber lie in $\mathcal{C}_{\geq 1} \cap \mathcal{C}_{\leq -1} = 0$, implying that the above map is an isomorphism. Therefore $\mathcal{C}^{\heartsuit}$ is additive and has kernels and cokernels, and the coimage is equivalent to the image. □

There are several conditions one can impose on a t-structure. We list some of them.

Remark 1.3.6. Let $\mathcal{C}$ be a stable category and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure.

(1) We will say that the t-structure is:
- *Left bounded* if $\mathcal{C} = \cup_{n \in \mathbb{Z}} \mathcal{C}_{\geq n}$.
- *Right bounded* if $\mathcal{C} = \cup_{n \in \mathbb{Z}} \mathcal{C}_{\leq n}$.
- *Bounded* if left and right bounded.

(2) We will say that the t-structure is:
- *Left separated* if $\bigcap_{n \in \mathbb{Z}} \mathcal{C}_{\geq n} \simeq 0$. In other terms, the t-structure is left separated if there are no ∞-connective objects.
- *Right separated* if $\bigcap_{n \in \mathbb{Z}} \mathcal{C}_{\leq n} \simeq 0$. In other terms, the t-structure is right separated if there are no ∞-coconnective objects.
- If both left and right separated hold, one often says that the t-structure is *non-degenerate.*

(3) Another set of conditions is completeness. For $x \in \mathcal{C}$, the maps $\tau_{\leq n} x \to \tau_{\leq n-1} x$ assemble into the Postnikov tower $\cdots \to \tau_{\leq 1} x \to \tau_{\leq 0} x \to \tau_{\leq -1} x \to \cdots$. We say that the t-structure is *left complete* if the canonical map

$$x \to \lim_{n \to +\infty} \tau_{\leq n} x$$

is an equivalence for every $x \in \mathcal{C}$. Dually, the maps $\tau_{\geq -n} x \to x$ factor through $\tau_{\geq -(n+1)} x$, yielding a direct system $\tau_{\geq 0} x \to \tau_{\geq -1} x \to \tau_{\geq -2} x \to \cdots$. We say that the t-structure is *right complete* if the canonical map

$$\mathrm{colim}_{n \to +\infty} \tau_{\geq -n} x \to x$$

is an equivalence for every $x \in \mathcal{C}$.

It is not hard to see that left complete implies left separated, and right complete implies right separated. The converse implications are not true in general; see [Lur17, Proposition 1.2.1.19]. Assume now that $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is presentable.

(1) We say that the t-structure is *compatible with filtered colimits* if $\mathcal{C}_{\geq 0}$ is closed under filtered colimits in $\mathcal{C}$. Equivalently, each truncation functor $\tau_{\geq n}$ preserves filtered colimits (and hence so does $\tau_{\leq n}$, since it fits into the truncation triangle).

(2) We say that the t-structure is *accessible* if $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is an accessible subcategory and the inclusion $i_{\geq 0} \colon \mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ is accessible (equivalently, $\tau_{\geq 0}$ is an accessible functor). In practice, accessible t-structures are the ones for which the connective part is generated by a set of objects.

Example 1.3.7. Let X be a quasi-compact quasi-separated scheme. Then $\mathrm{QCoh}(X)$ admits an accessible t-structure which is right and left complete, compatible with filtered colimits, and compatible with the tensor product (in the sense that the monoidal unit is connective and that the tensor product of two connective quasi-coherent sheaves is connective). Furthermore, it is compactly generated by the perfect connective objects $\mathrm{Perf}(X)_{\geq 0} = \mathrm{Perf}(X) \cap \mathrm{QCoh}(X)_{\geq 0}$. Finally, the heart $\mathrm{QCoh}(X)^{\heartsuit}$ identifies with the abelian 1-category of quasi-coherent sheaves on X. If X is noetherian (or more generally coherent), then the t-structure restricts to a bounded t-structure on $\mathrm{Coh}(X)$.

2. Idempotent-completion

In this chapter we study idempotent-completion for stable categories. We begin with retracts and idempotents, and we recall the universal property of the idempotent completion. We then discuss the resulting symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{perf}}$, together with the basic formalism of rigid 2-rings and their modules. As an application, we record the first representability result of these notes via the linear Yoneda embedding. We conclude with a brief discussion of lax additivity in the $(\infty, 2)$-categorical setting.

2.1. Retracts and idempotents.

We now fix terminology concerning retracts and idempotents in a (higher) category, and we record the basic relationship between retract data and idempotents that will be used repeatedly.

Definition 2.1.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a category.

(1) A *retract datum* in $\mathcal{C}$ consists of objects $x, y \in \mathcal{C}$ and morphisms $x \xrightarrow{i} y \xrightarrow{r} x$ together with a specified identification $r \circ i \simeq \mathrm{id}_x$. In this situation we say that x *is a retract of* y.

(2) An *idempotent* on $y \in \mathcal{C}$ is an endomorphism $e : y \to y$ together with a specified identification $e \circ e \simeq e$.

(3) Let $e : y \to y$ be an idempotent. A *splitting* of e is a retract datum $x \xrightarrow{i} y \xrightarrow{r} x$ with $r \circ i \simeq \mathrm{id}_x$ such that $i \circ r \simeq e$ as endomorphisms of y.

Notice that, contrary to 1-category theory, the splitting of an idempotent is an condition involving an infinite amout of data. Triangulated categories in which every idempontent splits are known as *Karoubian* categories. Here is the general definition.

Definition 2.1.2. A category $\mathcal{C} \in \mathrm{Cat}$ is *idempotent-complete* if every idempotent in $\mathcal{C}$ admits a splitting.

The basic properties of retracts and idempotent are summarized by the following.

Lemma 2.1.3. Then:

(1) Every retract determines an idempotent. Thus retract data canonically determine idempotents; a splitting is precisely the extra structure exhibiting a given idempotent as coming from a retract.

(2) Splitting of idempotents is unique.
(3) Retracts are preserved by functors.

Proof. Let $\mathcal{C}$ be a category. For (1), if $x \xrightarrow{i} y \xrightarrow{r} x$ is a retract datum, then $e \coloneqq i \circ r$ is an idempotent on y since

$$e \circ e = (i \circ r) \circ (i \circ r) \simeq i \circ (r \circ i) \circ r \simeq i \circ \mathrm{id}_x \circ r = i \circ r = e.$$

For (2), let $e : y \to y$ be an idempotent. Suppose e admits two splittings $x \xrightarrow{i} y \xrightarrow{r} x$ and $x' \xrightarrow{i'} y \xrightarrow{r'} x'$. Then $x \simeq x'$. Indeed, consider the composites $x \xrightarrow{i} y \xrightarrow{r'} x'$ and $x' \xrightarrow{i'} y \xrightarrow{r} x$. Using $r \circ i \simeq \mathrm{id}_x$, $r' \circ i' \simeq \mathrm{id}_{x'}$, and $i \circ r \simeq i' \circ r' \simeq e$, one checks that the two composites $x \to x' \to x$ and $x' \to x \to x'$ identify with id_x and $\mathrm{id}_{x'}$, respectively. Hence $x \simeq x'$.

For (3), let $f : \mathcal{C} \to \mathcal{D}$ be a functor and let x be a retract of y in $\mathcal{C}$. Apply F to a retract datum $x \xrightarrow{i} y \xrightarrow{r} x$ and to the identification $r \circ i \simeq \mathrm{id}_x$ to deduce that $F(x)$ is a retract of $F(y)$ in $\mathcal{D}$. □

Lemma 2.1.4. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. Then $\mathcal{C}$ is idempotent-complete if and only if every idempotent in $h\mathcal{C}$ splits.

Proof. This is immediate from the definition plus the fact that retracts in $\mathcal{C}$ are retracts in $h\mathcal{C}$. □

Retracts interact well with additive categories.

Remark 2.1.5. Recall that an additive category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ is closed under direct summands if whenever $x \oplus y \in \mathcal{C}$ then $x, y \in \mathcal{C}$.

The next result shows that retracts are not far away from direct summands.

Lemma 2.1.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ be an additive category and let $B \subseteq \mathcal{C}$ be a full subcategory. Then B is closed under direct summands if and only if it is closed under retracts.

Proof. For ($\Rightarrow$), let $x \to y \to x$ be a retract in $\mathcal{C}$ of $y \in B$. Then the diagram

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & x \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(x \to y) & \longrightarrow & 0 \end{array}$$

in which every square is exact, shows that $y \simeq x \oplus \mathrm{cofib}(x \to y)$, and since B is closed under direct summands, it follows that $x \in B$. Conversely ($\Leftarrow$), the retracts $x \to x \oplus y \to x$ and $y \to x \oplus y \to y$ show that $x, y \in B$. □

Lemma 2.1.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $B \subseteq \mathcal{C}$ be a full subcategory.
(1) If B is idempotent-complete, then it is closed under retracts.
(2) If $\mathcal{C}$ is idempotent-complete and B is closed under retracts, then B is idempotent-complete.

Proof. For (1), if $x \xrightarrow{i} y \xrightarrow{r} x$ is a retract in $\mathcal{C}$ of $y \in B$, then the composition $i \circ r : y \to y$ is idempotent, thus it splits in $\mathcal{B}$ as $y \xrightarrow{p} z \xrightarrow{q} y$ with $i \circ r \simeq p \circ q$ and $\mathrm{id}_y \simeq q \circ p$. Since (x, i, r) also satisfies these properties and since the splitting data is unique, it follows that $x \simeq z$ is in B. For (2), let $f : y \to y$ be an idempotent of B. Then f is idempotent in $\mathcal{C}$ and thus it splits as $x \xrightarrow{i} y \xrightarrow{r} x$ with $f \simeq i \circ r$ and $\mathrm{id}_x \simeq r \circ i$. In particular, since B is closed under retracts it follows that $x \in B$, making the splitting data of f in $\mathcal{C}$ a splitting data in B. □

Definition 2.1.8. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $B \subseteq \mathcal{C}$ be a full subcategory. The *thick closure* $\mathrm{Thick}_{\mathcal{C}}(B)$ is the smallest full stable subcategory of $\mathcal{C}$ closed under retracts containing B.

Remark 2.1.9. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $B \subseteq \mathcal{C}$ be a full subcategory. If $\mathcal{C}$ is idempotent-complete, then Lemma 2.1.7 implies that $\mathrm{Thick}_{\mathcal{C}}(B)$ is idempotent-complete.

We close this subsection with an example of idempotent-complete category.

Proposition 2.1.10. If $\mathcal{C}$ admits all small colimits, then $\mathcal{C}$ is idempotent complete.

Proof. See [Lur09, Corollary 4.4.5.16]. □

2.2. **The idempotent completion functor.** We now show how to turn a category into an idempotent one.

Construction 2.2.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. Let $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$ be the presheaf category and let $よ_{\mathcal{C}} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ be the Yoneda embedding. Define $\mathcal{C}^{\natural} \subseteq \mathrm{PSh}(\mathcal{C})$ to be the full subcategory spanned by retracts of representables, that is by objects P for which there exists $X \in \mathcal{C}$ and maps $P \to よ_{\mathcal{C}}(X) \to P$ whose composite is equivalent to id_P. We call $\mathcal{C}^{\natural}$ the *idempotent completion* of $\mathcal{C}$.

Proposition 2.2.2. Let $\mathcal{C} \in \mathrm{Cat}$ be a category. Then inclusion $よ_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\natural}$ is fully faithful and the category $\mathcal{C}^{\natural}$ is idempotent complete. Furthermore, precomposition with the Yoneda embedding induces an equivalence

$$(よ_{\mathcal{C}})^* : \mathrm{Fun}(\mathcal{C}^{\natural}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for every idempotent complete $\mathcal{D}$.

Proof. The Yoneda embedding is fully faithful, hence so is its restriction to $\mathcal{C}^{\natural}$. The presheaf category $\mathrm{PSh}(\mathcal{C})$ admits all small colimits, hence is idempotent complete by Proposition 2.1.10. Since $\mathcal{C}^{\natural}$ is, by definition, closed under retracts inside $\mathrm{PSh}(\mathcal{C})$, it is itself idempotent complete by Lemma 2.1.7.

For the universal property, let $\mathcal{D}$ be idempotent complete and let $F : \mathcal{C} \to \mathcal{D}$ be any functor. By left Kan extension along $よ_{\mathcal{C}}$ we obtain $\overline{F} : \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$, and since $\mathcal{D}$ is idempotent complete the functor $\overline{F}$ carries retracts of representables to retracts of the corresponding values of F. Thus $\overline{F}$ factors essentially uniquely through $\mathcal{C}^{\natural}$, yielding an extension $\mathcal{C}^{\natural} \to \mathcal{D}$. Conversely, any functor $\mathcal{C}^{\natural} \to \mathcal{D}$ is determined by restriction to $\mathcal{C}$ because every object of $\mathcal{C}^{\natural}$ is a retract of some $よ_{\mathcal{C}}(X)$. □

Remark 2.2.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ be an additive category. The functor $よ_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C}^{\natural}$ exhibits $\mathcal{C}$ as a *dense* subcategory of $\mathcal{C}^{\natural}$, that is, every object of $\mathcal{C}^{\natural}$ is a direct summand of an object of $\mathcal{C}$.

We now study idempontent-completeness related to stable categories.

Definition 2.2.4. We let $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{st}}$ denote the full subcategory spanned by the idempotent-complete stable categories.

Definition 2.2.5. Let $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}^{\mathrm{st}}$ be an exact functor. We will say that f is a *Karoubi equivalence* if the induced functor $f^{\natural} : \mathcal{C}^{\natural} \to \mathcal{D}^{\natural}$ is an equivalence.

In other terms, f is a Karoubi equivalence if it is fully faithful and its essential image is dense, in the sense of Remark 2.2.3, that is every object of $\mathcal{D}$ is a retract of some object in the essential image of f.

Proposition 2.2.6. There exists a Bousfield localization $(-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \rightleftarrows \mathrm{Cat}^{\mathrm{perf}} : \mathrm{incl}$ at the class of Karoubi equivalences.

Proof. Functoriality follows from the universal property in Proposition 2.2.2. The adjunction is obtained by applying the same universal property objectwise in $\mathrm{Cat}^{\mathrm{st}}$, noting that exact functors are preserved under the Kan-extension construction used in the proof of the above mentioned proposition. Finally, the localisation statement is a reformulation of the fact that f is inverted by $(-)^{\natural}$ if and only if f is a Karoubi equivalence. □

We record a convenient strengthening of the above (proved in [CDH+25, Proposition A.3.4]).

Remark 2.2.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $K_0(\mathcal{C})$ be its Grothendieck group. Define the *minimalisation* $\mathcal{C}_{\min} \subseteq \mathcal{C}$ to be the full stable subcategory spanned by objects X with $[X] = 0$ in $K_0(\mathcal{C})$. We call $\mathcal{C}$ *minimal* if $\mathcal{C}_{\min} = \mathcal{C}$.

Then localising $\mathrm{Cat}^{\mathrm{st}}$ at the class of Karoubi equivalences yields both a left and a right Bousfield localisation. The left-local objects are the minimal stable categories, and the right-local objects are the idempotent complete stable categories. Moreover, the left and right adjoints of this localisation are given by $\mathcal{C} \mapsto \mathcal{C}_{\min}$ and $\mathcal{C} \mapsto \mathcal{C}^{\natural}$, respectively. In particular, an exact functor is a Karoubi equivalence if and only if it induces an equivalence on minimalisations, and if and only if it induces an equivalence on idempotent completions. Consequently, the functor $(-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}$ preserves both limits and colimits.

We conclude with analysing the big variant.

Proposition 2.2.8. There exists an equivalence $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}} : (-)^{\omega}$ of categories.

Proof. It is clear that the ind-completion and taking compact objects sit into an adjunction $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}} : (-)^{\omega}$, thus it suffices to show that unit and counit are equivalences.

Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and consider the unit $\mathcal{C} \to (\mathrm{Ind}(\mathcal{C}))^{\omega}$. It suffices to show that every compact object of $\mathrm{Ind}(\mathcal{C})$ lies in the essential image of $よ_{\mathcal{C}}$. Let $X \in \mathrm{Ind}(\mathcal{C})$ be compact. By definition of $\mathrm{Ind}(\mathcal{C})$, the object X can be written as a filtered colimit $X \simeq \mathrm{colim}_{i \in I}\, よ_{\mathcal{C}}(c_i)$ with $c_i \in \mathcal{C}$. Consider the identity map $\mathrm{id}_X : X \to X \simeq \mathrm{colim}_i\, よ_{\mathcal{C}}(c_i)$. Since X is compact, id_X factors through some stage, so there exists $i \in I$ and a map $u : X \to よ_{\mathcal{C}}(c_i)$ such that the composite $X \xrightarrow{u} よ_{\mathcal{C}}(c_i) \to \mathrm{colim}_i\, よ_{\mathcal{C}}(c_i) \simeq X$ is equivalent to id_X. Thus X is a retract of $よ_{\mathcal{C}}(c_i)$ in $\mathrm{Ind}(\mathcal{C})$.

Because $よ_{\mathcal{C}}$ is fully faithful, retract data on $よ_{\mathcal{C}}(c_i)$ correspond to retract data on c_i in $\mathcal{C}$. Since $\mathcal{C}$ is idempotent-complete, this retract is represented by an object $c \in \mathcal{C}$ and an equivalence $X \simeq よ_{\mathcal{C}}(c)$. Therefore every compact object of $\mathrm{Ind}(\mathcal{C})$ lies in $よ_{\mathcal{C}}(\mathcal{C})$, and we conclude that the induced functor $\mathcal{C} \to (\mathrm{Ind}(\mathcal{C}))^{\omega}$ is essentially surjective. It is fully faithful since $よ_{\mathcal{C}}$ is fully faithful, hence it is an equivalence.

For the counit, let $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ and consider $\mathrm{Ind}(\mathcal{D}^{\omega}) \to \mathcal{D}$ The inclusion $i : \mathcal{D}^{\omega} \hookrightarrow \mathcal{D}$ extends (by the universal property of ind-completion) to a colimit-preserving functor $\varepsilon : \mathrm{Ind}(\mathcal{D}^{\omega}) \to \mathcal{D}$ whose restriction to $\mathcal{D}^{\omega}$ is i. It suffices to show that ε is an equivalence. First, ε is essentially surjective since $\mathcal{D}$ is compactly generated by $\mathcal{D}^{\omega}$, and ε preserves colimits and contains $\mathcal{D}^{\omega}$ in its image. For full faithfulness, it suffices to check it on compact generators: given $x, y \in \mathcal{D}^{\omega}$ the equivalence $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D}^{\omega})}(j(x), j(y)) \simeq \mathrm{Hom}_{\mathcal{D}^{\omega}}(x, y) \simeq \mathrm{Hom}_{\mathcal{D}}(x, y)$ shows that ε is fully faithful on the subcategory of compact objects. Since both source and target are generated under colimits by these compact objects and ε preserves colimits, it follows that ε is fully faithful on all objects. Hence ε is an equivalence. □

2.3. **The monoidal structure on $\mathbf{Cat}^{\mathbf{perf}}$ and a representability result.** The title is self-explanatory.

Remark 2.3.1. Lurie constructed a symmetric monoidal structure on Pr^{L} characterized by the following universal property: given two presentable categories $\mathcal{C}$ and $\mathcal{D}$, their Lurie tensor product is a presentable category $\mathcal{C} \otimes \mathcal{D}$ with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that for every presentable category $\mathcal{E}$, precomposition with it induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{L},\mathrm{L}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Here the superscript L, L denotes the full subcategory spanned by those functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits separately in each variable. It follows from this universal property that the symmetric monoidal structure is also closed. To be precise, for every pair of presentable categories $\mathcal{C}$ and $\mathcal{D}$, there exists a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L},\mathrm{L}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}))$$

in $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$, so that $\mathrm{Fun}^{\mathrm{L}}(-,-)$ exhibits the internal hom. Since $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Spc}, \mathcal{C}) \simeq \mathcal{C}$, it follows that the category of spaces Spc is the neutral element for the Lurie tensor product.

We will need a stable version of this observation.

Remark 2.3.2. Let $\Sigma^\infty_+ : \mathrm{Spc} \to \mathrm{Sp}$ be the suspension spectrum functor. Notice that a presentable category $\mathcal{C}$ is stable if and only if the canonical map $\mathrm{id}_{\mathcal{C}} \otimes \Sigma^\infty_+ : \mathcal{C} \to \mathcal{C} \otimes \mathrm{Sp}$ is an equivalence. In particular, since Sp is a stable category, the inverse of the equivalence $\mathrm{Sp} \to \mathrm{Sp} \otimes \mathrm{Sp}$ makes Sp into a commutative algebra of Pr^{L}. It follows that the category of stable presentable categories and left adjoints can be realized as $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \simeq \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})$ and that taking spectrum objects $- \otimes \mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ defines a symmetric monoidal functor.

Remark 2.3.3. By Proposition 2.2.8, there exists an equivalence $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}} : (-)^\omega$. The symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ may betransported along the above equivalence to a symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{perf}}$. Concretely, for $\mathcal{A}, \mathcal{B} \in \mathrm{Cat}^{\mathrm{perf}}$ define

$$\mathcal{A} \otimes \mathcal{B} := \big(\mathrm{Ind}(\mathcal{A}) \otimes \mathrm{Ind}(\mathcal{B})\big)^\omega \in \mathrm{Cat}^{\mathrm{perf}},$$

where $\otimes$ on the right-hand side denotes the tensor product in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. With this definition, $\otimes$ is characterized by the expected universal property: for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, restriction along $\mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex,ex}}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}\big(\mathcal{A}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{C})\big),$$

where $\mathrm{Fun}^{\mathrm{ex,ex}}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ denotes the full subcategory of functors which are exact in each variable. In particular, this equips $\mathrm{Cat}^{\mathrm{perf}}$ with the structure of a symmetric monoidal category for which compact spectra Sp^ω furnishes the monoidal unit.

It follows by construction that a commutative algebra object in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ can be identified with a small idempotent-complete stable category equipped with a symmetric monoidal structure whose tensor product is exact in each variable. This motivates the following.

Definition 2.3.4. A commutative algebra in $\mathrm{Cat}^{\mathrm{perf}}$ is called a *small* 2*-ring*.

We also remind the relevant features of commutative algebra objects and modules.

Remark 2.3.5. Let $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ be a commutative algebra object. Let $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be the category of modules over $\mathcal{A}$ in $\mathrm{Cat}^{\mathrm{perf}}$ and regard it as a symmetric monoidal category with the induced relative tensor product $\otimes_{\mathcal{A}}$. This symmetric monoidal structure is also closed, and the internal hom may be identified with the pullback

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(-,-) & \longrightarrow & \mathrm{Fun}^{\mathrm{ex}}(-,-) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{A}}(-,-) & \longrightarrow & \mathrm{Fun}(-,-) \end{array}$$

where $\mathrm{Fun}_{\mathcal{A}}(-,-)$ denotes the category of $\mathcal{A}$-linear functors. Objects of $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(-,-)$ will be called *$\mathcal{A}$-linear functors*. Similarly, let $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be the category of modules over $\mathrm{Ind}(\mathcal{A})$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and equip it with the relative tensor product $\otimes_{\mathrm{Ind}(\mathcal{A})}$. Again, this symmetric monoidal structure is closed, with internal hom $\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(-,-)$ defined similarly. Objects of this category will be called *$\mathrm{Ind}(\mathcal{A})$-linear functors*.

Since the ind-completion is symmetric monoidal, the definitions of linear functors immediately imply the following.

Lemma 2.3.6. Let $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ be a commutative algebra object and let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Then precomposition with the Yoneda embedding $よ : \mathcal{M} \hookrightarrow \mathrm{Ind}(\mathcal{M})$ induces an equivalence

$$よ^* : \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Ind}(\mathcal{M}), \mathcal{N}) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$$

of $\mathcal{A}$-modules for every $\mathcal{N} \in \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$.

Our next goal is to prove a representability result. We begin with the construction of the enriched Yoneda embedding.

Construction 2.3.7. Let $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ be a commutative algebra and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Let $x \in \mathcal{C}$ and consider the action functor $- \otimes x : \mathcal{A} \to \mathcal{C}$. By definition this functor preserves finite colimits and hence it admits an ind-right adjoint $\mathcal{C}(x, -) : \mathcal{C} \to \mathrm{Ind}(\mathcal{A})$. The functoriality of these cosntruction produces a functor

$$よ_{\mathcal{A}} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{A}))$$

which is given on objects by $x \mapsto \mathcal{C}(-, x)$.

In general the above functor is not fully-faithful; dualizability is exactly what is needed.

Definition 2.3.8. A commutative algebra in $\mathrm{Cat}^{\mathrm{perf}}$ is called *small rigid* 2*-ring* if every object in it is dualizable. We let $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ denote the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ spanned by the rigid commutative algebras.

We recall the following folkore result.

Lemma 2.3.9. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a rigid commutative algebra.

(1) Taking duals defines a symmetric monoidal equivalence $(-)^\vee : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$.
(2) Let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a presentable $\mathrm{Ind}(\mathcal{A})$-module. Then the action of $\mathcal{A}$ on $\mathcal{C}$ restricts to the full subcategory $\mathcal{C}^\omega \subseteq \mathcal{C}$ of compact objects.

Proof. Consider point (1). Since $\mathcal{A}$ is rigid, every object $a \in A$ admits a (chosen) dual $a^\vee$ together with evaluation and coevaluation maps $\mathrm{ev}_a : a \otimes a^\vee \to \mathbb{1}_{\mathcal{A}}$ and $\mathrm{coev}_a : \mathbb{1}_{\mathcal{A}} \to a^\vee \otimes a$ satisfying the triangle identities. Define the functor $(-)^\vee : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ by taking duals. Since for dualizable objects there are canonical equivalences $(a \otimes b)^\vee \simeq b^\vee \otimes a^\vee \simeq a^\vee \otimes b^\vee$, the functor $(-)^\vee$ is symmetric monoidal. Moreover, $(-)^\vee$ is adjoint to itself: the unit $\eta_a : a \to (a^\vee)^\vee$ and counit $\varepsilon_a : (a^\vee)^\vee \to a$ are the usual double-dual maps. Dualizability implies η_a (equivalently ε_a) is an equivalence for every a, hence $(-)^\vee$ is an equivalence of symmetric monoidal categories.

Consider point (2). Fix $a \in \mathcal{A}$. The action endofunctor $a \otimes - : \mathcal{C} \to \mathcal{C}$ preserves colimits (because the $\mathrm{Ind}(\mathcal{A})$-action does), hence is a left adjoint. Since a is dualizable in $\mathcal{A}$, the functor $a \otimes -$ has right adjoint $a^\vee \otimes -$. In particular, $a^\vee \otimes -$ preserves filtered colimits (indeed it preserves all colimits), so $a \otimes -$ preserves compact objects. In particular, the $\mathcal{A}$-action restricts to $\mathcal{C}^\omega \subseteq \mathcal{C}$. □

The following result has been proved many times in the literature. We recall the proof for completeness.

Lemma 2.3.10. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a rigid algebra and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ be an $\mathrm{Ind}(\mathcal{A})$-compactly generated module. Then $\mathcal{C}$ is dualizable, and the dual can be identified with $\mathcal{C}^\vee \simeq \mathrm{Ind}((\mathcal{C}^\omega)^{\mathrm{op}})$.

Proof. Set $\mathcal{C}^\vee = \mathrm{Ind}((\mathcal{C}^\omega)^{\mathrm{op}})$. First of all, point (2) of Lemma 2.3.9 implies that the $\mathrm{Ind}(\mathcal{A})$-action on $\mathcal{C}$ restricts to an $\mathcal{A}$-action on the small idempotent-complete stable category $\mathcal{C}^\omega$. Hence $(\mathcal{C}^\omega)^{\mathrm{op}}$ is canonically an $\mathcal{A}$-module as well, and therefore $\mathrm{Ind}((\mathcal{C}^\omega)^{\mathrm{op}})$ inherits a canonical $\mathrm{Ind}(\mathcal{A})$-module structure. In particular, $\mathcal{C}^\vee \in \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$. To prove dualizability of $\mathcal{C}$ with dual $\mathcal{C}^\vee$, it suffices to construct, for every module $\mathcal{B} \in \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$, a natural equivalence

$$\mathcal{C}^\vee \otimes_{\mathrm{Ind}(\mathcal{A})} \mathcal{B} \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}, \mathcal{B}),$$

where $\otimes_{\mathrm{Ind}(\mathcal{A})}$ denotes the relative Lurie tensor product. Because $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ is closed symmetric monoidal, for every $\mathcal{B}$ and $\mathcal{B}$ there is a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee \otimes_{\mathrm{Ind}(\mathcal{A})} \mathcal{B}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee,\ \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{B}, \mathcal{C})). \tag{1}$$

Since taking opposites is compatible with $\mathrm{Ind}(\mathcal{A})$-linearity, there is a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{B}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}}.$$

Plugging this into (1) and using the closed structure again (and currying-uncurrying) gives a chain of natural equivalences

$$\begin{aligned}\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee \otimes_{\mathrm{Ind}(\mathcal{A})} \mathcal{B}, \mathcal{C}) &\simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee, \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}}))\\ &\simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee, \mathcal{B}^{\mathrm{op}}))\\ &\simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{C}).\end{aligned}$$

Yoneda then implies that $\mathcal{C}^\vee \otimes_{\mathrm{Ind}(\mathcal{A})} \mathcal{B} \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}}$. By using Lemma 2.3.6 twice it follows that

$$\mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}^\vee, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}((\mathcal{C}^\omega), \mathcal{B}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathcal{C}, \mathcal{B})$$

thus concluding the proof. □

Corollary 2.3.11. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a rigid algebra and let $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be two $\mathcal{A}$-modules. Then there exists a natural equivalence

$$\mathrm{Ind}(\mathcal{M}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{N}) \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{M}, \mathrm{Ind}(\mathcal{N}))$$

in $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$.

Proof. There are equivalences

$$\begin{aligned}\mathrm{Ind}(\mathcal{M}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{N}) &\simeq \mathrm{Ind}(\mathcal{M}^{\mathrm{op}}) \otimes_{\mathrm{Ind}(\mathcal{A})} \mathrm{Ind}(\mathcal{N})\\ &\simeq \mathrm{Ind}(\mathcal{M})^\vee \otimes_{\mathrm{Ind}(\mathcal{A})} \mathrm{Ind}(\mathcal{N})\\ &\simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Ind}(\mathcal{M}), \mathrm{Ind}(\mathcal{N}))\\ &\simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{M}, \mathrm{Ind}(\mathcal{N})).\end{aligned}$$

Indeed the first one follows since $\mathrm{Ind} : \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Mod}_{\mathrm{Ind}(\mathcal{A})}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ is symmetric monoidal, the second one by Lemma 2.3.10 since $\mathcal{A}$ is rigid, the third one by definition of internal hom and the last one by Lemma 2.3.6. □

Remark 2.3.12. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a rigid algebra and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Then Corollary 2.3.11 implies that the functor $よ_{\mathcal{A}} : \mathcal{C} \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{A}))$ of Construction 2.3.7 is fully-faithful and exhibits the target as the ind-completion of the source.

This motivates the following.

Definition 2.3.13. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a rigid algebra and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. We will refer to the functor $よ_{\mathcal{A}} : \mathcal{C} \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{A}))$ as the *$\mathcal{A}$-linear Yoneda embedding of* $\mathcal{C}$.

The last ingredient for the above claimed representability result is the following characterization of dualizable modules.

Lemma 2.3.14. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$. Then $\mathcal{C}$ is dualizable if and only if the two maps

$$\mathrm{ev}_{\mathrm{Ind}(\mathcal{C})} : \mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}) \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes_{\mathcal{A}} \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{A}),$$

$$\mathrm{coev}_{\mathrm{Ind}(\mathcal{C})} : \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{C}) \otimes_{\mathcal{A}} \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}^{\mathrm{op}})$$

preserve compact objects. In that case, their restrictions to compact objects are the evaluation and coevaluation of a duality between $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$.

Proof. Assume that $\mathcal{C}$ is dualizable and let $\mathrm{ev}_{\mathcal{C}} : \mathcal{C}^{\vee} \otimes_{\mathcal{A}} \mathcal{C} \to \mathcal{A}$ exhibit $\mathcal{C}^{\vee}$ as dual to $\mathcal{C}$. Since Ind is symmetric monoidal, $\mathrm{Ind}(\mathrm{ev}_{\mathcal{C}})$ exhibits $\mathrm{Ind}(\mathcal{C}^{\vee})$ as dual to $\mathrm{Ind}(\mathcal{C})$. Uniqueness of duals [Lur17, Lemma 4.6.1.10] implies that $\mathrm{Ind}(\mathrm{ev}_{\mathcal{C}})$ may be identified with $\mathrm{ev}_{\mathrm{Ind}(\mathcal{C})}$, which therefore preserves compact objects. A dual argument shows that $\mathrm{coev}_{\mathrm{Ind}(\mathcal{C})}$ preserves compact objects. Conversely, if $\mathrm{ev}_{\mathrm{Ind}(\mathcal{C})}$ and $\mathrm{coev}_{\mathrm{Ind}()}$ preserve compact objects, then their restrictions to compact objects satisfy the triangle identities and hence exhibit $\mathcal{C}^{\mathrm{op}}$ as a dual of $\mathcal{C}$. □

It follows that dualizable modules over rigid 2-ring satisfy a representability result.

Proposition 2.3.15. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be a dualizable $\mathcal{A}$-module. Then there exists an equivalence

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$$

induced by the $\mathcal{A}$-linear Yoneda embedding.

Proof. Since in any closed monoidal category it is $\mathcal{C} \simeq \mathcal{C} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\vee}, \mathcal{A})$, the claim follows since $\mathcal{C}^{\vee} \simeq \mathcal{C}^{\mathrm{op}}$. □

To check that a module is dualizable is often useful to study two other properties.

Definition 2.3.16. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$. We say that $\mathcal{C}$ is:

(1) *Smooth* if the diagonal $\Delta \in \mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C})$ corresponding to the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is compact.
(2) *Proper* if, for all $x, y \in \mathcal{C}$, the mapping object $\mathcal{C}(x, y) \in \mathrm{Ind}(\mathcal{C})$ is compact.

Remark 2.3.17. To be more precise, the following refinement of Proposition 2.3.15 holds. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. By identifying $\mathcal{C}$ and $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ with full subcategories of $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{A}))$ it follows that:

(1) If $\mathcal{C}$ is proper, then $\mathcal{C} \subseteq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.
(2) If $\mathcal{C}$ is smooth, then $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \subseteq \mathcal{C}$.

Proposition 2.3.18. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$. Then $\mathcal{C}$ is dualizable if and only if it is smooth and proper, and the dual of $\mathcal{C}$ is $\mathcal{C}^{\mathrm{op}}$.

Proof. The proposition is a simple corollary of Lemma 2.3.14. Indeed, $\mathrm{ev}_{\mathrm{Ind}(\mathcal{C})}$ preserves compact objects if and only if $\mathcal{C}$ is proper. Similarly, $\mathrm{coev}_{\mathrm{Ind}(\mathcal{C})}$ preserves compact objects if and only if $\mathcal{C}$ is smooth. □

We will come back to the diagonal bimodule in Section 9.

2.4. **Lax additivity.** There are several models for $(\infty, 2)$-categories, ranging from strict 2-categories and bicategories to fully weak models such as complete 2-fold Segal spaces or scaled simplicial sets. For us, an $(\infty, 2)$-category will be a category enriched in Cat. In particular, an $(\infty, 2)$-*category* $\mathbb{C}$ may be thought as a space of objects together with an hom-category $\mathcal{C}(x, y)$, defined for every two objects $x, y \in \mathcal{C}$, plus a composition rule and identities, which are required to satisfy the usual conditions. Discarding non-invertible 2-morphisms yields an underlying category $\iota_1\mathbb{C}$.

Example 2.4.1. The main examples to keep in mind are:

(1) The $(\infty, 2)$-category **Cat** of small categories, functors and natural transformations. The underlying category $\iota_1\mathbf{Cat} \simeq \mathrm{Cat}$ is given the usual category of small categories
(2) The $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{L}}$ presentable categories, colimit-preserving functors and natural transformations. The underlying category of $\iota_1\mathbf{Pr}^{\mathrm{L}} \simeq \mathrm{Pr}^{\mathrm{L}}$ is given by the usual category of presentable categories and colimit-preserving functors. A similar statement holds for $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$.
(3) The $(\infty, 2)$-category $\mathbf{Cat}^{\mathrm{st}}$ of small stable categories, exact functors and natural transformations. Again, $\iota_1\mathbf{Cat}^{\mathrm{st}} \simeq \mathrm{Cat}^{\mathrm{st}}$.
(4) The $(\infty, 2)$-category $\mathbf{Cat}^{\mathrm{perf}}$ of small idempontent-complete stable categories, exact functors and natural transformations, and again $\iota_1\mathbf{Cat}^{\mathrm{perf}} \simeq \mathrm{Cat}^{\mathrm{perf}}$.

Notice that $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is actually a category enriched in $\widehat{\mathrm{Cat}}_{\mathrm{st}}^{\mathrm{colim}}$, the large category of cocomplete stable categories and colimit-preserving functors, whereas the last two examples are enriched in $\mathrm{Cat}^{\mathrm{st}}$.

The theory of $(\infty, 2)$-categorically allows to talk about *lax* commutativity data.

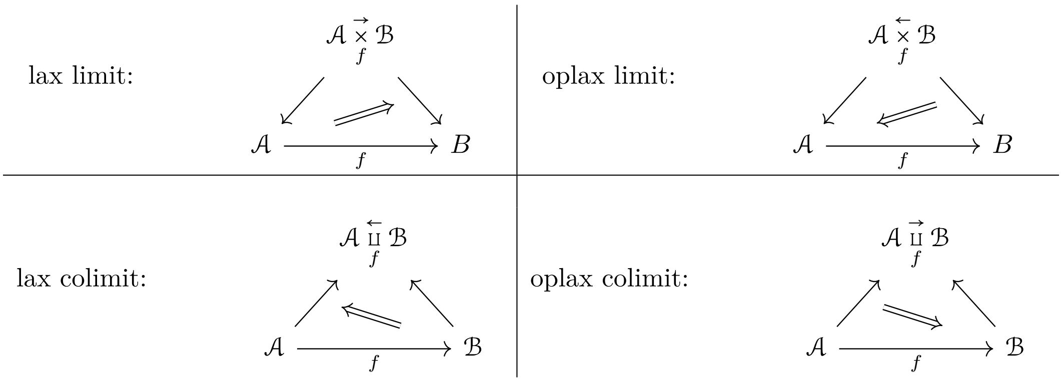
Remark 2.4.2. Let $\mathbb{C}$ be an $(\infty, 2)$-category and let I be a small category. Let $F : I \to \iota_1\mathcal{C}$ be a diagram and let $x \in \mathcal{C}$ be an object.

(1) The *category of lax cones from x to F* by
$$\mathrm{Cone}^{\mathrm{lax}}(x, F) \coloneqq \mathrm{Nat}^{\mathrm{lax}}(\mathrm{const}_x, F).$$
Unwinding the definition, an object of $\mathrm{Cone}^{\mathrm{lax}}(x, F)$ consists of the following assignements. For each object $i \in I$, a 1-morphism $\eta_i : x \to F(i)$, for each arrow $\alpha : i \to j$ in I, a 2-morphism $\eta_\alpha : F(\alpha) \circ \eta_i \Rightarrow \eta_j$ subject to the usual unit and composition coherences (for id_i and for composable $i \to j \to k$). Morphisms in $\mathrm{Cone}^{\mathrm{lax}}(x, F)$ are *modifications*, that is, families of 2-morphisms $\theta_i : \eta_i \Rightarrow \eta'_i$ compatible with the structure 2-morphisms.
(2) Similarly, the *category of oplax cones* is
$$\mathrm{Cone}^{\mathrm{oplax}}(x, F) \;\coloneqq\; \mathrm{Nat}^{\mathrm{oplax}}(\mathrm{const}_x, F),$$
defined by reversing the direction of the structure 2-morphisms (so $\eta_j \Rightarrow F(\alpha) \circ \eta_i$).
(3) Dually, we define *lax* and *oplax cocone categories* by
$$\mathrm{Cocone}^{\mathrm{lax}}(F, x) \;\coloneqq\; \mathrm{Nat}^{\mathrm{lax}}(F, \mathrm{const}_x), \qquad \mathrm{Cocone}^{\mathrm{oplax}}(F, x) \;\coloneqq\; \mathrm{Nat}^{\mathrm{oplax}}(F, \mathrm{const}_x).$$

A *lax limit* of F is an object $\lim^{\mathrm{lax}} F \in \mathcal{C}$ equipped with a distinguished lax cone $\lambda \in \mathrm{Cone}^{\mathrm{lax}}(\lim^{\mathrm{lax}} F, F)$ such that for every $x \in \mathcal{C}$, composition with λ induces an equivalence of categories
$$\mathbb{C}(x, \lim^{\mathrm{lax}} F) \to \mathrm{Cone}^{\mathrm{lax}}(x, F).$$
An *oplax limit* is defined by replacing $\mathrm{Cone}^{\mathrm{lax}}$ with $\mathrm{Cone}^{\mathrm{oplax}}$. Likewise, a *lax colimit* (respectively, *oplax colimit*) of F is an object $\mathrm{colim}^{\mathrm{lax}} F$ (respectively $\mathrm{colim}^{\mathrm{oplax}} F$) equipped with a lax (respectively oplax) cocone $\mu \in \mathrm{Cocone}^{\mathrm{lax}}(F, \mathrm{colim}^{\mathrm{lax}} F)$ such that for every $x \in \mathcal{C}$, composition with μ induces an equivalence
$$\mathbb{C}(\mathrm{colim}^{\mathrm{lax}} F, x) \to \mathrm{Cocone}^{\mathrm{lax}}(F, x) \qquad \text{(respectively with } \mathrm{Cocone}^{\mathrm{oplax}}\text{)}.$$

We now discuss the main example.

Example 2.4.3. Let $\mathbb{C}$ be a pointed $(\infty,2)$-category and let $f : \mathcal{A} \to \mathcal{B}$ be 1-morphism in $\mathbb{C}$. The (op)lax (co)limit of f correspond to the following diagrams:



As always, it is interesting to understand when these four lax universal constructions exist and are canonically equivalent.

The main results is then [CDW25, Corollary 4.15 and Theorem 4.19], which asserts that in an $(\infty,2)$-category enriched in categories with all colimits, lax limits and lax colimits coincide (when they exist) and are absolute, that is, preserved by the colimit-preserving functor on hom-categories.

Example 2.4.4. Lax limits in the $(\infty,2)$-category $\mathbf{Pr}^{\mathrm{L}}$ of presentable categories exist and are computed as underlying categories. Indeed, given a diagram $F : I \to \mathbf{Pr}^{\mathrm{L}}$, its lax limit is computed as sections of the Grothendieck construction $\lim^{\mathrm{lax}} F \simeq \mathrm{Fun}_I(I, \int_I F)$, since [Lur09, Proposition 5.5.3.17 and Proposition 5.5.3.3] imply that $\lim^{\mathrm{lax}} F$ presentable and a functor into it preserves colimits if and only if it does so after postcomposing with each pointwise evaluation map $\lim^{\mathrm{lax}} F \to F(i)$. It follows that $\mathbf{Pr}^{\mathrm{L}}$ has also lax colimit, which agree with lax limits, and thus it has all lax bilimits, which are furthermore absolute.

Since $\mathbf{Cat}^{\mathrm{st}}$ and $\mathbf{Cat}^{\mathrm{perf}}$ are not enriched in categories with all colimits, the main theorem of [CDW25] does not apply. The following definition fixes this finiteness issue, and captures the equivalence between lax limits and colimits.

Definition 2.4.5. Let $\mathbb{C}$ be an $(\infty,2)$-category. We will say that $\mathbb{C}$ is *(finitely) lax semiadditive* if:

(1) It is enriched in categories with (finite) colimits and functors preserving them.
(2) Every diagram indexed by a (finite) small category admits a lax limit and colimit, which agree.

We will furthermore say that $\mathbb{C}$ is *(finitely) lax additive* if, moreover, every hom-category is stable.

Remark 2.4.6. Let $\mathbb{C}$ be an $(\infty,2)$-category and denote by $\mathbb{C}^{\mathrm{op}}$ the $(\infty,2)$-category obtained from $\mathbb{C}$ by reversing the directions of the 1-morphisms, that is $\mathbb{C}^{\mathrm{op}}(x,y) = \mathbb{C}(y,x)$. If $\mathbb{C}$ is enriched in (stable) categories with (finite)colimits, then so is $\mathbb{C}^{\mathrm{op}}$. Moreover lax limits/colimits/bilimits in $\mathbb{C}^{\mathrm{op}}$ correspond to lax colimits/limits/bilimits in $\mathbb{C}$. Thus an $(\infty,2)$-category is (finitely) lax (semi)additive if and only if its opposite is (finitely) lax (semi)additive.

Warning 2.4.7. Notice that definition of (finitely) lax semiadditive $(\infty,2)$-category only involves lax limits and colimits, since they are required to exist and coincide. It does not say anything about oplax limits and colimits. In particular, it is possible to define the notion of *(finitely) oplax semiadditive* $(\infty,2)$-category: that is, an $(\infty,2)$-category enriched in categories with finite limits and limit-preserving functors that admit (finite) oplax bilimits.

Remark 2.4.8. The major feature of (finitely) lax additive $(\infty, 2)$-category, is that all the constructions of Example 2.4.3 coincide, thus giving the notion of *lax sum*.

Example 2.4.9 ([CDW25, Example 5.7])**.** The $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is lax additive. Indeed, since it is enriched in stable categories with all colimits and colimit-preserving functors, it suffices to note that the argument of Example 2.4.4 may be refined to show that $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ admits all lax limits (and hence all lax bilimits). Notice that $\mathbf{Pr}^{\mathrm{R}}_{\mathrm{st}}$, appropriately defined, is only finitely lax additive, since composition with a right adjoint functor is exact but does not preserve arbitrary colimits.

Example 2.4.10 ([CDW25, Example 5.7])**.** The $(\infty, 2)$-category $\mathbf{Cat}^{\mathrm{st}}$ is finitely lax additive. Indeed, being enriched in $\mathrm{Cat}^{\mathrm{st}}$, it suffices to show that it admits lax bilimits. This can be proved via the matrix calculus of [CDW25]: one restricts to indexing categories S for which all mapping spaces are *finite* spaces. For such S, the colimits appearing in the matrix formulas (colimits indexed by the mapping spaces) reduce to finite colimits, hence are available in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$. Consequently, the matrix calculus applies to show that every finite diagram in $\mathrm{Cat}^{\mathrm{st}}$ admits a lax bilimit, and that lax limits and lax colimits canonically agree.

Proposition 2.4.11. The $(\infty, 2)$-category $\mathbf{Cat}^{\mathrm{perf}}$ is finitely lax additive.

Proof. Again it suffices to show that it admits lax bilimits. Notice that the adjunction of Proposition 2.2.2 extends to an $(\infty, 2)$-adjunction $(-)^{\natural} : \mathbf{Cat}^{\mathrm{st}} \rightleftarrows \mathbf{Cat}^{\mathrm{perf}} : \mathrm{incl}$ where the right adjoint induces an equivalence on hom-categories (it is "2-fully faithful"). In particular, once $\mathbf{Cat}^{\mathrm{perf}}$ has lax limits, it will have lax colimits (essentially because if it has them, then they are preserved by the inclusion, but in $\mathbf{Cat}^{\mathrm{st}}$ they are lax bilimits, in particular lax colimits, which are then preserved by $(-)^{\natural}$ and since the composition is the identity, this provides the required cocone.). Thus let S be a finite category and let $F : S \to \mathbf{Cat}^{\mathrm{perf}}$ be a diagram. As a functor with values in $\mathbf{Cat}^{\mathrm{st}}$, the diagram F admits a lax limit object $\lim^{\mathrm{lax}}_S F \in \mathbf{Cat}^{\mathrm{st}}$. It suffices to show that $\lim^{\mathrm{lax}}_S F$ is idempontent-complete. By using the section model for lax limits, an endomorphism $e : x \to x$ in $\lim^{\mathrm{lax}}_S F$ is the same as a family of endomorphisms $e_s : x_s \to x_s$ in $F(s)$ compatible with the structure maps. Assume e is idempotent. Since each $F(s) \in \mathrm{Cat}^{\mathrm{perf}}$ is idempotent-complete, choose splittings

$$x_s \xrightarrow{p_s} y_s \xrightarrow{i_s} x_s, \qquad p_s i_s = \mathrm{id}_{y_s}, \quad i_s p_s = e_s.$$

For each arrow $\alpha : s \to t$ in S, let $\varphi_\alpha : F(\alpha)(x_s) \to x_t$ be the structure map of x. Define a structure map for y by

$$\psi_\alpha := p_t \circ \varphi_\alpha \circ X(\alpha)(i_s) : F(\alpha)(y_s) \to y_t.$$

The compatibility of the family (e_s) with φ_α implies $e_t \circ \varphi_\alpha \simeq \varphi_\alpha \circ F(\alpha)(e_s)$, and this identity is exactly what is needed to check that the maps ψ_α satisfy the same coherences as the φ_α. Thus (y_s, ψ_α) defines an object $y \in \lim^{\mathrm{lax}}_S F$ and the maps (p_s) and (i_s) assemble to morphisms $x \to y \to x$ in L splitting e. Hence every idempotent in $\lim^{\mathrm{lax}}_S F$ splits, so $\lim^{\mathrm{lax}}_S F \in \mathrm{Cat}^{\mathrm{perf}}$. □

We now describe the lax sum for small stable categories.

Example 2.4.12. The $(\infty, 2)$-category $\mathbf{Cat}^{\mathrm{st}}$ is finitely lax additive. Let $f : \mathcal{A} \to \mathcal{B}$ be an exact functor in $\mathrm{Cat}^{\mathrm{st}}$ and let $\mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B}$ be the lax sum of f. Then the underlying category may be described as the category of sections of the Grothendieck construction

$$\begin{array}{ccc} \int_{\Delta^1} f & \longrightarrow & \mathrm{Cat}_* \\ \big\downarrow & & \big\downarrow \\ \Delta^1 & \xrightarrow[{[1] \mapsto f}]{} & \mathrm{Cat}. \end{array}$$

More precisely, as

$$\mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} = \{(a, b, \eta) \text{ such that } a \in \mathcal{A},\, b \in \mathcal{B} \text{ and } \eta : f(a) \to b\}.$$

In particular, the various (op)lax (co)cones can be described as follows.

(1) For the lax limit cone, the structural maps are given by the projections, and the natural transformation filling the diagram is given by $f(a) \to b$.
(2) The oplax limit cone is given by the functors $(a, b, \eta) \mapsto a$ and $(a, b, \eta) \mapsto \mathrm{fib}(\eta)$, whereas the natural transformation is given by $(a, b, \eta) \mapsto (\mathrm{fib}(\eta) \to f(a))$.
(3) The lax colimit cocone corresponds to the functors $a \mapsto (a, f(a), \mathrm{id}_{f(a)})$ and $b \mapsto (0, b, 0 \to b)$ whereas the natural transformation is given by $a \mapsto (a, f(a), \mathrm{id}_{f(a)})$.
(4) The oplax colimit cocone is given by $a \mapsto (a, 0, f(a) \to 0)$ and $b \mapsto (0, \Sigma b, 0 \to \Sigma b)$, and natural transformation $a \mapsto (a \to 0, 0 \to \Sigma f(a), f(a) \to 0 \to \Sigma f(a))$.

We leave the reader to check the explicit computations.

As we will see insubsection 3.3, lax additivity is related to *semiorthogonal decompositions*.

3. Verdier sequences, semiorthogonal decompositions and recollements

In this chapter we discuss Verdier sequences and their various refinements. We begin with the basic notion of Verdier quotient and with practical criteria for recognizing Verdier inclusions and Verdier projections. We then study split Verdier sequences, semiorthogonal decompositions, and stable recollements, emphasizing the relations between these notions. We conclude with a collection of examples that will serve as a model for the applications considered later in the text.

3.1. Verdier sequences.

We begin with the notion of a Verdier sequence, following [CDH+25, Appendix A].

Definition 3.1.1. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a composable pair in $\mathrm{Cat}^{\mathrm{st}}$ whose composite vanishes. We say that it is a *Verdier sequence* if it is both a fibre sequence and a cofibre sequence in $\mathrm{Cat}^{\mathrm{st}}$. In this case we refer to f as a *Verdier inclusion* and to p as the *Verdier projection* (or *Verdier quotient*).

We now unwind this definition. Fibres in $\mathrm{Cat}^{\mathrm{st}}$ are easy to describe.

Remark 3.1.2. Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor in $\mathrm{Cat}^{\mathrm{st}}$. Its *fibre* in $\mathrm{Cat}^{\mathrm{st}}$ is computed as the kernel $\ker(f) \subseteq \mathcal{C}$, that is the full stable subcategory spanned by those objects $c \in \mathcal{C}$ such that $f(c) \simeq 0$ in $\mathcal{D}$.

Cofibres in $\mathrm{Cat}^{\mathrm{st}}$ are described by Verdier quotients.

Remark 3.1.3. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$. A morphism $d \to d'$ in $\mathcal{D}$ is an *equivalence modulo* $\mathcal{C}$ if its fibre (equivalently, its cofibre) lies in the smallest stable subcategory of $\mathcal{D}$ spanned by the essential image of f. Denote by mod the class of all equivalences modulo $\mathcal{C}$. Then the *Verdier quotient of* $\mathcal{D}$ *by* $\mathcal{C}$ is defined as the localisation $\mathcal{D}/\mathcal{C} = \mathcal{D}[\mathrm{mod}^{-1}]$.

Notation 3.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $B \subseteq \mathcal{C}$ be a full subcategory. We will denote by $\mathrm{thick}_{\mathcal{C}}(B)$ the *thick closure*, that is to the smallest stable subcategory of $\mathcal{C}$ which contains B and is closed under retracts.

Lemma 3.1.5. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$ and let $q : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ be the Verdier quotient. Then $\ker(q)$ is equal to the thick closure of $\mathrm{im}(f)$.

Proof. By definition, a morphism u in $\mathcal{D}$ lies in the class mod of equivalences modulo $\mathcal{C}$ if and only if $\mathrm{cofib}(u) \in \mathrm{im}(f)$. An object $x \in \mathcal{D}$ satisfies $q(x) \simeq 0$ if and only if the morphism $0 \to x$ becomes an equivalence in the localisation, that is $0 \to x \in \mathrm{mod}$. But $\mathrm{cofib}(0 \to x) \simeq x$, hence $0 \to d \in \mathrm{mod}$ if and only if $x \in \mathrm{im}(f)$. Thus the full subcategory of $\mathcal{D}$ on the objects mapping to 0 in $\mathcal{D}/\mathcal{C}$ is precisely the thick closure of $\mathrm{im}(f)$. □

We now look at criterions for identifying Verdier sequences. There are essentially two situations in practice, corresponding to the data of the (possibly) Verdier inclusion and of the (possibly) Verdier projection. We begin with the former.

Proposition 3.1.6. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$, and let $q : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ be the Verdier quotient. Then the following are equivalent:

(1) The functor f is a Verdier inclusion.
(2) The induced map $\mathcal{C} \to \ker(\mathcal{D} \to \mathcal{D}/\mathcal{C})$ is an equivalence.
(3) The functor f is fully faithful and its essential image is closed under retracts in $\mathcal{D}$.

Proof. Consider (1) ⇒ (2). If f is a Verdier inclusion, there exists a Verdier sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$. Since it is a fibre sequence in Cat, the induced functor $\mathcal{C} \to \ker(p)$ is an equivalence. Since it is also a cofibre sequence, $\mathcal{E}$ has the universal property of the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$, so $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$ and under this identification $\ker(p) \simeq \ker(q)$. Hence $\mathcal{C} \to \ker(q)$ is an equivalence.

Consider (2) ⇒ (1). Assume $\mathcal{C} \to \ker(q)$ is an equivalence. Then $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a fibre sequence in $\mathrm{Cat}^{\mathrm{st}}$ by definition of kernel. On the other hand, q is by construction the cofibre of the inclusion of its kernel (it is initial among exact functors out of $\mathcal{D}$ which send $\mathcal{C}$ to 0), hence the same sequence is also a cofibre sequence. Therefore f is a Verdier inclusion.

Consider (2) ⇒ (3). The kernel $\ker(q) \subseteq \mathcal{D}$ is a full subcategory, hence the equivalence $\mathcal{C} \simeq \ker(q)$ identifies f with the inclusion of a full subcategory, so f is fully faithful. Moreover, $\ker(q)$ is closed under retracts in $\mathcal{D}$ (if x is a retract of y and $q(y) \simeq 0$, then $q(x)$ is a retract of 0, hence $q(x) \simeq 0$), so the essential image of f is closed under retracts.

Consider (3) ⇒ (2). Assume f is fully faithful and its essential image is closed under retracts. Since f is exact, its essential image is a stable subcategory of $\mathcal{D}$: it contains 0, is closed under shifts, and if $f(x) \to f(y)$ is any morphism in $\mathcal{D}$ then by full faithfulness it comes from a morphism $x \to y$ in $\mathcal{C}$, so its cofibre is $f(\mathrm{cofib}(x \to y))$ and again lies in the essential image. Together with closure under retracts and with Lemma 3.1.5, this shows, that $\ker(q) = \mathrm{im}(f)$, or, in other terms, that $\mathcal{C} \to \ker(q)$ is an equivalence. □

For Verdier projection we have the following.

Proposition 3.1.7. Let $p : \mathcal{D} \to \mathcal{E}$ be in $\mathrm{Cat}^{\mathrm{st}}$, and let $\ker(p) \subseteq \mathcal{D}$ denote the full stable subcategory on the objects d with $p(d) \simeq 0$. Then the following are equivalent:

(1) The functor p is a Verdier projection.
(2) The induced map $\mathcal{D}/\ker(p) \to \mathcal{E}$ is an equivalence.
(3) The functor p exhibits $\mathcal{E}$ as the localisation of $\mathcal{D}$ at the maps it takes to equivalences.

Proof. Let W be the class of maps p takes to equivalences. Consider (1) ⇒ (2). If p is a Verdier projection, then by definition there is a Verdier sequence $\ker(p) \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$. Since it is a cofibre sequence in $\mathrm{Cat}^{\mathrm{st}}$, the functor p has the universal property of the Verdier quotient of $\mathcal{D}$ by $\ker(p)$. Equivalently, $\mathcal{E} \simeq \mathcal{D}/\ker(p)$ and the induced map $\mathcal{D}/\ker(p) \to \mathcal{E}$ is an equivalence.

Consider (2) ⇒ (1). Assume $\mathcal{D}/\ker(p) \to \mathcal{E}$ is an equivalence. By construction, the composite $\ker(p) \to \mathcal{D} \to \mathcal{D}/\ker(p)$ is null, hence so is $\ker(p) \to \mathcal{D} \xrightarrow{p} \mathcal{E}$. Moreover, the sequence $\ker(p) \to \mathcal{D} \to \mathcal{D}/\ker(p)$ is a cofibre sequence in $\mathrm{Cat}^{\mathrm{st}}$ (by the defining property of Verdier quotients), and it is also a fibre sequence because the fibre of $\mathcal{D} \to \mathcal{D}/\ker(p)$ is the kernel $\ker(\mathcal{D} \to \mathcal{D}/\ker(p)) = \ker(p)$. Transporting along the equivalence $\mathcal{D}/\ker(p) \simeq \mathcal{E}$ shows that $\ker(p) \to \mathcal{D} \xrightarrow{p} \mathcal{E}$ is both a fibre and a cofibre sequence, so p is a Verdier projection.

Consider (2) ⇒ (3). Assume $\mathcal{E} \simeq \mathcal{D}/\ker(p)$ via the induced map. Let $q : \mathcal{D} \to \mathcal{D}/\ker(p)$ be the quotient functor. Then q is, by definition, a localisation of $\mathcal{D}$ at the class mod of equivalences modulo $\ker(p)$. In particular, q is a localisation at *the* class of maps it sends to equivalences. Transporting along the equivalence $\mathcal{D}/\ker(p) \simeq \mathcal{E}$, it follows that p is a localisation at the maps it sends to equivalences, that is, at W.

Consider (3) ⇒ (2). Assume p exhibits $\mathcal{E}$ as the localisation $\mathcal{D}[W^{-1}]$. By definition, p inverts every map in W, and it is initial among functors out of $\mathcal{D}$ with this property. Since every morphism $c \to 0$ with $c \in \ker(p)$ lies in W (its image is an equivalence because $p(c) \simeq 0$),any functor $\mathcal{D} \to \mathcal{X}$ which kills $\ker(p)$ factors essentially uniquely through p. This is exactly the universal property characterising the Verdier quotient $\mathcal{D}/\ker(p)$, hence $\mathcal{D}/\ker(p) \simeq \mathcal{E}$. □

Putting everything together we deduce the following.

Corollary 3.1.8. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a composable pair in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composite $p \circ f \simeq 0$. Then the following are equivalent:

(1) The sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a Verdier sequence.
(2) The functor f is fully faithful and its essential image is closed under retracts in $\mathcal{D}$, and the functor p exhibits $\mathcal{E}$ as the Verdier quotient $\mathcal{D}/\mathcal{C}$.
(3) The functor p is a localisation, and f exhibits $\mathcal{C}$ as the kernel of p, that is the induced map $\mathcal{C} \to \ker(p)$ is an equivalence.

Proof. The equivalence (1) ⇔ (2) is Proposition 3.1.6 and (1) ⇔ (3) is Proposition 3.1.7. □

The next observation (which appears as [CDH+25, Proposition A.1.5]) will be quite useful in the following.

Remark 3.1.9. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$. Then p satisfies the *calculus of fractions*. In other terms, the mapping spectra in the localization may be computed as

$$\mathrm{colim}_{(z \to y) \in \mathcal{C}_{/y}} \hom_{\mathcal{D}}(x, \mathrm{cofib}(z \to y)) \to \hom_{\mathcal{E}}(p(x), p(y)).$$

Applying $\Omega^\infty : \mathrm{Sp} \to \mathrm{Spc}$ gives back the usual formula for computing the mapping space in a localization admitting calculus of fractions. In particular, a morphism $p(x) \to p(y)$ in $\mathcal{E}$ may be represented by a *cospan* $x \to w \leftarrow y$ in $\mathcal{D}$ whose right leg $w \leftarrow y$ is an equivalence modulo $\mathcal{C} = \ker(p)$, that is its fibre (or equivalently cofibre) lies in $\mathcal{C} = \ker(p)$. The localisation sends then a cospan $x \to w \leftarrow y$ to the composition $p(w \to y)^{-1} \circ p(x \to w) : p(x) \to p(y)$.

3.2. Split Verdier sequences.

We now discuss the notion of split Verdier sequence.

Definition 3.2.1. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$. We will say that:

(1) The sequence is *right split* if p admits a right adjoint,
(2) The sequence is *left split* if p admits a left adjoint,
(3) The sequence is *split* if p admits both a left and a right adjoint.

In this case we refer to f as a (right or left) *split Verdier inclusion* and to p as a (right or left)*split Verdier projection*.

As we already did, we now study conditions for split Verdier projections. We first notice that left (respectively, right) splitness is indeed a splitting condition.

Corollary 3.2.2. An exact functor $p : \mathcal{D} \to \mathcal{E}$ is a (left or right) split Verdier projection if and only if it admits fully faithful (left or right) adjoints.

Proof. It suffices to prove the left-adjoint statement; the right-adjoint statement follows by passing to opposite categories. For ($\Rightarrow$), assume that p is a right split Verdier projection. Thus p fits into a Verdier sequence $\ker(p) \to \mathcal{D} \xrightarrow{p} \mathcal{E}$ and admits a right adjoint p^R with counit $\varepsilon : p \circ p^R \to \mathrm{id}_{\mathcal{E}}$ and unit $\eta : \mathrm{id}_{\mathcal{D}} \to p^R \circ p$. Fix $d \in \mathcal{D}$. The composite $\varepsilon_{p(d)} \circ p(\eta_d) = \mathrm{id}_{p(d)}$ shows that $p(\eta_d) : p(d) \to pp^R p(d)$ admits a retraction. In a stable category, any map admitting a retraction has vanishing fibre, hence $\mathrm{fib}(p(\eta_d)) \simeq 0$. Since p is exact, $p(\mathrm{fib}(\eta_d)) \simeq \mathrm{fib}(p(\eta_d)) \simeq 0$, so $\mathrm{fib}(\eta_d) \in \ker(p)$. As $\ker(p)$ is a stable subcategory, also $\mathrm{cofib}(\eta_d) \simeq \Sigma\mathrm{fib}(\eta_d)$ lies in $\ker(p)$, hence $p(\mathrm{cofib}(\eta_d)) \simeq 0$, that is $\mathrm{cofib}(p(\eta_d)) \simeq 0$. Now $\varepsilon_{p(d)}$ admits a section $p(\eta_d)$, so in a stable category one has $\mathrm{fib}(\varepsilon_{p(d)}) \simeq \mathrm{cofib}(p(\eta_d))$. Therefore $\mathrm{fib}(\varepsilon_{p(d)}) \simeq 0$, so $\varepsilon_{p(d)}$ is an equivalence. Since p is essentially surjective (being a Verdier projection), it follows that ε is an equivalence on all objects of $\mathcal{E}$. Hence p^R is fully faithful.

Conversely ($\Leftarrow$), assume p admits a fully faithful right adjoint $i : \mathcal{E} \to \mathcal{D}$. Then the counit $pi \to \mathrm{id}_{\mathcal{E}}$ is an equivalence. Set $\mathcal{C} := \ker(p) \subseteq \mathcal{D}$. For a morphism u in $\mathcal{D}$, exactness implies that $p(u)$ is an equivalence if and only if $p(\mathrm{cofib}(u)) \simeq \mathrm{cofib}(p(u)) \simeq 0$, that is $\mathrm{cofib}(u) \in \mathcal{C}$. Since $\mathcal{C}$ is stable and closed under retracts, these are precisely the equivalences modulo $\mathcal{C}$. Thus p exhibits $\mathcal{E}$ as the Verdier quotient $\mathcal{D}/\mathcal{C}$, so p is a Verdier projection. As it also admits a right adjoint, it is right split. ☐

Remark 3.2.3. Put differently, a left (respectively, right) split Verdier projection is precisely a right (respectively, left) Bousfield localisation functor among stable categories, and a split Verdier projection is precisely a functor which is both a left and a right Bousfield localisation.

We now notice that, for a Verdier sequence, splitting conditions on the Verdier projection are equivalent to splitting conditions for the Verdier inclusion. We start with a general result.

Lemma 3.2.4. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a composable pair in $\mathrm{Cat}^{\mathrm{ex}}$ with vanishing composite $p \circ f \simeq 0$. Then the following are equivalent:

1. The sequence is a left (respectively, right) split Verdier sequence.
2. The sequence is a fibre sequence, and p admits a fully faithful left (respectively, right) adjoint q.
3. The sequence is a cofibre sequence, and f is fully faithful and admits a left (respectively, right) adjoint g.

Furthermore, when these equivalent conditions hold, the sequence $\mathcal{E} \xrightarrow{q} \mathcal{D} \xrightarrow{g} \mathcal{C}$ obtained by passing to left (resp. right) adjoints is a right (resp. left) split Verdier sequence.

Proof. It suffices to prove the left-adjoint statement; the right-adjoint statement follows by passing to opposite categories. Consider (1) $\Rightarrow$ (2). If $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a left split Verdier sequence, then it is in particular a fibre sequence, and p admits a left adjoint. Since p is a Verdier projection, Corollary 3.2.2 implies that this left adjoint is fully faithful. Consider (2) $\Rightarrow$ (3). Assume the sequence is a fibre sequence and that $q \dashv p$ with q fully faithful. Since the sequence is a fibre sequence, the induced map $\mathcal{C} \to \ker(p)$ is an equivalence; in particular f is fully faithful. Let $\varepsilon : q \circ p \to \mathrm{id}_{\mathcal{D}}$ be the counit. For $d \in \mathcal{D}$ set $k(d) := \mathrm{cofib}(\varepsilon_d) \in \mathcal{D}$. Applying p and using exactness gives $p(k(d)) \simeq \mathrm{cofib}(p(\varepsilon_d))$. Because q is fully faithful, the unit $\mathrm{id}_{\mathcal{E}} \to p \circ q$ is an equivalence, hence $p(\varepsilon_d) : pqp(d) \to p(d)$ is an equivalence; therefore $p(k(d)) \simeq 0$, so $k(d) \in \ker(p) \simeq \mathcal{C}$. Let $g : \mathcal{D} \to \mathcal{C}$ be the (essentially unique) exact functor characterised by $f(g(d)) \simeq k(d)$.

We claim that $g \dashv f$. Indeed, for $c \in \mathcal{C}$ the cofiber sequence $qp(d) \to d \to fg(d)$ induces a fibre sequence $\hom_{\mathcal{D}}(fg(d), f(c)) \to \hom_{\mathcal{D}}(d, f(c)) \to \hom_{\mathcal{D}}(qp(d), f(c))$. The last term vanishes since $\hom_{\mathcal{D}}(q(e), f(c)) \simeq \hom_{\mathcal{E}}(e, pf(c)) \simeq 0$ (using $pf \simeq 0$), hence $\hom_{\mathcal{D}}(d, f(c)) \simeq \hom_{\mathcal{D}}(fg(d), f(c)) \simeq \hom_{\mathcal{C}}(g(d), c)$, as desired.

It remains to show that the sequence is a cofibre sequence. Let $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ be the Verdier quotient. Since p is exact and $\mathcal{C} \simeq \ker(p)$, a morphism u in $\mathcal{D}$ becomes an equivalence under p if and only if $\mathrm{cofib}(u) \in \ker(p) \simeq \mathcal{C}$, so p inverts the equivalences modulo $\mathcal{C}$ and therefore factors as $p = \bar{p} \circ \pi$ for a functor $\bar{p} : \mathcal{D}/\mathcal{C} \to \mathcal{E}$. Similarly q induces $\bar{q} := \pi q : \mathcal{E} \to \mathcal{D}/\mathcal{C}$. Then $\bar{p}\bar{q} \simeq pq \simeq \mathrm{id}_{\mathcal{E}}$. On the other hand, for each d the morphism $\varepsilon_d : qp(d) \to d$ has cofiber $fg(d) \in \mathcal{C}$, hence $\pi(\varepsilon_d)$ is an equivalence, so $\pi qp(d) \simeq \pi(d)$, that is $\bar{q}\bar{p}\pi(d) \simeq \pi(d)$. Since π is essentially surjective, it follows that $\bar{q}\bar{p} \simeq \mathrm{id}_{\mathcal{D}/\mathcal{C}}$. Thus $\bar{p}$ is an equivalence, so $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$ and the sequence is a cofibre sequence. This proves (3).

Consider (3) ⇒ (2). Assume the sequence is a cofibre sequence and that $g \dashv f$ with f fully faithful. Let $\eta : \mathrm{id}_{\mathcal{D}} \to fg$ be the unit. Define $k(d) := \mathrm{fib}(\eta_d)$. Since $gf \simeq \mathrm{id}_{\mathcal{C}}$, the unit $\eta_{f(c)} : f(c) \to fgf(c)$ is an equivalence, so $k(f(c)) \simeq 0$ for all c. By the universal property of the cofibre $\mathcal{E}$, the assignment $d \mapsto k(d)$ therefore factors through p: there exists $q : \mathcal{E} \to \mathcal{D}$ and a natural equivalence $q\,p \simeq k$. The fibre inclusions $qp(d) \to d$ assemble into a natural transformation $\varepsilon : qp \to \mathrm{id}_{\mathcal{D}}$.

Applying p to the fibre sequence $qp(d) \to d \to fg(d)$ and using $pf \simeq 0$ yields that $p(\varepsilon_d) : pqp(d) \to p(d)$ is an equivalence. Since p is essentially surjective (it is a Verdier projection in a cofibre sequence), this implies $pq \simeq \mathrm{id}_{\mathcal{E}}$. Let $\alpha : \mathrm{id}_{\mathcal{E}} \to pq$ be the inverse equivalence. Then $(q, p, \alpha, \varepsilon)$ exhibits an adjunction $q \dashv p$ with unit α and counit ε, and the unit is an equivalence, hence q is fully faithful.

Finally, the fibre sequence condition follows by identifying the kernel of p. If $d \in \mathcal{D}$ satisfies $p(d) \simeq 0$, then $qp(d) \simeq 0$, hence $\eta_d : d \to fg(d)$ has trivial fibre and is therefore an equivalence. Thus $d \simeq f(g(d))$ lies in the essential image of f. The reverse inclusion $\mathrm{im}(f) \subseteq \ker(p)$ holds since $pf \simeq 0$. Hence $\ker(p) = \mathrm{im}(f)$, and because f is fully faithful we get an equivalence $\mathcal{C} \simeq \ker(p)$, that is, the sequence is a fibre sequence. This proves (2); since (2) + (3) implies (1) trivially, the claim follows. □

We deduce the following useful result.

Corollary 3.2.5. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$. Then p admits a left (respectively, right) adjoint if and only if f admits a left (respectively, right) adjoint.

Proof. It suffices to prove the left-adjoint statement; the right-adjoint statement follows by passing to opposite categories. For (⇒), assume p admits a left adjoint q. Since p is a Verdier projection, the sequence is a fibre sequence, hence $\mathcal{C} \simeq \ker(p)$ and in particular f is fully faithful. By Corollary 3.2.2, the existence of a left adjoint forces q to be fully faithful. The implication (2) ⇒ (3) of Lemma 3.2.4 implies then that f admits a left adjoint. For (⇐), assume that f admits a left adjoint g. In a Verdier sequence, p exhibits $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$. Since f is fully faithful and has a left adjoint, the implication (3) ⇒ (2) of Lemma 3.2.4 implies that p admits a left adjoint. □

Remark 3.2.6. In diagrams, split Verdier sequences appear as

$$\mathcal{C} \underset{}{\overset{\overset{\perp}{\longleftarrow}}{\longrightarrow}} \mathcal{D} \overset{\overset{\perp}{\longleftarrow}}{\longrightarrow} \mathcal{E} \qquad\qquad \mathcal{C} \underset{\underset{\perp}{\longleftarrow}}{\longrightarrow} \mathcal{D} \underset{\underset{\perp}{\longleftarrow}}{\longrightarrow} \mathcal{E}$$

And, by Lemma 3.2.4, to show that the sequences are idneed left (respectively, right) split Verdier, it suffices to check that the functor $\mathcal{D} \to \mathcal{E}$ has a fully-faithful left (respectively, right) adjoint. This is an extremely useful criterion in practice.

3.3. **Semiorthogonal decompositions.** We now discuss semiorthogonal decompositions and their mutations. Conceptually, a semiorthogonal decomposition is a one-sided splitting of a Verdier sequence.

Notation 3.3.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be a full stable subcategory. We define the *right orthogonal* and *left orthogonal* of $\mathcal{A}$ as

$$\mathcal{A}^{\perp} := \{x \in \mathcal{D} \mid \hom_{\mathcal{C}}(a, x) \simeq 0 \text{ for all } a \in \mathcal{C}\}, \qquad {}^{\perp}\mathcal{A} := \{x \in \mathcal{C} \mid \hom_{\mathcal{C}}(x, a) \simeq 0 \text{ for all } a \in \mathcal{A}\}.$$

Notice that $\mathcal{A} \subseteq$.

Remark 3.3.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be a full stable subcategory. It is easy to see that $\mathcal{A}^{\perp}$ and ${}^{\perp}\mathcal{A}$ are full stable subcategories of $\mathcal{D}$, which are closed under retracts in $\mathcal{D}$.

Definition 3.3.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ be full stable subcategories. We say that $(\mathcal{A}, \mathcal{B})$ is a *semiorthogonal pair* if $\hom_{\mathcal{D}}(b, a) \simeq 0$ for all $a \in \mathcal{A},\ b \in \mathcal{B}$.

Equivalently, $(\mathcal{A}, \mathcal{B})$ is a *semiorthogonal pair* if $\mathcal{B} \subseteq {}^{\perp}\mathcal{A}$ (or $\mathcal{A} \subseteq \mathcal{B}^{\perp}$). A convinient way to impose equality is given by decompositions:

Definition 3.3.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. A *(two-step) semiorthogonal decomposition* of $\mathcal{D}$ is a pair of full stable subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ closed under retracts such that:

(1) The pair $(\mathcal{A}, \mathcal{B})$ is semiorthogonal.
(2) The pair $(\mathcal{A}, \mathcal{B})$ is a decomposition, in the sense that for every $c \in \mathcal{C}$ there exists an exact sequence $b \to c \to a$ with $b \in \mathcal{B}$ and $a \in \mathcal{A}$.

In this case we write $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$. Notice tha in a semiorthogonal decomposition, "the second category appears first in the defining properties".

Remark 3.3.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and $\langle \mathcal{A}, \mathcal{B} \rangle$ a semiorthogonal decomposition. In particular, condition (2) implies that $\mathcal{C}$ is generated as a thick subcategory by $\mathcal{A}$ and $\mathcal{B}$: indeed every object of $\mathcal{C}$ is an extension of an object of $\mathcal{A}$ by an object of $\mathcal{B}$, and closure under shifts and retracts propagates this to the thick closure.

The main point is that semiorthogonal decompositions are equivalent to one-sided split Verdier sequences.

Lemma 3.3.6. Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a right split Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$ and let $q : E \to D$ be the fully faithful right adjoint of p. Set $\mathcal{A} := q(E) \subseteq \mathcal{D}$ and $\mathcal{B} := i(\mathcal{C}) = \ker(p) \subseteq \mathcal{D}$. Then the pair $(\mathcal{A}, \mathcal{B})$ is semiorthogonal decomposition.

Proof. Since $p \dashv q$, for $c \in \mathcal{C}$ and $e \in \mathcal{E}$ it follows $\hom_{\mathcal{D}}(i(c), q(e)) \simeq \hom_{\mathcal{E}}(pi(c), e) \simeq 0$, proving (1). For (2), consider the unit $\eta_d : d \to qp(d)$ of the adjunction $p \dashv q$ and let $b_d := \mathrm{fib}(\eta_d)$. Exactness of p gives $p(b_d) \simeq \mathrm{fib}(p(\eta_d)) \simeq 0$, hence $b_d \in \ker(p) = \mathcal{B}$. Also $qp(d) \in \mathcal{A}$ by definition. Thus $b_d \to d \to qp(d)$ is the desired fibre sequence. ☐

Lemma 3.3.7. Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then there exists a *right split* Verdier sequence

$$\mathcal{B} \to \mathcal{D} \xrightarrow{p_{\mathcal{A}}} \mathcal{A}$$

whose right adjoint is the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$.

Proof. For every $d \in \mathcal{D}$ choose a fibre sequence $b \to d \to d$ as in the definition of semiorthogonal decomposition. Semiorthogonality implies that for any $a \in \mathcal{A}$,composition with $d \to a$ induces an equivalence

$$\hom_{\mathcal{D}}(a, a) \simeq \hom_{\mathcal{D}}(d, a),$$

since $\hom_{\mathcal{D}}(b, a) \simeq 0$. This identifies a functorially as the reflection of d into $\mathcal{A}$, hence defines an exact functor $p_{\mathcal{A}} : \mathcal{D} \to \mathcal{A}$ left adjoint to the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$. Its kernel is precisely the full subcategory of objects mapping to 0, that is, by those d with $a_d \simeq 0$, equivalently those $d \simeq b_d \in \mathcal{B}$. Thus $\ker(p_{\mathcal{A}}) = \mathcal{B}$. By Corollary 3.2.2, $p_{\mathcal{A}}$ is a right split Verdier projection, hence $\mathcal{B} \to \mathcal{D} \to \mathcal{A}$ is a right split Verdier sequence. Finally, $\mathcal{B} \subseteq {}^{\perp}\mathcal{A}$ by semiorthogonality, and the converse inclusion follows from the universal property of $p_{\mathcal{A}}$. ☐

Corollary 3.3.8. Semiorthogonal decompositions are equivalent to right split Verdier sequences.

Proof. Follows from Lemma 3.3.7 and Lemma 3.3.6. ☐

Remark 3.3.9. Dually, a *left split* Verdier sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ with fully faithful left adjoint $q : \mathcal{E} \to \mathcal{D}$ determines a semiorthogonal decomposition

$$\mathcal{D} = \langle \ker(p), q(\mathcal{E}) \rangle,$$

and conversely any semiorthogonal decomposition yields such a left split Verdier sequence by passing to opposite categories.

We now discuss mutations: a tool to construct new semiorthogonal decompositions from old ones.

Definition 3.3.10. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be a full stable subcategory. We say that $\mathcal{A}$ is *left admissible* if the inclusion $i_* : \mathcal{A} \hookrightarrow \mathcal{C}$ admits a left adjoint i^*, and *right admissible* if it admits a right adjoint $i^!$. We say that $\mathcal{A}$ is *admissible* if it is both left and right admissible.

Remark 3.3.11. By Lemma 3.3.7, in any semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ the subcategory $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible. If the associated Verdier sequence is split (both adjoints), then both $\mathcal{A}$ and $\mathcal{B}$ are admissible.

Construction 3.3.12. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be a full stable subcategory with inclusion i.

(1) Assume that $\mathcal{A}$ is right admissible, and let $i_* \dashv i^!$. We define the *left mutation functor* through $\mathcal{A}$ to be the exact endofunctor

$$\mathbf{L}_{\mathcal{A}} : \mathcal{C} \to \mathcal{C}, \qquad \mathbf{L}_{\mathcal{A}}(c) := \mathrm{cofib}\big(i_* i^!(c) \to c\big),$$

where $i_* \circ i^! \to \mathrm{id}_{\mathcal{C}}$ is the counit of $i_* \dashv i^!$.

(2) Assume that $\mathcal{A}$ is left admissible, and let $i^* \dashv i_*$. We define the *right mutation functor* through $\mathcal{A}$ is the exact endofunctor

$$\mathbf{R}_{\mathcal{A}} : \mathcal{C} \to \mathcal{C}, \qquad \mathbf{R}_{\mathcal{A}}(c) := \mathrm{fib}\big(c \to i_* \, i^*(c)\big),$$

where $\mathrm{id}_{\mathcal{C}} \to i \circ i^*$ is the unit of $i^* \dashv i_*$.

Notice the exchange of left and right. There is the following: a right admissible subcategory has a a right adjoint, and the left mutation will move the subcategory to the left, a left admissible subcategory has a a left adjoint, and the right mutation will move the subcategory to the right.

Lemma 3.3.13. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be full stable subcategory.

(1) If $\mathcal{A}$ is right admissible, then there exists an exact sequence $i_* i^! \to \mathrm{id}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$ and $\mathbf{L}_{\mathcal{A}}(\mathcal{C}) \subseteq \mathcal{A}^{\perp}$.

(2) If $\mathcal{A}$ is left admissible, then there exists an exact sequence $\mathbf{R}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{C}} \to i_* i^*$ and $\mathbf{R}_{\mathcal{A}}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{A}$.

Proof. It suffices to prove (1) since (2) follows by passing to opposite categories. Assume that $i_* \dashv i^!$. The exact sequence is by definition of $\mathbf{L}_{\mathcal{A}}$. Let $c \in \mathcal{C}$. Given $a \in \mathcal{A}$, applying $\hom_{\mathcal{C}}(i_*(a), -)$ to the exact sequence $i_* i^!(c) \to c \to \mathbf{L}_{\mathcal{A}}(c)$ produces the exact sequence

$$\hom_{\mathcal{C}}(i_*(a), i_* i^!(c)) \to \hom_{\mathcal{C}}(i_*(a), c) \to \hom_{\mathcal{C}}(i_*(a), \mathbf{L}_{\mathcal{A}}(c)).$$

But now the first map identifies with the equivalence

$$\hom_{\mathcal{C}}(i_*(a), i_* i^!(c)) \simeq \hom_{\mathcal{C}}(a, i^!(c)) \to \hom_{\mathcal{C}}(i_*(a), c)$$

via fully-faithfulness of i_* and adjunction. □

Construction 3.3.14. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{A} \subseteq \mathcal{C}$ be an admissible full stable subcategory. Given a full stable subcategory $\mathcal{B} \subseteq \mathcal{C}$, we define

$$\mathbf{L}_{\mathcal{A}}(\mathcal{B}) := \mathrm{Thick}\big(\{\mathbf{L}_{\mathcal{A}}(b) \mid b \in \mathcal{B}\}\big), \qquad \mathbf{R}_{\mathcal{A}}(\mathcal{B}) := \mathrm{Thick}\big(\{\mathbf{R}_{\mathcal{A}}(b) \mid b \in \mathcal{B}\}\big).$$

Proposition 3.3.15. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and $\langle \mathcal{A}, \mathcal{B} \rangle$ a semiorthogonal decomposition.

(1) If $\mathcal{A}$ is right admissible, then $\mathcal{C} = \langle \mathbf{L}_{\mathcal{A}}(\mathcal{B}), \mathcal{A}\rangle$ is again a semiorthogonal decomposition and the restriction $\mathbf{L}_{\mathcal{A}}|_{\mathcal{B}} : \mathcal{B} \to \mathbf{L}_{\mathcal{A}}(\mathcal{B})$ is an equivalence.
(2) If $\mathcal{B}$ is left admissible, then $\mathcal{C} = \langle \mathcal{B}, \mathbf{R}_{\mathcal{B}}(\mathcal{A})\rangle$ is again a semiorthogonal decomposition and the restriction $\mathbf{R}_{\mathcal{B}}|_{\mathcal{A}} : \mathcal{A} \to \mathbf{L}_{\mathcal{B}}(\mathcal{A})$ is an equivalence.

Proof. It suffices to prove (1) since (2) follows by passing to opposite categories. Let $i_* \dashv i^!$ for $i_* : \mathcal{A} \hookrightarrow \mathcal{C}$ the inclusion. To show that $\mathcal{C} = \langle \mathbf{L}_{\mathcal{A}}(\mathcal{B}), \mathcal{A}\rangle$ is a semiorthogonal decomposition it suffices to check the definition. Lemma 3.3.13 implies that $\mathbf{L}_{\mathcal{A}}(\mathcal{C}) \subseteq \mathcal{A}^{\perp}$, which implies immediately that $\mathbf{L}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{A}^{\perp}$; in other terms $\hom_{\mathcal{C}}(\mathcal{A}, \mathbf{L}_{\mathcal{A}}(\mathcal{B})) \simeq 0$, thus the orthogonality condition. For the existence of decompositions, standing the exact sequence $i_*i^!(c) \to c \to \mathbf{L}_{\mathcal{A}}(c)$, it suffice to check that $i_*i^!(c)$ belongs to $\mathcal{A}$, which is obvius up to the fully-faithful emebdding $i_* : \mathcal{A} \hookrightarrow \mathcal{C}$.

For the second claim, it suffices to show that $i_*i^!(b) \simeq 0$ for $b \in \mathcal{B}$. But for every $a \in \mathcal{A}$ it is $\hom_{\mathcal{A}}(a, i^!(b)) \simeq \hom_{\mathcal{C}}(i_*(a), b) \simeq 0$, so that $i^!(b) \simeq 0$ in $\mathcal{A}$, thus $i_*i^!(b) \simeq 0$. □

Definition 3.3.16. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and $\langle \mathcal{A}, \mathcal{B}\rangle$ a semiorthogonal decomposition.

(1) We will say that the semiorthogonal decomposition $\mathcal{C} = \langle \mathbf{L}_{\mathcal{A}}(\mathcal{B}), \mathcal{A}\rangle$ is obtained by $\langle \mathcal{A}, \mathcal{B}\rangle$ via a *left mutation*.
(2) We will say that the semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{B}, \mathbf{R}_{\mathcal{B}}(\mathcal{A})\rangle$ is obtained by $\langle \mathcal{A}, \mathcal{B}\rangle$ via a *right mutation*.

Lax sums are particular examples of mutations:

Proposition 3.3.17. Let $f : \mathcal{A} \to \mathcal{B}$ be in $\mathrm{Cat}^{\mathrm{st}}$. Then:

(1) There exists a semiorthogonal decomposition $\mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \simeq \langle \mathcal{A}, \mathcal{B}\rangle$ induced by the lax limit cone.
(2) There exists a semiorthogonal decomposition $\mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \simeq \langle \mathcal{B}, (a, f(a), \mathrm{id}_{f(a)})\rangle$ induced by the oplax limit cone.

Furthermore, a right mutation of the first decomposition produces the second one, and a left mutation of the second one produces the first one.

Proof. Consider the lax limit cone explained in Example 2.4.12. To fix the notation, let $p : \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \to \mathcal{B}$ be the projections and let

$$\ker(p) = \{(0, b, 0 \to b)\} \simeq \mathcal{B}, \qquad \ker(q) = \{(a, 0, f(a) \to 0)\} \simeq \mathcal{A}$$

be the kernels. Now, for orthogonality, it suffices to check that there is exactly one morphism $(0, b, 0 \to b) \to (a, 0, f(a) \to 0)$. This is obvious, since $0 \to a$ and $b \to 0$ are the zero ones. For the existence of decompositions, let $(a, b, f(a) \to b) \in \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B}$. Then $(0, b, 0 \to b) \to (a, b, f(a) \to b) \to (a, 0, f(a) \to 0)$ is exact, since it is after projecting to the components. Thus (1). Consider now the oplax limit cone, given by $p : \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \to \mathcal{A}$ defined by $(a, b, f(a) \to b) \mapsto a$ and $r : \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \to \mathcal{B}$ defined by $(a, b, f(a) \to b) \mapsto \mathrm{fib}(f(a) \to b)$. The kernel of r is then given by $\ker(r) = \{(a, f(a), \mathrm{id}_{f(a)})\}$. For orthogonality, it suffices to notice that the only map $(a, f(a), \mathrm{id}_{f(a)}) \to (0, b, 0 \to b)$ is the zero one. The existence of decompositions is straightforward. Thus (2). The claim on mutation follows since the left and right adjoint to the inclusion $i_* : \mathcal{B} \hookrightarrow \mathcal{A} \overset{\leftrightarrow}{\oplus}_f \mathcal{B} \to \mathcal{A}$ are given by $i^* : (a, b, f(a) \to b) \mapsto \mathrm{cofib}(f(a) \to b)$ and $i^! = q$. □

3.4. **Stable recollements.** Stable recollements are a further specification of semiorthogonal decompositions (and particular t-structures whose connective and coconnective halfs are stable).

Definition 3.4.1. Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cat}^{\mathrm{st}}$ and suppose we are given fully faithful exact inclusions $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{D}$ and $i_{\mathcal{E}} : \mathcal{E} \hookrightarrow \mathcal{D}$. We say that $\mathcal{D}$ is a *stable recollement* of the pair $(\mathcal{C}, \mathcal{E})$ if the following hold:

(1) There are adjunctions $L_{\mathcal{C}} \dashv i_{\mathcal{C}}$ and $L_{\mathcal{E}} \dashv i_{\mathcal{E}}$.

(2) The composite $L_{\mathcal{E}} \circ i_{\mathcal{C}} \simeq 0$ (equivalently, $L_{\mathcal{E}}$ vanishes on $\mathcal{C}$).
(3) The functors $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$ are jointly conservative.

Remark 3.4.2. In a picture, a stable recollement can be depicted as

$$\mathcal{C} \underset{\perp}{\overset{L_{\mathcal{C}}}{\leftrightarrows}} \mathcal{D} \underset{\perp}{\overset{L_{\mathcal{E}}}{\rightleftarrows}} \mathcal{E}$$

in which the functors going ot of $\mathcal{D}$ are jointly convervative.

The first result identifies stable recollement with split Verdier sequences.

Proposition 3.4.3. If $\mathcal{D}$ is a stable recollement of $\mathcal{C}$ and $\mathcal{E}$, then the sequence $\mathcal{C} \xrightarrow{i_{\mathcal{C}}} \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{st}}$ is a split Verdier sequence. Conversely, if $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a split Verdier sequence, then $\mathcal{D}$ is a stable recollement of the stable subcategories $f(\mathcal{C})$ and $q(\mathcal{E})$, where q denotes the right adjoint of p.

Proof. Consider ($\Rightarrow$). Assume $\mathcal{D}$ is a stable recollement of $\mathcal{C}$ and $\mathcal{E}$, with inclusions $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{D}$ and $i_{\mathcal{E}} : \mathcal{E} \hookrightarrow \mathcal{D}$, left adjoints $L_{\mathcal{C}} \dashv i_{\mathcal{C}}$ and $L_{\mathcal{E}} \dashv i_{\mathcal{E}}$, vanishing composite $L_{\mathcal{E}}\, i_{\mathcal{C}} \simeq 0$, and $L_{\mathcal{C}}, L_{\mathcal{E}}$ jointly conservative. Notice that $\ker(L_{\mathcal{E}}) \simeq \mathcal{C}$ inside $\mathcal{D}$. The inclusion $i_{\mathcal{C}}(\mathcal{C}) \subseteq \ker(L_{\mathcal{E}})$ is immediate from $L_{\mathcal{E}}\, i_{\mathcal{C}} \simeq 0$. For the converse, let $d \in \mathcal{D}$ satisfy $L_{\mathcal{E}}(d) \simeq 0$ and consider the unit $\eta_d : d \to i_{\mathcal{C}} L_{\mathcal{C}}(d)$ of the adjunction $L_{\mathcal{C}} \dashv i_{\mathcal{C}}$. Let $k = \mathrm{fib}(\eta_d)$ and notice that $L_{\mathcal{C}}(k) \simeq 0$ by the triangle identities. Applying $L_{\mathcal{E}}$ shows $L_{\mathcal{E}}(k) \simeq 0$ since $L_{\mathcal{E}}(\eta_d) : L_{\mathcal{E}}(d) \to L_{\mathcal{E}} i_{\mathcal{C}} L_{\mathcal{C}}(d) \simeq 0$ is an equivalence (since $L_{\mathcal{E}}(d) \simeq 0$). Joint conservativity now forces $k \simeq 0$, hence η_d is an equivalence and $d \simeq i_{\mathcal{C}} L_{\mathcal{C}}(d)$ lies in the essential image of $i_{\mathcal{C}}$. Thus $\ker(L_{\mathcal{E}}) = i_{\mathcal{C}}(\mathcal{C})$, so the sequence $\mathcal{C} \xrightarrow{i_{\mathcal{C}}} \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$ is a fibre sequence in $\mathrm{Cat}^{\mathrm{st}}$. Since $L_{\mathcal{E}}$ has a right adjoint, it follows by Lemma 3.2.4 that the sequence is right split Verdier. Since $i_{\mathcal{C}}$ admits a left adjoint, Corollary 3.2.5 implies that $L_{\mathcal{E}}$ has also a left adjoint, thus proving splitness.

Conversely ($\Leftarrow$), assume $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a split Verdier sequence, and let q be the right adjoint of p. Then f is fully faithful (Verdier inclusion) and q is fully faithful (since p is a split Verdier projection). Hence $f(\mathcal{C})$ and $q(\mathcal{E})$ are stable subcategories of $\mathcal{D}$ with inclusions f and q, proving the first condition for stable recollement. Let $L_{\mathcal{E}} = p : \mathcal{D} \to \mathcal{E}$, which is left adjoint to q by definition of q. Moreover, in a split Verdier sequence the inclusion f admits a left adjoint; denote it by $g : \mathcal{D} \to \mathcal{C}$. Then $L_{\mathcal{E}}\, f = pf \simeq 0$, so condition the second condition of stable recollement holds for the pair $(f(\mathcal{C}), q(\mathcal{E}))$. It remains to check joint conservativity of g and p. For every $d \in \mathcal{D}$, the split Verdier structure gives a functorial exact sequence $qp(d) \to d \to fg(d)$ in $\mathcal{D}$. If $p(d) \simeq 0$ then $qp(d) \simeq 0$, and if also $g(d) \simeq 0$ then $fg(d) \simeq 0$; hence $d \simeq 0$. Thus g and p are jointly conservative, and $\mathcal{D}$ is a stable recollement of $f(\mathcal{C})$ and $q(\mathcal{E})$. ☐

We summarize the situation.

Remark 3.4.4. In particular, it follows that any of the diagrams of Verider sequence

$$\mathcal{C} \underset{\perp}{\overset{L_{\mathcal{C}}}{\leftrightarrows}} \mathcal{D} \underset{\perp}{\overset{L_{\mathcal{E}}}{\rightleftarrows}} \mathcal{E}, \qquad \mathcal{C} \underset{\perp}{\overset{\perp}{\leftrightarrows}} \mathcal{D} \longrightarrow \mathcal{E} \qquad \text{and} \qquad \mathcal{C} \longrightarrow \mathcal{D} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \mathcal{E}$$

in which the first one corresponds to a stable recollement, can be completed to a diagram

$$\mathcal{C} \underset{\perp}{\overset{L_{\mathcal{C}}}{\leftrightarrows}} \mathcal{D} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \mathcal{E},$$

in which both the top and the bottom left pointing maps also form Verdier sequences. In particular, to provide a stable recollement it is convinient to provide a split Verdier sequence, which in turn is the same of giving a functor $\mathcal{D} \to \mathcal{E}$ which has (fully-faithful) adjoints on both sides.

Remark 3.4.5. Let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a stable recollement. Then the pair $(i_{\mathcal{C}}(\mathcal{C}), i_{\mathcal{E}}(\mathcal{E}))$ defines a t-structure on $\mathcal{D}$. Indeed, stability under (de)suspensions is clear, orthogonality follows since

$$\mathrm{Hom}_{\mathcal{D}}(i_{\mathcal{C}}(\mathcal{C}), i_{\mathcal{E}}(\mathcal{E})) \simeq \mathrm{Hom}_{\mathcal{D}}(L_{\mathcal{E}} i_{\mathcal{C}}(\mathcal{C}), \mathcal{E}) \simeq 0$$

being $L_{\mathcal{E}} i_{\mathcal{C}} \simeq 0$. For the existence of decompositions, it suffices to notice that the sequence $i_{\mathcal{C}} L_{\mathcal{C}} \to \mathrm{id}_{\mathcal{D}} \to i_{\mathcal{E}} L_{\mathcal{E}}$ is exact.

3.5. **Examples.** We now discuss some examples of Verdier sequences and stable recollements in the context of schemes. We begin with an observation on pullback and pushforward.

Remark 3.5.1. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z.

(1) Consider the adjoint functors $j^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(U) : j_*$. Then j^* is t-exact and j_* is fully-faithful and and t-exact. In particular, j_* identifies with the *extension by zero*. In general j_* does not preserve coherent complexes.
(2) Consider the adjoint functors $i^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(Z) : i_*$. Then i_* is fully-faithful and t-exact. In general, the pullback functor i^* is not left t-exact, since the inclusion $i : Z \hookrightarrow X$ is not of finite tor-amplitude.

Notation 3.5.2. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. We let $\mathrm{QCoh}_Z(X)$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by those quasi-coherent sheaves $F \in \mathrm{QCoh}(X)$ such that $\pi_n(F) \in \mathrm{QCoh}(X)^{\heartsuit}$ is supported on Z (that is, their restriction to U vanishes). It is not hard to see that $\mathrm{QCoh}_Z(X)$ defines a stable subcategory of $\mathrm{QCoh}(X)$.

Proposition 3.5.3. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then there exists a right split Verdier sequence

$$\mathrm{QCoh}_Z(X) \to \mathrm{QCoh}(X) \xrightarrow{j^*} \mathrm{QCoh}(U)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proof. Since the right adjoint j_* is fully-faithful, Lemma 3.2.4 implies the existence of a right split Verdier sequence $\ker(j^*) \to \mathrm{QCoh}(X) \xrightarrow{j^*} \mathrm{QCoh}(U)$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Thus the claim will be proved if $\mathrm{QCoh}_Z(X) \simeq \ker(j^*)$. Notice, first of all, that $\pi_n(j^*(F)) \simeq j^*\pi_n(F)$ for every $F \in \mathrm{QCoh}(X)$. This follows since j^* is t-exact[2]. For $(\subseteq)$, let $F \in \mathrm{QCoh}_Z(X)$. Then $\pi_n(F)$ is supported on Z, so that $j^*\pi_n(F) \simeq 0$ in $\mathrm{QCoh}(U)^{\heartsuit}$ for every $n \in \mathbb{Z}$. But this implies that $\pi_n(j^*(F)) \simeq j^*\pi_n(F) \simeq 0$, and since the t-structure on $\mathrm{QCoh}(U)$ is non-degenerate by Example 1.3.7, it follows that $j^*F \simeq 0$, hence $F \in \ker(j^*)$. Conversely $(\supseteq)$, if $F \in \ker(j^*)$ then $j^*F \simeq 0$, so that $\pi_n(j^*F) \simeq j^*\pi_n(F) \simeq 0$, implying that $F \in \mathrm{QCoh}_Z(X)$. □

The above right split Verdier sequence restricts to perfect complexes, after idempotent-completion.

Corollary 3.5.4. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then there exists a Verdier sequence

$$\mathrm{Perf}_Z(X) \to \mathrm{Perf}(X) \xrightarrow{j^*} \mathrm{Perf}(U)$$

in $\mathrm{Cat}^{\mathrm{st}}$, after idempotent-competion.

[2] In particular, this is exactly the point where the proof of the analogue claim for closed immersions fails.

Proof. It suffices to prove that $(\mathrm{Perf}(X)/\mathrm{Perf}_Z(X))^\natural \to \mathrm{Perf}(U)$ is an equivalence. This follows from the Neeman–Thomason localization theorem [Nee92, Theorem 2.1], which identifies the compact objects in the quotient as the idempotent completion of the Verdier quotient on compacts. □

In the case of noetherian schemes we have a Verdier sequence for coherent complexes. We need a technical result.

Lemma 3.5.5. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ carry bounded t-structures and that f is t-exact. If $f^\heartsuit : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is an equivalence, then f is an equivalence.

Proof. Let $x \in \mathcal{C}$ and define the amplitude of x to be

$$\mathrm{amp}_{\mathcal{C}}(x) = \sup\{n \in \mathbb{Z} \mid \pi_n^{\mathcal{C}}(x) \neq 0\} - \inf\{n \in \mathbb{Z} \mid \pi_n^{\mathcal{C}}(x) \neq 0\}.$$

Notice that $\mathrm{amp}_{\mathcal{C}}(x) \in \mathbb{N}$ being the t-structure on $\mathcal{C}$ bounded. Give a similar definition for the amplitude of an object of $\mathcal{D}$. Now to the actual proof. Notice first of all that f is conservative. Indeed, if $f(x) \simeq 0$, then $0 \cong \pi_n^{\mathcal{D}}(f(x)) \cong f^\heartsuit \pi_n^{\mathcal{C}}(x)$ implies that $\pi_n^{\mathcal{C}}(x) \cong 0$ for every $n \in \mathbb{Z}$, being $f^\heartsuit$ conservative. In particular, x must be zero, since the t-structure on $\mathcal{C}$ is bounded, so that f is conservative. To prove fully-faithfulness and essential surjectivity and fully-faithfulness separately it is convinient to separated the arguments (since they involve complicated induction arguments).

(1) Fully-faithfulness amounts to show that the map $\hom_{\mathcal{C}}(x,y) \to \hom_{\mathcal{D}}(f(x), f(y))$ is an equivalence for every $x, y \in \mathcal{C}$, or, equivalently, that $\mathrm{Hom}_{h\mathcal{C}}(x, \Sigma^n y) \to \mathrm{Hom}_{h\mathcal{D}}(f(x), \Sigma^n f(y))$ is an isomorphism for every $n \in \mathbb{Z}$. Assume first that $x \in \mathcal{C}^\heartsuit$ lies in the heart. The proog goes by induction on the cohomological amplitude of y. If y has amplitude 0, then $y \simeq \Sigma^m b$ for some $b \in \mathcal{C}^\heartsuit$ and some $m \in \mathbb{Z}$. Hence $\mathrm{Hom}_{\mathcal{C}}(a, \Sigma^n y) \simeq \mathrm{Hom}_{\mathcal{C}}(a, \Sigma^{m+n} b)$. If $m+n<0$, then this group vanishes by orthogonality of the t-structure, and similarly on $\mathcal{D}$. If $m+n=0$, this is exactly

$$\mathrm{Hom}_{\mathcal{C}^\heartsuit}(a,b) \to \mathrm{Hom}_{\mathcal{D}^\heartsuit}(f(a), f(b)),$$

which is an isomorphism since $f^\heartsuit$ is an equivalence. If $m+n>0$, then

$$\mathrm{Hom}_{\mathcal{C}}(a, \Sigma^{m+n} b) \to \mathrm{Hom}_{\mathcal{D}}(f(a), \Sigma^{m+n} f(b))$$

is an isomorphism by the identification of higher extensions in the heart with morphisms into positive suspensions, together with the fact that f is exact, t-exact, and induces an equivalence on hearts. Assume now that y has positive amplitude, and let m be the largest integer such that $H^m(y) \neq 0$. Consider the truncation triangle

$$\tau_{\leq m-1} y \to y \to \Sigma^{-m} \pi_m^{\mathcal{C}}(y).$$

Applying $\mathrm{Hom}_{\mathcal{C}}(a, \Sigma^n -)$ and $\mathrm{Hom}_{\mathcal{D}}(f(a), \Sigma^n -)$ yields a morphism of long exact sequences. By induction, the comparison maps are isomorphisms for $\tau_{\leq m-1} y$, since it has smaller amplitude, and also for $\Sigma^{-m}\pi_m^{\mathcal{C}}(y)$, by the amplitude 0 case. It follows from the five lemma that $\mathrm{Hom}_{\mathcal{C}}(a, \Sigma^n y) \to \mathrm{Hom}_{\mathcal{D}}(f(a), \Sigma^n f(y))$ is an isomorphism for every $n \in \mathbb{Z}$.

Now to the general case. Let $x \in \mathcal{C}$ be arbitrary. The proof goes by induction on the cohomological amplitude of x, in that for every $y \in \mathcal{C}$ and every $m \in \mathbb{Z}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(x, \Sigma^m y) \to \mathrm{Hom}_{\mathcal{D}}(f(x), \Sigma^m f(y))$$

is an isomorphism. If x has amplitude 0, then $x \simeq \Sigma^n a$ for some $a \in \mathcal{C}^\heartsuit$. In this case $\mathrm{Hom}_{\mathcal{C}}(x, \Sigma^m y) \simeq \mathrm{Hom}_{\mathcal{C}}(a, \Sigma^{m-n} y)$ and similarly on $\mathcal{D}$, so the claim follows from the previous paragraph. Assume now that x has positive amplitude, and let n be the largest integer such that $\pi_n^{\mathcal{C}}(x) \neq 0$. Consider the exact sequence

$$\tau_{\leq n-1} x \to x \to \Sigma^{-n} \pi_n^{\mathcal{C}}(x).$$

Applying $\mathrm{Hom}_{\mathcal{C}}(-, \Sigma^m y)$ and $\mathrm{Hom}_{\mathcal{D}}(-, \Sigma^m f(y))$ yields again a morphism of long exact sequences. By induction on the amplitude of x, the comparison maps are isomorphisms for

$\tau_{\leq n-1}x$, and by the special case already proved they are also isomorphisms for $\Sigma^{-n}H^n(x)$. Therefore the middle map $\mathrm{Hom}_{\mathcal{C}}(x, \Sigma^m y) \to \mathrm{Hom}_{\mathcal{D}}(f(x), \Sigma^m f(y))$ is an isomorphism for every $m \in \mathbb{Z}$. It follows that for every $x, y \in \mathcal{C}$ the natural map $\mathrm{hom}_{\mathcal{C}}(x, y) \to \mathrm{hom}_{\mathcal{D}}(f(x), f(y))$ is an equivalence. Thus f is fully faithful.

(2) For essential surjectivity, let $d \in \mathcal{D}$. The proof goes by induction on $\mathrm{amp}_{\mathcal{D}}(y)$. If $\mathrm{amp}_{\mathcal{D}}(y) = 0$ then $d \in \mathcal{D}^{\heartsuit}$ and, by essentially surjectivity of $f^{\heartsuit}$, there exists $c \in \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$ such that $f(c) = f^{\heartsuit}(c) \simeq d$, thus prooving the base case. For the inductive step, let n be the greates integer such that $\pi_n^{\mathcal{D}}(y) \neq 0$ and consider the exact sequence

$$\tau_{\leq n-1}y \to y \to \Sigma^{-n}\pi_n^{\mathcal{D}}(y).$$

By induction there exists $c' \in \mathcal{C}$ such that $f(c') \simeq \tau_{\leq n-1}y$ and by essential surjectivity on the heart there exists $c'' \in \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$ such that $f(c'') \simeq \Sigma^{-n}\pi_n(y)$. Since the morphism $\Sigma^{-n}\pi_n^{\mathcal{D}}(y) \to \Sigma\tau_{\leq n-1}y$ is equivalent to a morphism $f(c'') \to \Sigma f(c') \simeq f(\Sigma c')$, it lifts uniquely (by the fully-faithfulness proved before) to a morphism $c'' \to \Sigma c'$ in $\mathcal{C}$. The fibre $c \to c'' \to \Sigma c'$ shows then that $f(c) \simeq y$, thus proving essential surjectivity.

Putting everything together, the claim follows. □

Proposition 3.5.6. Let X be a noetherian scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then there exists a Verdier sequence

$$\mathrm{Coh}_Z(X) \to \mathrm{Coh}(X) \xrightarrow{j^*} \mathrm{Coh}(U)$$

in $\mathrm{Cat}^{\mathrm{perf}}$.

Proof. Notice first that, since j is flat being an open immersion, the functor $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$ is t-exact, and hence preserves coherent complexes, thus restricting to a functor $j^* : \mathrm{Coh}(X) \to \mathrm{Coh}(U)$. The goal is to apply Lemma 3.5.5. Notice first that the bounded t-structure on $\mathrm{Coh}(X)$ restricts to a bounded t-structure on $\mathrm{Coh}_Z(X)$ (essentially because if a complex on X has cohomology supported on Z then its truncations have cohomology supported on Z). In particular, the quotient $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X)$ has a bounded t-structure, and the induced functor $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \to \mathrm{Coh}(U)$ is t-exact (see Exercise E.3.2). Thus the claim follows from Exercise E.3.3. □

Remark 3.5.7. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. By Corollary 3.2.2 (and its proof) it follows that the inclusion $\iota_* : \mathrm{QCoh}_Z(X) \to \mathrm{QCoh}(X)$ admits a right adjoint Γ_Z, called the *global section on* Z, which is given by $\Gamma_Z(F) \simeq \mathrm{fib}(F \to j_* j^* F)$ for $F \in \mathrm{QCoh}(X)$. In particular, the situation is summarised by the diagram

$$\mathrm{QCoh}_Z(X) \underset{\Gamma_Z}{\overset{\iota_*}{\rightleftarrows}} \mathrm{QCoh}(X) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{QCoh}(U).$$

In general, the above diagram cannot be extended to a stable recollement, since the restriction j^* does not admit a left adjoint $j_!$ at the level of derived categories of quasi-coherent sheaves. This is tightly related to the Grothendieck-Neeman duality [BDS16, Theorem 1.7]: the functor j^* admits a left adjoint $j_!$ if and only if Grothendieck-Neeman duality holds for $j : U \hookrightarrow X$, that is, if and only if j is quasi-perfect (see [LN07, Definition 1.1]), and this deos not happen often.

Nonetheless, for schemes there is a *wrong way recollement*, proved in [Jor05].

Proposition 3.5.8. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into a quasi-compact open U and closed Z. Then there exists a stable recollement

$$\mathrm{QCoh}(U) \underset{\longleftarrow}{\overset{j^*}{\underset{\longleftarrow}{\xrightarrow{\;j_*\;}}}} \mathrm{QCoh}(X) \underset{\longleftarrow}{\overset{\iota_*}{\underset{\longleftarrow}{\xrightarrow{\;\Gamma_Z\;}}}} \mathrm{QCoh}_Z(X).$$

The reason why it is a wrong way recollement is because, in general, the open part is on the right of the diagram.

4. Calculus of subcategories

In this chapter we introduce a calculus of subcategories of a stable category. We first study the basic operations on subcategories, such as extension, retract closure, and inverse or direct image along exact functors. We then pass to the finite constructions given by coproduct closures and thickenings, and to their "big" variants obtained by allowing arbitrary coproducts. The chapter ends with the notion of generation time, which will be one of the main numerical invariants used in the rest of the notes.

4.1. **Basic operations.**

Notation 4.1.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. We will denote by $\mathrm{Sub}(\mathcal{C})$ the large set of full subcategories of $\mathcal{C}$, and we will regard it as a poset ordered by inclusion. Given $A, B \in \mathrm{Sub}(\mathcal{C})$, the operations $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ equip $\mathrm{Sub}(\mathcal{C})$ with the structure of a distributive lattice, for which the empty subcategory and $\mathcal{C}$ furnish the initial and terminal element. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$. Then the *inverse image* defines a functor $f^{-1} : \mathrm{Sub}(\mathcal{D}) \to \mathrm{Sub}(\mathcal{C})$ via $A \mapsto f^{-1}(A)$ defined in the obvious way. Notice that there exists a double adjunction

$$\mathrm{Sub}(\mathcal{D}) \underset{\underset{\forall_f}{\longleftarrow}}{\overset{\exists_f}{\underset{\longleftarrow}{\xrightarrow{\;f^{-1}\;}}}} \mathrm{Sub}(\mathcal{C})$$

thus making f^{-1} a morhpism of lattices (since it preserves all meets and all joins). The *existential image* $\exists_f$ is defined by $B \mapsto f(B)$ and the *universal image* $\forall_f$ is defined by $B \mapsto \bigvee\{A \in \mathrm{Sub}(\mathcal{D})$ such that $f^{-1}(A) \subseteq B\}$. It follows that the assignment $\mathcal{C} \mapsto \mathrm{Sub}(\mathcal{C})$ may be regarded as a functor $(\mathrm{Cat}^{\mathrm{st}})^{\mathrm{op}} \to \mathrm{Lattices}$ with values in the category of large distributive lattices.

Remark 4.1.2. We will understand distributive lattices as 1-categories with contractible mapping spaces, that have all limits and colimits, for which arbitrary colimits distribute over limits.

The "calculus of subcategories" consists of studying the possible operations defined on the distributive lattice of subcategories. We begin with introducing a monoidal structure on $\mathrm{Sub}(\mathcal{C})$.

Definition 4.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A, B \subseteq \mathcal{C}$ be two full subcategories. The *extension of A with B* is the full subcategory $A * B$ spanned by those objects $x \in \mathcal{C}$ for which there is an exact sequence $a \to x \to b$ with $a \in A$ and $b \in B$.

Remark 4.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A, B \subseteq \mathcal{C}$ be two full subcategories. If $0 \in \mathcal{A}$, then the convolution $A * B$ contains B via the extension $0 \to b \to b$ for $b \in B$.

Remark 4.1.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and regard the extension of subcategories as an operation $- * - : \mathrm{Sub}(\mathcal{C}) \times \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$. Let 0 denote the zero subcategory (that is, the subcategory containing only the zero object). Then:

(1) The triple $(\mathrm{Sub}(\mathcal{C}), *, 0)$ defines a monoidal category for which the tensor product preserves colimits in each variable separately.

(2) The extension prodcut is compatible with taking opposites, in that $(A * B)^{\mathrm{op}} = B^{\mathrm{op}} * A^{\mathrm{op}}$ for every pair of full subcategories $A, B \subseteq \mathcal{C}$.

Indeed, for the first part of (1) it suffices to notice that the octahedral axiom furnishes $A * (B * C) = (A * B) * C$ for every triple of subcategories $A, B, C \subseteq \mathcal{C}$ and that $A * 0 = A = 0 * A$. For the second part, it is just a computation. Point (2) is trivial.

Remark 4.1.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A \in \mathrm{Sub}(\mathcal{C})$ be a full subcategory. Let $\lambda_A, \rho_A : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ be the functors defined by

$$B \mapsto \lambda_A(B) = \{x \in \mathcal{C} \,|\, \{x\} * A \subseteq B\} \qquad B \mapsto \rho_A(B) = \{x \in \mathcal{C} \,|\, A * \{x\} \subseteq B\}$$

Then a trivial computation shows that $\lambda_A \dashv A * -$ and $A * - \dashv \rho_A$. It follows that the monoidal structure is both left and right closed.

The following result summarise everything: we have defined a residuated lattice.

Corollary 4.1.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{Sub}(\mathcal{C})$ is complete and cocomplete 1-category with contractible mapping spaces for which arbitrary colimits distribute over finite limits. Furthermore, with the monoidal structure given by the extension product, $\mathrm{Sub}(\mathcal{C})$ is left and right closed.

We now study the functoriality of this observation.

Lemma 4.1.8. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$.

(1) The functor f^{-1} is lax monoidal.
(2) The functor $\exists_f$ is oplax monoidal. If f is a Verdier inclusion, then $\exists_f$ is symmetric monoidal.

Proof. The first claim amounts to show that $f^{-1}(A) * f^{-1}(B) \subseteq f^{-1}(A * B)$ for $A, B \in \mathrm{Sub}(\mathcal{D})$, which is trivial. In particular, since the right adjoint f^{-1} is lax monoidal, the left adjoint $\exists_f$ is oplax monoidal, that is $\exists_f(A * B) \subseteq \exists_f(A) * \exists_f(B)$ for $A, B \in \mathrm{Sub}(\mathcal{C})$. The last claim follows from Proposition 3.1.6. □

We now introduce a *closure operator* on $\mathrm{Sub}(\mathcal{C})$. Recall that a closure operator on a poset $(P, \leq)$ is a functor $\mathrm{cl} : P \to P$ which is *extensive* (that is, $x \leq \mathrm{cl}(x)$), *monotone* (that is $x \leq y$ implies $\mathrm{cl}(x) \leq \mathrm{cl}(y)$) and *idempotent* (that is $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$). Recall also that an *interior operator* on a poset $(P, \leq)$ is a functor $\mathrm{in} : P \to P$ which is *intensive* (that is $\mathrm{in}(x) \leq x$), monotone and idempotent.

Definition 4.1.9. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $B \subseteq \mathcal{C}$ be a full subcategory. We define:

(1) The *retract closure* $\mathrm{smd}(B)$ of B is the smallest full subcategory of $\mathcal{C}$ closed under retracts and containing B.
(2) The *retract interior* $\mathrm{smd}^{\flat}(B)$ of B is the largest full subcategory of B closed under retracts.

Remark 4.1.10. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then the assignments $B \mapsto \mathrm{smd}(B)$ and $B \mapsto \mathrm{smd}^{\flat}(B)$ define a closure $\mathrm{smd} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ and an interior operator $\mathrm{smd}^{\flat} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$. Notice furthermore that they extend to an adjunction $\mathrm{smd} : \mathrm{Sub}(\mathcal{C}) \rightleftarrows \mathrm{Sub}(\mathcal{C}) : \mathrm{smd}^{\flat}$. In particular, the retract closure preserves all joins.

Lemma 4.1.11. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{smd} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ is lax monoidal.

Proof. Let $A, B \subseteq \mathcal{C}$ be two full subcategories. The claim is $\mathrm{smd}(A) * \mathrm{smd}(B) \subseteq \mathrm{smd}(A * B)$. Pick $y \in \mathrm{smd}(A) * \mathrm{smd}(B)$. Then there is a cofibre sequence $a' \to y \to b'$ with $a' \in \mathrm{smd}(A)$ and $b' \in \mathrm{smd}(B)$. Choose retract data $a' \to a \to a'$ and $b' \to b \to b'$ with $a \in A$ and $b \in B$. Using functoriality of exact sequences, one checks that y is a retract of an object y' sitting in a exact sequence $a \to y' \to b$, hence $y' \in A * B$ and therefore $y \in \mathrm{smd}(A * B)$. □

We now study the interaction between the retract closure and the existential and inverse image.

Lemma 4.1.12. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$.

(1) Then $\exists_f \circ \mathrm{smd} \subseteq \mathrm{smd} \circ \exists_f$.
(2) If f is a Verdier inclusion, then $\exists_f \circ \mathrm{smd} = \mathrm{smd} \circ \exists_f$.
(3) Then $\mathrm{smd} \circ f^{-1} \subseteq f^{-1} \circ \mathrm{smd}$.
(4) If f is a Verdier projection, then $f^{-1} \circ \mathrm{smd} \subseteq \ker(f) * f^{-1}(-) * \ker(f)$.

Proof. Let $A \in \mathrm{Sub}(\mathcal{C})$ be a full subcategory. Consider (1) and let $x \in \mathrm{smd}(A)$. Then there exists a retract $x \to a \to x$ with $a \in A$. Apply f to get that $f(x)$ is a retract of $f(a)$, so that $\exists_f \circ \mathrm{smd} \subseteq \mathrm{smd} \circ \exists_f$.

Consider (2). It follows from Proposition 3.1.6: since the essential image of f is closed under retracts, any such diagram $x \to f(a) \to x$ must have $x \simeq f(y)$, and since f is fully-faithful, the above lifts to a retract $x \to a \to x$, thus showing that x is the image of a retract of an object of A.

Consider (3). Let $B \in \mathrm{Sub}(\mathcal{D})$ and let $x \in \mathrm{smd}(f^{-1}(B))$. By definition, there exists $y \in f^{-1}(B)$ and maps $x \to y \to x$ whose composite is id_x. Applying f, the object $f(x)$ is a retract of $f(y) \in B$, hence $f(x) \in \mathrm{smd}(B)$. Therefore $x \in f^{-1}(\mathrm{smd}(B))$.

Consider (4) let $x \in f^{-1}(\mathrm{smd}(B))$ for $B \in \mathrm{Sub}(\mathcal{D})$. Thus $q(x) \in \mathrm{smd}(B)$ and there exists a retract $f(x) \to b \to f(x)$. Choose $y \in \mathcal{C}$ such that $f(y) \simeq b$; then $y \in f^{-1}(B)$. By the calculus of fractions for Verdier projections Remark 3.1.9, the morphisms $f(x) \to b \simeq f(y)$ and $f(y) \simeq b \to f(x)$ may be represented by roofs, in that there exist morphisms $x \to w \leftarrow y$ and $y \to v \leftarrow x$ in $\mathcal{C}$ whose right leg $w \leftarrow y$ and $v \leftarrow x$ is an equivalence modulo $\ker(f)$, that is its fibre (or equivalently cofibre) lies in $\ker(f)$. Then x fits into an exact sequence $\mathrm{fib}(x \to w) \to x \to w$ and in particular $x \in \ker(f) * \{w\}$. On the other side, the exact sequence $y \to w \to \mathrm{cofib}(y \to w)$ shows $w \in \{y\} * \ker(f) \subseteq f^{-1}(B) * \ker(f)$. Putting everything together shows $x \in \ker(f) * \{w\} \subseteq \ker(f) * f^{-1}(B) * \ker(f)$, which implies the claim. Notice that the same result would have been obtained starting from $y \to v \leftarrow x$. ☐

4.2. **The finite picture.** We now introduce some finite operations. We begin with the closure under shifts.

Notation 4.2.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Given a full subcategory $B \subseteq \mathcal{C}$, we write $\mathbb{Z}(B)$ for the strictly full subcategory spanned by all shifts $\Sigma^i b$ with $i \in \mathbb{Z}$ and $b \in B$. Notice that $\mathbb{Z} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ is a closure operator that commutes with all joins and meets. Notice that it is oplax monoidal, in the sense that $\mathbb{Z}(A * B) \subseteq \mathbb{Z}(A) * \mathbb{Z}(B)$. It is not hard to see that $\mathbb{Z}(A) * \mathbb{Z}(B) = \mathbb{Z}(A * \mathbb{Z}(B)) = \mathbb{Z}(\mathbb{Z}(A) * B)$, so if one of the input is shift closed then $\mathbb{Z}$ is symmetric monoidal.

Definition 4.2.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. Define $\mathrm{add}(A) \subseteq \mathcal{C}$ to be the full subcategory generated from A under finite coproduct. We define, inductively, the following full subcategories:

(1) The 1-*coproduct closure of* A as $\mathrm{coprod}_1(A) := \mathrm{add}(A)$.
(2) The n-*coproduct closure of* A as $\mathrm{coprod}_{n+1}(A) := \mathrm{coprod}_1(A) * \mathrm{coprod}_n(A)$.
(3) The *coproduct closure of* A as $\mathrm{coprod}(A) := \bigcup_{n \geq 1} \mathrm{coprod}_n(A)$.

By construction, $\mathrm{coprod}_n(A)$ is *not* closed under retracts in general. This is deliberate: it isolates the "pure n-step cell" part. In the following we will also denote by $\mathrm{coprod}_0(A) = 0$ the zero subcategory.

Lemma 4.2.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{coprod}_n : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ is a functor for every $n \in \mathbb{N}$, even infinite.

Proof. The first claim is that $\mathrm{coprod}_n(A) \subseteq \mathrm{coprod}_n(B)$ whenever $A \subseteq B$. This follows easily for $n = 1$ by Notation 4.2.1 and the definition of add. The general setp follows then by induction, once it is remembered that $*$ preserves inclusions in each variable separately. The case where n is infinite follows by functoriality of joins. ☐

Remark 4.2.4. Notice that $\mathrm{coprod}_1, \mathrm{coprod} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ are closure operators, whereas the n-corpoduct closure is not. Furthermore, the functors coprod_n, for $n \in \mathbb{N}$, and coprod do not preserve arbitrary joins or meets (but it preserves directed joins) since mixed extensions are available, and are not (op)lax monoidal.

We still have a calculus for the n-coproducts.

Lemma 4.2.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then:

(1) The filtration is increasing, $\mathrm{coprod}_n \subseteq \mathrm{coprod}_{n+1}$.
(2) The filtration is strictly additive, $\mathrm{coprod}_n * \mathrm{coprod}_m = \mathrm{coprod}_{n+m}$.
(3) The coproduct closure is a fixed point, $\mathrm{coprod} = \mathrm{coprod} * \mathrm{coprod}$.
(4) The filtration is strictly multiplicative, $\mathrm{coprod}_n(\mathrm{coprod}_m) = \mathrm{coprod}_{nm}$.
(5) Let $A \in \mathrm{Sub}(\mathcal{C})$ and $x \in \mathrm{coprod}_n(A)$. Then there exists a filtration $0 = x_0 \to x_1 \to \cdots \to x_n = x$ where each successive cofibre lies in $\mathrm{coprod}_1(A)$.

Proof. For (1), notice that $0 \in \mathrm{coprod}_1$, so that $\mathrm{coprod}_n \subseteq \mathrm{coprod}_{n+1}$ by Remark 4.1.4. For (2), it suffices to use the associativity of $*$. For (3), the inclusion ($\subseteq$) is always true, so consider ($\supseteq$). Given $x \in \mathrm{coprod} * \mathrm{coprod}$ there exists a triangle $y \to x \to z$ with $y \in \mathrm{coprod}_n$ and $z \in \mathrm{coprod}_m$, so that $x \in \mathrm{coprod}_{n+m} \subseteq \mathrm{coprod}$. Point (5) can be then proved by induction. □

We also have the following functoriality of (the shifts and of) the n-coproducts.

Lemma 4.2.6. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$ and let $\ker(f)$ be the kernel. Then:

(1) The *existential image commutes* $\mathbb{Z}$, in that $\exists_f \circ \mathbb{Z} = \mathbb{Z} \circ \exists_f$.
(2) The *inverse image commutes* $\mathbb{Z}$, in that $f^{-1} \circ \mathbb{Z} = \mathbb{Z} \circ f^{-1}$.
(3) The *existential image preserves* coprod_n for every $n \in \mathbb{N}$, even infinite, in that $\exists_f \circ \mathrm{coprod}_n \subseteq \mathrm{coprod}_n \circ \exists_f$. Equality holds if f is a Verdier inclusion.
(4) The *inverse image preserves* coprod_n *up to a kernel* for every $n \in \mathbb{N}$, even infinite, in that $\mathrm{coprod}_n(f^{-1}(-) \cup \ker(f)) \subseteq f^{-1}(\mathrm{coprod}_n(-))$.

Proof. Let $A \in \mathrm{Sub}(\mathcal{C})$ be a full subcategory. Then an element belongs to $\exists_f(\mathbb{Z}(A))$ if and only if it is $f(\Sigma^n a)$ for some $n \in \mathbb{Z}$ and $a \in A$, and since f is exact, if and only if it is $\Sigma^n f(a)$, that is, if and only if is in $\mathbb{Z}(\exists_f(A))$. This proves (1). The proof of (2) is similar. For (3), notice that in the case $n = 1$ it is actually $\exists_f \circ \mathrm{coprod}_1 = \mathrm{coprod}_1 \circ \exists_f$ since f is exact. For the general case, use induction and point (2) of Lemma 4.1.12. Consider now (4) and let $B \in \mathrm{Sub}(\mathcal{D})$ be a full subcategory. Since the counit of the existential-inverse image furnishes an inclusion $\exists_f\big(f^{-1}(B)\big) \subseteq B$ and since $\exists_f(\ker(f)) \subseteq \{0\}$ by kernel arguments, it is $\exists_f(f^{-1}(B) \cup \ker(f)) \subseteq B \cup \{0\}$ via functoriality of the existential image. By using that the existential image preserves coprod_n it follows that

$$\exists_f\big(\mathrm{coprod}_n(f^{-1}(B) \cup \ker(f))\big) \subseteq \mathrm{coprod}_n\big(\exists_f(f^{-1}(B) \cup \ker(f))\big) \subseteq \mathrm{coprod}_n(B \cup \{0\}).$$

Since $0 \in \mathrm{coprod}_1(B)$, it follows that $\mathrm{coprod}_n(B \cup \{0\}) = \mathrm{coprod}_n(B)$ for all $n \geq 1$. Hence

$$\exists_f\big(\mathrm{coprod}_n(f^{-1}(B))\big) \subseteq \mathrm{coprod}_n(B).$$

Unwinding the definition of $\exists_f$, this means: for every $x \in \mathrm{coprod}_n(f^{-1}(B) \cup \ker(f))$ one has $f(x) \in \mathrm{coprod}_n(B)$, that is, $x \in f^{-1}(\mathrm{coprod}_n(B))$. This proves the first inclusion. The statement for $\mathrm{coprod} = \bigcup_{n \geq 1} \mathrm{coprod}_n$ follows by taking unions over n. □

Taking closure under retracts in each step leads to the following.

Definition 4.2.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. We define:

(1) The 1-*thickening of* A as $\mathrm{thick}_1(A) := \mathrm{smd}(\mathrm{coprod}_1(\mathbb{Z}(A)))$
(2) The n-*thickening of* A as $\mathrm{thick}_n(A) := \mathrm{smd}(\mathrm{thick}_1(A) * \mathrm{thick}_{n-1}(A))$.
(3) The *thickening of* A as $\mathrm{thick}(A) := \bigcup_{n \in \mathbb{N}} \mathrm{thick}_n(A)$.

In other terms, the 1-thickening of A is the smallest full subcategory of $\mathcal{C}$ closed under shifts, finite coproducts and retracts containing A and the the n-thickening as the closure under retracts of the extension product of the 1 and $(n-1)$-thickening of A. In the following we will also denote by $\mathrm{thick}_0(A) = 0$ the zero subcategory.

Lemma 4.2.8. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{thick}_n : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ is a functor for every $n \in \mathbb{N}$, even infinite.

Proof. The claim is that $\mathrm{thick}_n(A) \subseteq \mathrm{thick}_n(B)$ whenever $A \subseteq B$. The proog goes by induction, the case $n = 0, 1$ being trivial. Assume the claim for $n-1$. Then $\mathrm{thick}_1(A) \subseteq \mathrm{thick}_1(B)$ and $\mathrm{thick}_{n-1}(A) \subseteq \mathrm{thick}_{n-1}(B)$. In particular,

$$\mathrm{thick}_1(A) * \mathrm{thick}_{n-1}(A) \subseteq \mathrm{thick}_1(A) * \mathrm{thick}_{n-1}(B) \subseteq \mathrm{thick}_1(B) * \mathrm{thick}_{n-1}(B)$$

since the extension product preserves inclusions. The claim then follows since smd is monotone. □

The following result explains the calculus for thickenings.

Lemma 4.2.9. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category.

1. The filtration is increasing, $\mathrm{thick}_n \subseteq \mathrm{thick}_{n+1}$.
2. The filtration is additive $\mathrm{thick}_m * \mathrm{thick}_n \subseteq \mathrm{thick}_{m+n}$.
   It becomes an equality after retract closure, $\mathrm{thick}_{n+m} = \mathrm{smd}(\mathrm{thick}_n * \mathrm{thick}_m)$.
3. The thickening is a fixed point, $\mathrm{thick} * \mathrm{thick} = \mathrm{thick}$.
4. The filtration is strictly multiplicative, $\mathrm{thick}_n(\mathrm{thick}_m) = \mathrm{thick}_{nm}$.
5. Let $A \in \mathrm{Sub}(\mathcal{C})$ and $x \in \mathrm{thick}_n(A)$. Then there exists a filtration $0 = x_0 \to x_1 \to \cdots \to x_n = x$ where each successive cofibre lies in $\mathrm{thick}_1(A)$.

The filtration in (4) is called the *cell filtration* of x.

Proof. The proof goes exactly as in Lemma 4.2.5. □

We have the following converse to the cell filtration.

Corollary 4.2.10. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{thick}_n = \mathrm{smd}(\mathrm{thick}_1^{*n})$.

Proof. It follows from point (2) of Lemma 4.2.9. □

Lemma 4.2.11. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{thick}_n = \mathrm{smd}\,\mathrm{coprod}_n$ for every $n \in \mathbb{N}$, even infinite.

We also have the following functoriality for n-thickenings.

Lemma 4.2.12. Let $f : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Cat}^{\mathrm{st}}$. Then:

1. The *existential image preserves* thick_n for every $n \in \mathbb{N}$, even infinite, in that $\exists_f(\mathrm{thick}_n(-)) \subseteq \mathrm{thick}_n(\exists_f(-))$. Equality holds if f is a Verdier inclusion.
2. The *inverse image preserves* thick_n *up to a kernel* for every $n \in \mathbb{N}$, even infinite, in that $\mathrm{thick}_n(f^{-1}(-) \cup \ker(f)) \subseteq f^{-1}(\mathrm{thick}_n(-))$.

Proof. Let $A \in \mathrm{Sub}(\mathcal{C})$ and let $x \in \mathrm{thick}_n(A)$. Then by Lemma 4.2.9 there exists a filtration $0 = x_0 \to x_1 \to \cdots \to x_n = x$ whose successive cofibres lie in $\mathrm{thick}_1(A)$. Since f is exact, the successive cofibres, after applying f, all lie in $\exists_f(\mathrm{thick}_1(A)) \subseteq \mathrm{thick}_1(\exists_f(A))$. Here the last inclusion follows from Lemma 4.2.6 and point (2) of Lemma 4.1.12. The claim for infinite n follows since $\exists_f$ preserves joins, being a left adjoint. For (2), the proof follows from Lemma 4.2.6 plus the fact that f^{-1} commutes with $\mathbb{Z}$ by Lemma 4.2.6 and with the retract closure by point (3) of Lemma 4.1.12. □

There exists an interesting functoriality of n-coproducts and thickening for (left, respectively right split) Verdier projections.

Proposition 4.2.13. Let $q:\mathcal{D}\to\mathcal{E}$ be a Verdier projection with kernel $\ker(q)$.

(1) If $q\dashv q^{\mathrm{R}}$ has a right adjoint, then $q^{-1}=\ker(q)*\exists_{q^{\mathrm{R}}}$ as functors $\mathrm{Sub}(\mathcal{E})\to\mathrm{Sub}(\mathcal{D})$.
(2) Dually, if $q^{\mathrm{L}}\dashv q$ has a left adjoint, then $q^{-1}=\exists_{q^{\mathrm{L}}}*\ker(q)$ as functors $\mathrm{Sub}(\mathcal{E})\to\mathrm{Sub}(\mathcal{D})$.

In one of this cases, there are inclusions

$$q^{-1}\circ\mathrm{coprod}_n\subseteq\mathrm{coprod}_{n+1}(q^{-1}\cup\ker(q))\qquad\text{and}\qquad q^{-1}\circ\mathrm{thick}_n\subseteq\mathrm{thick}_{n+1}(q^{-1}\cup\ker(q))$$

of functors $\mathrm{Sub}(\mathcal{E})\to\mathrm{Sub}(\mathcal{D})$.

Proof. Let $E\in\mathrm{Sub}(\mathcal{E})$ and consider (1). If $x\in q^{-1}(E)$, then the exact sequence $s_x\to x\to q^{\mathrm{R}}q(x)$ with $s_x\in\ker(q)$ shows that $x\in\ker(q)*\exists_{q^{\mathrm{R}}}(E)$. Conversely, given $x\in\ker(q)*\exists_{q^{\mathrm{R}}}(E)$, then there exists an exact sequence $k\to x\to q^{\mathrm{R}}(e)$. Applying q yields $0\to q(x)\to qq^{\mathrm{R}}(e)\simeq e$ being q^{R} fully-faithful, thus $x\in q^{-1}(E)$. The proof of (2) is similar. Consider now the last statement, and assume (1) (the proof in case of (2) is analogous). Then

$$\begin{aligned}
q^{-1}(\mathrm{coprod}_n(E))&=\ker(q)*\exists_{q^{\mathrm{R}}}(\mathrm{coprod}_n(E))\\
&\subseteq\ker(q)*\mathrm{coprod}_n(\exists_{q^{\mathrm{R}}}(E))\\
&\subseteq\ker(q)*\mathrm{coprod}_n(q^{-1}(E))\\
&\subseteq\ker(q)*\mathrm{coprod}_n(\ker(q)\cup q^{-1}(E))\\
&\subseteq(\ker(q)\cup q^{-1}(E))*\mathrm{coprod}_n(\ker(q)\cup q^{-1}(E))\\
&\subseteq\mathrm{coprod}_{n+1}(\ker(q)\cup q^{-1}(E)).
\end{aligned}$$

Here the first equality is point (1), the second inclusion is Lemma 4.2.6, the third one is the inclusion $\exists_{q^{\mathrm{R}}}(E)\subseteq q^{-1}(E)$ (which follows since q^{R} is fully-faithful), the fourth and fifth one follow since $q^{-1}(E)\subseteq\ker(q)\cup q^{-1}(E)$ and the last one is by definition of coprod_n. The statement with thickening is analogous. □

The 1-thickening of a full subcategory is not stable in general, since it lacks cofibres (but it contains the zero object).

Example 4.2.14. Let $R\in\mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring and consider the category of perfect modules Perf_R. Consider the 1-thickening $\mathrm{thick}_1(R)$ of R. Pick $t\in R$ and notice that the multiplication by $t:R\to R$ lies in $\mathrm{thick}_1(R)$ of R. However, the cofibre is given by the quotient R/t, and if t is not a unit or a zero divisor then $R/t\notin\mathrm{thick}_1(R)$.

In the limit $n\to\infty$, the n-thickening becomes more and more stable.

Lemma 4.2.15. Let $\mathcal{C}\in\mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A\subseteq\mathcal{C}$ be a full subcategory. Then the thickening $\mathrm{thick}(A)$ is stable and coincide with the thick closure of Notation 3.1.4 (thus justifying the notation).

Proof. Indeed, notice that $\mathrm{thick}(A)$ contains the zero object and is closed under shifts (because each $\mathrm{thick}_n(A)$ is). For cofibres, pick $x\to y$ in $\mathrm{thick}(A)$, and assume that $x\in\mathrm{thick}_n(A)$ and $y\in\mathrm{thick}_m(A)$ for $n,m\in\mathbb{N}$. Then, by rotating, the cofibre sequence $y\to\mathrm{cofib}\to\Sigma x$ has $\Sigma x\in\mathrm{thick}_n(A)$ and $y\in\mathrm{thick}_m(A)$, and Lemma 4.2.9 implies that $\mathrm{cofib}\in\mathrm{thick}_m(A)*\mathrm{thick}_n(A)$, that is $\mathrm{cofib}\in\mathrm{thick}(A)$. Closure under retracts follows since every n-thickening of A is closed under retracts. Finally, since $\mathrm{thick}(A)$ is stable, closed under retract and contains A, it follows that $\mathrm{thick}_{\mathcal{C}}(A)\subseteq\mathrm{thick}(A)$. Conversely, if D is a stable subcategory of $\mathcal{C}$ closed under retract and contains A, then the operations used to construct $\mathrm{thick}(A)$ preserve D, so that $\mathrm{thick}(A)\subseteq D$, thus proving $\mathrm{thick}(A)\subseteq\mathrm{thick}_{\mathcal{C}}(A)$. □

4.3. **The big picture.** It is useful to record a variant of the previous constructions which allows arbitrary corpoducts.

Definition 4.3.1. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. We define $\mathrm{Add}(A)$ be the full subcategory generated from A under arbitrary coproducts. We define, inductively, the following full subcategories:

(1) The *big* 1*-coproduct closure of* A as $\mathrm{Coprod}_1(A) \coloneqq \mathrm{Add}(\mathbb{Z}A)$.
(2) The *big* n*-coproduct closure of* A as $\mathrm{Coprod}_n(A) \coloneqq \mathrm{Coprod}_1(A) * \mathrm{Coprod}_{n-1}(A)$.
(3) The *big coproduct closure of* A as the smallest full subcategory $\mathrm{Coprod}(A)$ containing A and closed under $\mathrm{Add}(-)$ and $*$.
(4) The *big* 1*-thickening of* A as $\mathrm{Thick}_1(A) \coloneqq \mathrm{smd}(\mathrm{Add}(\mathbb{Z}A))$.
(5) The *big* n*-thickening of* A as $\mathrm{Thick}_n(A) \coloneqq \mathrm{smd}(\mathrm{Thick}_1(A) * \mathrm{Thick}_{n-1}(A))$.
(6) The *big thickening of* A as $\mathrm{Thick}(A) = \cup_{n\in\mathbb{N}} \mathrm{Thick}_n(A)$.

Warning 4.3.2. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. Notice that the definition of the big coproduct closure of A of Definition 4.3.1 differs from the definition of coproduct closure of Definition 4.2.2. The reason behind the discrepacy is the following. Since $\mathrm{coprod}_1(A)$ is defined using *finite* coproducts, the union $\mathrm{coprod}(A) = \bigcup_{n\geq 1} \mathrm{coprod}_n(A)$ is automatically closed under finite coproducts: given finitely many objects $x_i \in \mathrm{coprod}_{n_i}(A)$, they all lie in $\mathrm{coprod}_N(A)$ for $N = \max_i n_i$, hence $\bigoplus_i x_i \in \mathrm{coprod}_N(A) \subseteq \mathrm{coprod}(A)$. In contrast, with arbitrary coproducts one may have a family $(x_i)_{i\in I}$ such that each $x_i \in \mathrm{Coprod}_{n_i}(A)$ but the integers n_i are unbounded. Then there need not exist a single N with $x_i \in \mathrm{Coprod}_N(A)$ for all i, so the naive union $\bigcup_{n\in\mathbb{N}} \mathrm{Coprod}_n(A)$ is *not* guaranteed to be closed under arbitrary coproducts.

Recall now the following.

Definition 4.3.3. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. We will say that A is *localizing* if it is stable and closed under filtered colimits. We denote by $\mathrm{Loc}_{\mathcal{C}} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ the localizing operator.

Remark 4.3.4. Notice that a localizing subcategory is always closed under small colimits. Furthermore, the Eilenberg-swindle implies that a localizing subcategory is always thick. Such a category is also closed under extensions.

The next result shows that the big thickening is localizing.

Lemma 4.3.5. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be presentable stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. Then $\mathrm{Thick}(A)$ is localizing and $\mathrm{Thick}(A) = \mathrm{Loc}_{\mathcal{C}}(A)$.

Proof. By construction $\mathrm{Thick}(A)$ is stable and closed under coproducts; closure under all small colimits follows since a stable subcategory closed under all coproducts is closed under all small colimits. Minimality is proved as in Lemma 4.2.15. □

We also recall a "windowed" version of Definition 4.2.7, following [Nee25, Reminder 0.12].

Construction 4.3.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $A \subseteq \mathcal{C}$ be a full subcategory. Given integers $p \leq q$, we define the $[p,q]$*-window of* A as the full subcategory $A[p,q] = \{\Sigma^{-i}a \text{ for } a \in A,\ p \leq i \leq q\}$. We may then plug the $[p,q]$-window of A instead of the shift closure of A as input of Definition 4.2.7 and Definition 4.3.1, thus we may define:

(1) The 1*-thickening of the window* $A[p,q]$ as $\mathrm{thick}_1(A[p,q]) \coloneqq \mathrm{smd}(\mathrm{add}(A[p,q]))$.
(2) The n*-thickening of the window* $A[p,q]$ as $\mathrm{thick}_n(A[p,q]) \coloneqq \mathrm{smd}(\mathrm{thick}_1(A[p,q]) * \mathrm{thick}_{n-1}(A[p,q]))$.
(3) The *thickening of the window* $A[p,q]$ as $\mathrm{thick}(A[p,q]) \coloneqq \cup_{n\in\mathbb{N}} \mathrm{thick}_n(A[p,q])$.

Furthermore, if $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is presentable, then we may define:

(1) The *big* 1*-thickening of the window* $A[p,q]$ as $\mathrm{Thick}_1(A[p,q]) \coloneqq \mathrm{smd}(\mathrm{Add}(A[p,q]))$.
(2) The *big* n*-thickening of the window* $A[p,q]$ as $\mathrm{Thick}_n(A[p,q]) \coloneqq \mathrm{smd}(\mathrm{Thick}_1(A[p,q]) * \mathrm{Thick}_{n-1}(A[p,q]))$.
(3) The *big thickening of the window* $A[p,q]$ as $\mathrm{Thick}(A[p,q]) \coloneqq \cup_{n\in\mathbb{N}} \mathrm{Thick}_n(A[p,q])$.

4.4. **The generation time.**

Definition 4.4.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{G} \subseteq \mathcal{C}$ be a full subcategory. Given a subcategory $D \subseteq \mathcal{C}$, we define the *generation time of* D *by* $\mathcal{G}$ as

$$\text{⏱}_{\mathcal{G}}(D) = \inf\{n \in \mathbb{N} \text{ such that } D \subseteq \mathrm{thick}_n(\mathcal{G})\}.$$

Notice that $\text{⏱}_{\mathcal{G}}(D) \in \mathbb{N} \cup \{+\infty\}$ since the infimum of the empty set is $+\infty$.

We recall the max-plus structure on the extend natural numbers.

Remark 4.4.2. Let $\mathbb{N}_\infty \coloneqq \mathbb{N} \cup \{\infty\}$ be ordered by the usual $\leq$, and define the max-plus operations $n \oplus m \coloneqq \max(n,m)$ and $n \otimes m \coloneqq n+m$ (with the convention $n+\infty = \infty$). Then $(\mathbb{N}_\infty, \leq)$ is a total order, hence a distributive lattice, and in fact it is complete: for every $S \subseteq \mathbb{N}_\infty$ the join $\bigvee S = \sup S$ and the meet $\bigwedge S = \inf S$ exist in $\mathbb{N}_\infty$, with $\bigvee \varnothing = 0$ and $\bigwedge \varnothing = \infty$. The multiplication $\otimes = +$ is monotone in each variable and preserves nonempty joins: for every $n \in \mathbb{N}_\infty$ and every nonempty family $\{m_i\}_{i\in I}$ one has $n \otimes (\bigoplus_{i\in I} m_i) = n + \sup_{i\in I} m_i = \sup_{i\in I}(n+m_i) = \bigoplus_{i\in I}(n \otimes m_i)$ (and similarly in the other variable). Notice that $\otimes$ does not preserve arbitrary joins, because it fails to preserve the empty join: for every $n > 0$, it is $n \otimes (\bigvee \varnothing) = n \otimes 0 = n \neq 0 = \bigvee \varnothing = \bigvee(n \otimes \varnothing)$. In particular, $(\mathbb{N}_\infty, \oplus, \otimes)$ is not a (unital) quantale and is not monoidal closed.

It follows that $\mathbb{N}_\infty$ is a complete and cocomplete 1-category with contractible mapping spaces for which arbitrary colimits distribute over finite limits, and it has a symmetric monoidal structure which preserves non-empty colimits in each variable separately.

We recall all the properties of the generation time.

Proposition 4.4.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{G} \subseteq \mathcal{C}$ be a full subcategory. Then the generation time by $\mathcal{G}$ defines an oplax colimit preserving functor $\text{⏱}_{\mathcal{G}} : \mathrm{Sub}(\mathcal{C}) \to \mathbb{N}_\infty$.

Proof. First of all, to show that $\text{⏱}_{\mathcal{G}}$ defines a functor it suffices to check that if $D, E \in \mathrm{Sub}(\mathcal{C})$ are full subcategories such that $D \subseteq E$, then

$$\text{⏱}_{\mathcal{G}}(D) \leq \text{⏱}_{\mathcal{G}}(E).$$

If the generation time of E is infinite, then there is nothing to pove. If it is finite, say equal to $n \in \mathbb{N}$, then the claim follows since $D \subseteq E \subseteq \mathrm{thick}_n(\mathcal{G})$. To show that the generation time by $\mathcal{G}$ preserves all colimits it suffices to construct a right adjoint to it. Define $n \mapsto \mathrm{thick}_n(\mathcal{G})$ for $n \in \mathbb{N}$ and $\infty \mapsto \mathcal{C}$. The equivalence on mapping spaces reads as $E \subseteq \mathrm{thick}_n(\mathcal{G})$ if and only if $\text{⏱}_{\mathcal{G}}(E) \leq n$ for $n \in \mathbb{N}_\infty$. This is clearly true for finite n, since if E is contained in the n-thickening of $\mathcal{G}$ then its generation time is bounded by n and viceversa, and for $n = \infty$ it is trivially true. Finally, to show that it is is oplax monoidal, it suffices to show that it preserves the monoidal unit (and this is trivial since $\text{⏱}_{\mathcal{G}}(0) = 0$) and that

$$\text{⏱}_{\mathcal{G}}(D * E) \leq \text{⏱}_{\mathcal{G}}(D) + \text{⏱}_{\mathcal{G}}(E).$$

For this inequality, if one of the generation times on the right is infinite, then the claim is trival. Assume therefore that the generation times of D and E by $\mathcal{G}$ are finite, say n and m. Then $D \subseteq \mathrm{thick}_n(\mathcal{G})$ and $E \subseteq \mathrm{thick}_m(\mathcal{G})$. Lemma 4.2.9 implies then that $D * E \subseteq \mathrm{thick}_n(\mathcal{G}) * \mathrm{thick}_m(\mathcal{G}) \subseteq \mathrm{thick}_{n+m}(\mathcal{G})$, thus proving the claim. □

Remark 4.4.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{G} \subseteq \mathcal{C}$ be a full subcategory. The preservation of colimits means explicitly that

$$\text{⏱}_{\mathcal{G}}(\cup_{i\in I} E_i) = \max_{i\in I} \text{⏱}_{\mathcal{G}}(E_i)$$

for every family $(E_i)_{i\in I} \subseteq \mathrm{Sub}(\mathcal{C})$ of subcategories.

On the side of the chosed generating subcategory we have the following.

Lemma 4.4.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $E \subseteq \mathcal{C}$ be a full subcategory. Then $\text{⏰}_{-}(E) : \mathrm{Sub}(\mathcal{C})^{\mathrm{op}} \to \mathbb{N}_{\infty}$ is a functor.

Proof. The claim is that whenever $\mathcal{G} \subseteq \mathcal{H}$ then $\text{⏰}_{\mathcal{H}}(D) \leq \text{⏰}_{\mathcal{G}}(D)$. If the right term is infinite then there is nothing to prove. Assume therefore that it is finite, equal to n. Then $D \subseteq \mathrm{thick}_n(\mathcal{G}) \subseteq \mathrm{thick}_n(\mathcal{H})$, where the last inclusion follows from Lemma 4.2.8. □

Remark 4.4.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then the generation time defines a functor $\text{⏰}_{-}(-) : \mathrm{Sub}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Sub}(\mathcal{C}) \to \mathbb{N}_{\infty}$, that is, a map of posets.

We record other useful properties.

Lemma 4.4.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{G} \in \mathrm{Sub}(\mathcal{C})^{\mathrm{op}}$ and $D \in \mathrm{Sub}(\mathcal{C})$ be full subcategories. Then:

(1) The generation time by $\mathcal{G}$ is insensitve to the shift and retract closure. In other terms,
$$\text{⏰}_{\mathcal{G}}(\mathbb{Z}(D)) = \text{⏰}_{\mathcal{G}}(D) \qquad \text{and} \qquad \text{⏰}_{\mathcal{G}}(\mathrm{smd}(D)) = \text{⏰}_{\mathcal{G}}(D).$$

(2) The generation time by $\mathcal{G}$ has thickening bound. In other terms, if $\text{⏰}_{\mathcal{G}}(D) \leq n < \infty$, then for every $k \geq 1$ one has
$$\text{⏰}_{\mathcal{G}}(\mathrm{thick}_k(D)) \leq kn.$$

(3) The generation time by $\mathcal{G}$ satisfies the Cell-filtration criterion. In other terms For $n \in \mathbb{N}$, it is $\text{⏰}_{\mathcal{G}}(D) \leq n$ if and only if every $x \in D$ admits a cell filtration, that is a diagram $0 = x_0 \to x_1 \to \cdots \to x_n = x$ whose successive cofibres lie in $\mathrm{thick}_1(G)$.

Proof. Point (1) follows since the thickenings $\mathrm{thick}_n(\mathcal{G})$ are closed under shifts and retracts. For (2) notice that if $D \subseteq \mathrm{thick}_n(\mathcal{G})$ then $\mathrm{thick}_k(D) \subseteq \mathrm{thick}_k \mathrm{thick}_n(\mathcal{G}) \subseteq \mathrm{thick}_{kn}(\mathcal{G})$ by Lemma 4.2.9. For (3) assume first that $\text{⏰}_{\mathcal{G}}(D) \leq n$. Then $D \subseteq \mathrm{thick}_n(\mathcal{G})$, and thus every $x \in \mathcal{D}$ admits a cell filtration by Lemma 4.2.9. Conversely, if $x \in D$ admits such filtration, then Corollary 4.2.10 implies $x \in \mathrm{thick}_n(\mathcal{G})$ and thus $\text{⏰}_{\mathcal{G}}(D) \leq n$. □

We now prove some useful inequality. The first one is the “existential inequality”.

Proposition 4.4.8. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{st}}$ and let $\mathcal{G} \in \mathrm{Sub}(\mathcal{C})^{\mathrm{op}}$ and $E \in \mathrm{Sub}(\mathcal{C})$ be full subcategories. Then $\text{⏰}_{\exists_f(\mathcal{G})}(\exists_f(E)) \leq \text{⏰}_{\mathcal{G}}(E)$.

Proof. If the right hand side is infinite then there is nothing to prove. Assume therefore that it is finite, say equal to n. Then $E \subseteq \mathrm{thick}_n(\mathcal{G})$ and the functoriality of $\exists_f$ together with **??** imply $\exists_f(E) \subseteq \exists_f(\mathrm{thick}_n(\mathcal{G})) \subseteq \mathrm{thick}_n(\exists_f(\mathcal{G}))$. Thus $\text{⏰}_{\exists_f(\mathcal{G})}(\exists_f(E)) \leq n = \text{⏰}_{\mathcal{G}}(E)$. □

We now prove the “adjunction inequality”.

Proposition 4.4.9. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{st}}$ and let $\mathcal{H} \in \mathrm{Sub}(\mathcal{D})^{\mathrm{op}}$ and $E \in \mathrm{Sub}(\mathcal{C})$ be full subcategories. Then $\text{⏰}_{\mathcal{H}}(\exists_f(E)) \leq \text{⏰}_{f^{-1}(\mathcal{H})}(E)$.

Proof. If the right term is infinite there is nothing to prove. assume that it is finite, say equal to n. Then $E \subseteq \mathrm{thick}_n(f^{-1}(\mathcal{H}))$, thus by applying $\exists_f$ it follows that $\exists_f(E) \subseteq \exists_f \mathrm{thick}_n(f^{-1}(\mathcal{H})) \subseteq \mathrm{thick}_n(\exists_f f^{-1}(\mathcal{H})) \subseteq \mathrm{thick}_n(\mathcal{H})$. Here the first inclusion follows since $\exists_f$ is a functor, the second one by Lemma 4.2.12 and the last one by using the counit $\exists_f \circ f^{-1} \to \mathrm{id}_{\mathrm{Sub}(\mathcal{D})}$. Thus $\text{⏰}_{\mathcal{H}}(\exists_f(E)) \leq n = \text{⏰}_{f^{-1}(\mathcal{H})}(E)$. □

Corollary 4.4.10. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{st}}$ and let $\mathcal{H} \in \mathrm{Sub}(\mathcal{D})^{\mathrm{op}}$ and $K \in \mathrm{Sub}(\mathcal{D})$. Then:

(1) It is $⏰_{\mathcal{H}}(\exists_f f^{-1}(K)) \le ⏰_{f^{-1}(\mathcal{H})}(f^{-1}(K))$.
(2) If K is f-saturated, in that $K = \exists_f f^{-1}(K)$, then $⏰_{\mathcal{H}}(K) \le ⏰_{f^{-1}(\mathcal{H})}(f^{-1}(K))$.

Proof. It follows from Proposition 4.4.9. □

Proposition 4.4.11. Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$ and let $K := \ker(p) \simeq \mathrm{im}(i) \subseteq \mathcal{D}$. Then:

(1) ($\exists_p p^{-1} = \mathrm{id}$ *and the kernel is invisible*) For all $H \in \mathrm{Sub}(\mathcal{E})$ one has

$$\exists_p\big(p^{-1}(H)\big) = H, \qquad \exists_p(K) = 0, \qquad \exists_p(A \cup K) = \exists_p(A) \quad (A \in \mathrm{Sub}(\mathcal{D})).$$

In particular, for all $G, H \in \mathrm{Sub}(\mathcal{E})$,

$$⏰_G(H) \;\le\; ⏰_{p^{-1}(G)}\big(p^{-1}(H)\big).$$

(2) (*Pullback is controlled up to the kernel*) For all $G, H \in \mathrm{Sub}(\mathcal{E})$,

$$⏰_{p^{-1}(G)*K}\big(p^{-1}(H)\big) \;\le\; ⏰_G(H).$$

Consequently,

$$⏰_{p^{-1}(G)\cup K}\big(p^{-1}(H)\big) \;\le\; 2⏰_G(H).$$

(3) Assume the Verdier sequence is right split. Then for every $H \in \mathrm{Sub}(\mathcal{E})$,

$$p^{-1}(H) = K * \exists_q(H).$$

In particular, for all $G, H \in \mathrm{Sub}(\mathcal{E})$,

$$⏰_{K\cup\exists_q(G)}\big(p^{-1}(H)\big) \;\le\; 1 + ⏰_G(H).$$

Dually, if the sequence is *left split* with fully faithful left adjoint $\ell : \mathcal{E} \to \mathcal{D}$, then

$$p^{-1}(H) = \exists_\ell(H) * K, \qquad ⏰_{K\cup\exists_\ell(G)}\big(p^{-1}(H)\big) \;\le\; 1 + ⏰_G(H).$$

Proof. **??**. Let $n := ⏰_A(B)$, so $B \subseteq \langle A\rangle_n$. Since p is exact, the induced map $\exists_p$ preserves shifts and cofibres, hence extensions; moreover it preserves retracts. Therefore $\exists_p(\langle A\rangle_n) \subseteq \langle \exists_p(A)\rangle_n$. Applying $\exists_p$ to the inclusion $B \subseteq \langle A\rangle_n$ yields $\exists_p(B) \subseteq \langle \exists_p(A)\rangle_n$, so $⏰_{\exists_p(A)}(\exists_p(B)) \le n$.

??. The functor p is essentially surjective (it exhibits $\mathcal{E}$ as the Verdier quotient $\mathcal{D}/K$), so every object of H admits a lift in $p^{-1}(H)$; this gives $H \subseteq \exists_p(p^{-1}H)$, while the counit of the adjunction on subcategories gives $\exists_p(p^{-1}H) \subseteq H$, hence equality. Also p vanishes on K, so $\exists_p(K) = 0$; thus $\exists_p(A \cup K) = \exists_p(A)$.

For the inequality, let $n := ⏰_{p^{-1}(G)}(p^{-1}(H))$, so $p^{-1}(H) \subseteq \langle p^{-1}(G)\rangle_n$. Applying $\exists_p$ and using the previous paragraph gives

$$H = \exists_p(p^{-1}H) \subseteq \exists_p\big(\langle p^{-1}(G)\rangle_n\big) \subseteq \langle \exists_p(p^{-1}G)\rangle_n \subseteq \langle G\rangle_n,$$

so $⏰_G(H) \le n$.

??. Let $n := ⏰_G(H)$, so $H \subseteq \langle G\rangle_n$. Apply p^{-1} to get $p^{-1}(H) \subseteq p^{-1}(\langle G\rangle_n)$. For a Verdier projection one has the pullback–thickening control

$$p^{-1}(\langle G\rangle_n) \subseteq \langle p^{-1}(G) * K\rangle_n$$

(precisely the content of the "kernel correction" statement for Verdier projections used earlier in §5.2–§5.4). Hence $p^{-1}(H) \subseteq \langle p^{-1}(G) * K\rangle_n$, proving $⏰_{p^{-1}(G)*K}(p^{-1}(H)) \le n$.

For the second bound, note that $p^{-1}(G) \subseteq p^{-1}(G) \cup K$ and $K \subseteq p^{-1}(G) \cup K$, hence $p^{-1}(G) * K \subseteq \langle p^{-1}(G) \cup K\rangle_2$. Therefore

$$p^{-1}(H) \subseteq \langle p^{-1}(G) * K\rangle_n \subseteq \Big\langle \langle p^{-1}(G) \cup K\rangle_2 \Big\rangle_n \subseteq \langle p^{-1}(G) \cup K\rangle_{2n},$$

so $⏰_{p^{-1}(G)\cup K}(p^{-1}(H)) \le 2n$.

??. Assume q exists and is fully faithful. For every $d \in \mathcal{D}$, the unit/counit exhibit a fibre sequence

$$b_d \to d \to q\,p(d)$$

with $b_d \in K$ (since $p(b_d) = 0$). If $d \in p^{-1}(H)$ then $p(d) \in H$, hence $q\,p(d) \in \exists_q(H)$ and the fibre sequence shows $d \in K * \exists_q(H)$. Conversely, if $d \in K * \exists_q(H)$ then $p(d)$ is an extension of 0 by an object of H, hence $p(d) \in H$. Thus $p^{-1}(H) = K * \exists_q(H)$.

Now let $n := \text{⏰}_G(H)$, so $H \subseteq \langle G \rangle_n$. Exactness of q implies $\exists_q(H) \subseteq \langle \exists_q(G) \rangle_n$. Therefore

$$p^{-1}(H) = K * \exists_q(H) \subseteq K * \langle \exists_q(G) \rangle_n \subseteq \langle K \cup \exists_q(G) \rangle_{n+1},$$

and the claimed +1 bound follows. The left split statement is dual. □

5. Generators

In this chapter we discuss the various notions of generators for stable categories. We begin by recalling the basic taxonomy, namely weak, classical and strong generation, and by relating these notions to the calculus of thickenings developed in the previous chapter. We then prove a gluing result for weak generators along compactly generated localizations, with the goal of producing generators in practice. As an application, we show that if X is a quasi-compact quasi-separated scheme,then $\mathrm{QCoh}(X)$ admits a single compact generator. We conclude with resolutions of the diagonal for smooth schemes, which will later serve as the bridge between generation, strong generation, and the diagonal dimension.

5.1. **Taxonomy of generators.** We begin with some nomenclature.

Definition 5.1.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{G} \in \mathrm{Sub}(\mathcal{C})$ be a full subcategory. We will say that:

1. The subcategory $\mathcal{G}$ *weakly generates* if $\hom_{\mathcal{C}}(\mathcal{G}, -) : \mathcal{C} \to \mathrm{Sp}$ detects zero objects.
2. The subcategory $\mathcal{G}$ *classical generates* if $\mathcal{C} = \mathrm{thick}(\mathcal{G})$.
3. The subcategory $\mathcal{G}$ *strongly generates* if $\mathcal{C} = \mathrm{thick}_n(\mathcal{G})$ for some $n \in \mathbb{N}$.

If $\mathcal{G}$ is such a generating subcategory, we will say that $\mathcal{C}$ is *weakly generated, classically generated, strongly generated*, by $\mathcal{G}$.

Example 5.1.2. Every stable category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ admits a (weak, classical, strong) generating family: just take $\mathcal{G} = \mathcal{C}$. In particular, to make the definition more useful in practice, it is useful to restrict the notion to subcategories of finite cardinalities (and hence to a single object). In this case, $\mathcal{G} = \{G\}$ and we will just say that G *weakly, classically, strongly generates* $\mathcal{C}$. We will extensively study this situation in Section 8.

Warning 5.1.3. In the literature the adverb *weakly* is often suppressed, so that one just says that $\mathcal{C}$ is generated by $\mathcal{G}$.

Lemma 5.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then:

1. Every strong generatoring subcategory classically generates, and every classical generating subcategory weakly generates.
2. If $\mathcal{C}$ has a strong generator, then every classical generator is a strong generator.

Proof. Consider (1). The first implication is obvious. For the second one, assume that $\mathcal{G} \in \mathrm{Sub}(\mathcal{C})$ is a classical generating subcategory and let $x \in \mathcal{C}$ be such that $\hom_{\mathcal{C}}(g, x) \simeq 0$ is the zero spectrum for evvery $g \in \mathcal{G}$. Let $\mathcal{X} \subseteq \mathcal{C}$ be the full subcategory spanned by those y such that $\hom_{\mathcal{C}}(y, x) \simeq 0$ is the zero spectrum. Then $\mathcal{X}$ is stable, closed under retracts and contains $\mathcal{G}$. Thus $\mathrm{thick}(\mathcal{G}) \subseteq \mathcal{X}$, that is $\mathcal{C} = \mathcal{X}$ since $\mathcal{G}$ classically generates. In particular, the identity of x is zero, thus showing $x \simeq 0$. For (2), assume that $\mathcal{C} = \mathrm{thick}_n(G')$ and let G be a classical generator. Then $G' \in \mathcal{C} = \mathrm{thick}(G)$ must belong to an m-thickening. Then

$$\mathcal{C} = \mathrm{thick}_n(G') = \mathrm{smd}(\mathrm{thick}_1(G')^{*n}) = \mathrm{smd}(\mathrm{thick}_m(G)^{*n}) = \mathrm{thick}_{nm}(G)$$

is also strongly generated by G. ☐

Remark 5.1.5. Notice that point (2) of Lemma 5.1.4 cannot be extended to arbitrary subcategories, since it will lead to the equivalence between strongly and classically generated categories, which is known to be false (see the next examples). With the notation of the lemma, the candidate proof breaks since arbitrary objects of $\mathcal{G}'$ belong to different n-thickenings $\mathrm{thick}_n(\mathcal{G})$.

We now relate Definition 5.1.1 to the standard notion of compact generation. The point is that compact generation is a presentable version of weak generation, while on compact objects it becomes classical generation.

Lemma 5.1.6. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ be a compactly generated stable category and let $\mathcal{G} \subseteq \mathcal{C}^\omega$ be a small subcategory of compact objects. Then the compact objects of $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$ are precisely the thick closure of $\mathcal{G}$ in $\mathcal{C}^\omega$, that is $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})^\omega = \mathrm{Thick}_{\mathcal{C}^\omega}(\mathcal{G})$.

Proof. Notice first of all that $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}) \subseteq \mathrm{Loc}_{\mathcal{C}}(\mathcal{G})^\omega$. Indeed, since compact objects are closed under finite colimits and retracts, $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$ is a small stable idempotent-complete full subcategory of $\mathcal{C}^\omega$. Moreover, if $x \in \mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$ and $\{y_i\}_{i\in I}$ is a filtered diagram in $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$, then

$$\hom_{\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})}(x, \mathrm{colim}_i\, y_i) \simeq \hom_{\mathcal{C}}(x, \mathrm{colim}_i\, y_i) \simeq \mathrm{colim}_i \hom_{\mathcal{C}}(x, y_i) \simeq \mathrm{colim}_i \hom_{\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})}(x, y_i),$$

because colimits in $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$ are computed in $\mathcal{C}$. Hence every object of $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$ is compact in $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$, so that the claimed inclusion follows.

Now by ind-extending the above inclusion and by using the universal property of the ind-completion, it follows the existence of a a colimit-preserving functor $F : \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})) \to \mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$. Its essential image is a localizing subcategory of $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$ containing $\mathcal{G}$, hence all of $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$. This shows that F is essentially surjective. For fully-faithfulness, fix $x \in \mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$ and let $\mathcal{U}_x \subseteq \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))$ be the full subcategory spanned by those y such that

$$\hom_{\mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))}(x, y) \to \hom_{\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})}(F(x), F(y))$$

is an equivalence. Since x is compact in $\mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))$, the object $F(x)$ is compact in $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$, and F preserves colimits, the subcategory $\mathcal{U}_x$ is localizing. Since it clearly contains $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$, being $F|_{\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})}$ the inclusion, it follows $\mathcal{U}_x = \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))$. So F is fully faithful on morphisms out of objects of $\mathrm{thick}_{\mathcal{C}^\omega}$. Now fix $y \in \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega})$, and let $\mathcal{V}_y \subseteq \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega})$ be the full subcategory spanned by those x such that

$$\hom_{\mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))}(x, y) \to \hom_{\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})}(F(x), F(y))$$

is an equivalence. Since both functors in the variable x send colimits to limits, $\mathcal{V}_y$ is again a localizing subcategory. By the previous paragraph, it contains $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$. Hence $\mathcal{V}_y = \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))$. This proves that F is fully faithful. Taking compact objects

$$\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})^\omega \simeq \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{C}))^\omega = \mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}),$$

proves then the claim. ☐

Corollary 5.1.7. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Suppose there exists a small full subcategory $\mathcal{G} \subseteq \mathcal{C}^\omega$ of compact objects such that $\mathcal{C} = \mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$. Then $\mathcal{C}$ is compactly generated.

Proof. By the previous result it follows that $\mathcal{C} = \mathrm{Loc}_{\mathcal{C}}(\mathcal{G}) \simeq \mathrm{Ind}(\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}))$ is compactly generated by $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G})$ and the inclusion $\mathrm{thick}_{\mathcal{C}^\omega}(\mathcal{G}) \subseteq \mathcal{C}^\omega$ extends to an equality. ☐

We deduce that a compact generator of a compactly generated stable category is a classical generator of the subcategory of compact objects.

Corollary 5.1.8. Let $\mathcal{C}$ be a compactly generated stable category and let $G \in \mathcal{C}^\omega$ be a compact generator. Then $\mathcal{C}^\omega = \mathrm{thick}(G)$.

Example 5.1.9. Classical generation does not imply strong generation in general. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^\heartsuit$ be a classical ring and consider the category of perfect complexes Perf_R. Then R generates Mod_R by Corollary 5.1.8 so that it classically generates Perf_R. However, it strongly generates Perf_R if and only if R is of finite global dimension (as we will see in Corollary 8.4.6).

Example 5.1.10. In the category of compact spectra the sphere spectrum is a classical generator which is not strong. This will be a consequence of the cell filtration and the existence of "ghosts" (as we will see in Example 8.4.3).

The next result shows that localizing generation implies weak generation.

Lemma 5.1.11. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $\mathcal{G} \subseteq \mathcal{C}$ be a full subcategory such that $\mathcal{C} = \mathrm{Loc}(\mathcal{G})$. Then $\mathcal{G}$ weakly generates $\mathcal{C}$.

Proof. Fix $x \in \mathcal{C}$ and consider the full subcategory $\mathcal{C}_x := \{y \in \mathcal{C} \mid \hom_{\mathcal{C}}(y,x) \simeq 0\}$. Since $\hom_{\mathcal{C}}(-,x) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is exact, the category $\mathcal{C}_x$ is stable. Moreover, since $\hom_{\mathcal{C}}(-,c)$ sends colimits in $\mathcal{C}$ to limits in Sp, it follows that $\mathcal{C}_x$ is closed under small colimits. Thus $\mathcal{C}_x$ is a localizing subcategory of $\mathcal{C}$. If $\hom_{\mathcal{C}}(g,x) \simeq 0$ for all $g \in \mathcal{G}$, then $\mathcal{G} \subseteq \mathcal{C}_x$, hence $\mathrm{Loc}(\mathcal{G}) \subseteq \mathcal{C}_x$. By assumption $\mathrm{Loc}(\mathcal{G}) = \mathcal{C}$, so $\mathcal{C}_x = \mathcal{C}$. In particular $\hom_{\mathcal{C}}(x,x) \simeq 0$, hence $\mathrm{id}_x \simeq 0$ and therefore $x \simeq 0$. □

5.2. **How to produce weak generators.** We now discuss a categorical result on compactly generated localizations.

Remark 5.2.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Recall that a localization of $\mathcal{C}$ is an adjunction $L : \mathcal{C} \rightleftarrows \mathrm{Local}(\mathcal{C}) : i$ in which the right adjoint is fully faithful. By [Lur09, Proposition 5.2.7.4], the above localization is equivalent to the datum of the idempotent exact endofunctor $\ell := iL : \mathcal{C} \to \mathcal{C}$ together with the unit $\eta : \mathrm{id}_{\mathcal{C}} \to \ell$. Indeed, the source of the right adjoint is equivalent to the full subcategory of *local objects*, that is of those objects $x \in \mathcal{C}$ for which $\eta_x : x \to \ell(x)$ is an equivalence.

Remark 5.2.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Let $\ell : \mathcal{C} \to \mathcal{C}$ be a localization functor. The kernel of the localization functor forms a stable subcategory $\ker(\ell) \subseteq \mathcal{C}$ whose objects are called *acyclic*. Since $\mathcal{C}$ is stable, for every $x \in \mathcal{C}$ the unit $\eta_x : x \to \ell x$ extends to an exact sequence $\gamma(x) \to x \to \ell(x)$, where $\gamma(x) \in \ker(\ell)$ and $\ell(x)$ is local. Thus every object decomposes into an acyclic part and a local part. In particular, the local objects are precisely the right orthogonal of acyclic

$$\mathrm{Local}(\mathcal{C}) = \ker(\ell)^\perp := \{x \in \mathcal{C} \mid \hom_{\mathcal{C}}(k,x) = 0 \text{ for every } k \in \ker(\ell)\}.$$

ndeed, if x is local and $k \in \ker(\ell)$, then every map $k \to x$ is null-homotopic, since applying ℓ kills k and fixes x. Conversely, if $x \in \ker(\ell)^\perp$, then in the exact sequence $\gamma(x) \to x \to \ell(x)$ the first map vanishes, because $\gamma(x) \in \ker(\ell)$. Hence $\gamma(x) \simeq 0$, so that $x \simeq \ell(x)$ is local. Similarly, one also has

$$\ker(\ell) = {}^\perp\mathrm{Local}(\mathcal{C}) := \{x \in \mathcal{C} \mid \hom_{\mathcal{C}}(x,y) = 0 \text{ for every } y \in \mathrm{Local}(\mathcal{C})\}.$$

In particular, Lemma 3.3.6 implies the existence of a semi-orthogonal decomposition $\ker(\ell) \rightleftarrows \mathcal{C} \rightleftarrows \mathrm{Local}(\mathcal{C})$.

Remark 5.2.3. Let $\ell : \mathcal{C} \to \mathcal{C}$ be a localization functor, and set $\mathcal{K} := \ker(\ell)$. Suppose that $\mathcal{S} \subseteq \mathcal{K}$ weakly generates $\mathcal{K}$ and that $\mathcal{T} \subseteq \mathrm{Local}(\mathcal{C})$ weakly generates the local objects. Then $\mathcal{S} \cup \mathcal{T}$ weakly generates $\mathcal{C}$.

Indeed, let $x \in \mathcal{C}$ be an object such that $\hom_{\mathcal{C}}(s, \Sigma^n x) = 0$ and $\hom_{\mathcal{C}}(t, \Sigma^n x) = 0$ for every $s \in \mathcal{S}$, $t \in \mathcal{T}$ and $n \in \mathbb{Z}$. Since $\mathcal{T}$ weakly generates $\mathrm{Local}(\mathcal{C})$, it follows that $\ell x \simeq 0$. Hence $x \simeq \gamma x \in \mathcal{K}$. Since $\mathcal{S}$ weakly generates $\mathcal{K}$, one concludes that $x \simeq 0$.

We deduce the following "one-object criterion" for local objects.

Lemma 5.2.4. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $L : \mathcal{C} \to \mathcal{C}$ be a localization. Assume there exists an object $G \in \mathcal{C}$ such that $\ker(L) = \mathrm{Loc}(G)$. Then, for $x \in \mathcal{C}$, the following are equivalent:

(1) The object x is L-local.
(2) It is $\hom_{\mathcal{C}}(G, x) \simeq 0$.

Proof. By Remark 5.2.2 it is $\mathrm{Local}(\mathcal{C}) = \ker(L)^{\perp}$. If $\ker(L) = \mathrm{Loc}(G)$, then the condition $\hom(G, x) \simeq 0$ implies $\hom(y, x) = 0$ for all $y \in \mathrm{Loc}(G)$, because $\hom(-, x)$ sends colimits to limits and is exact. Hence $x \in \ker(L)^{\perp}$ and is local. The converse is immediate since $G \in \ker(L)$. □

We recall the following behaviour of compact objects under a smashing localization (that is, a localization in which the left adjoint preserves comapct objects).

Lemma 5.2.5. Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction in $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ with R fully faithful (so L preserves compact objects and R preserves colimits). Then every compact object of $\mathcal{D}$ is a direct summand of $L(c)$ for some compact $c \in \mathcal{C}$.

Proof. Consider the localizing closure generated by $\mathrm{Loc}_{\mathcal{D}}(L(\mathcal{C}^{\omega}))$. Then $\mathrm{Loc}_{\mathcal{D}}(L(\mathcal{C}^{\omega})) = \mathcal{D}$. Indeed, for any $d \in \mathcal{D}$ it is $d \simeq L(R(d))$ since R is fully faithful. Since $\mathcal{C}$ is compactly generated, $R(d)$ lies in the localizing subcategory of $\mathcal{C}$ generated by $\mathcal{C}^{\omega}$, hence $R(d)$ can be written as a filtered colimit of objects of $\mathcal{C}^{\omega}$. Applying L, and using that L preserves colimits, it follows that $d \simeq L(R(d))$ belongs to the localizing subcategory of $\mathcal{D}$ generated by $L(\mathcal{C}^{\omega})$, that is $d \in \mathrm{Loc}_{\mathcal{D}}(L(\mathcal{C}^{\omega}))$. Thus $\mathrm{Loc}_{\mathcal{D}}(L(\mathcal{C}^{\omega})) = \mathcal{D}$.

Now let $\mathrm{thick}(L(\mathcal{C}^{\omega})) \subseteq \mathcal{D}$ denote the thick subcategory generated by $L(\mathcal{C}^{\omega})$. Since $\mathcal{D}$ is compactly generated and $\mathrm{Loc}_{\mathcal{D}}(\mathcal{C}^{\omega}) = \mathcal{D}$, every compact object $d \in \mathcal{D}^{\omega}$ belongs to $\mathrm{thick}(L(\mathcal{C}^{\omega}))$. By construction of the thick closure, this means precisely that d is a retract of some object k obtained from a finite number of objects in the essential image of $L|_{\mathcal{C}^{\omega}}$ by iterating finite direct sums and cofibres. Since L is exact and preserves finite direct sums, the same finite construction can be performed in $\mathcal{C}^{\omega}$, yielding a compact object $c \in \mathcal{C}^{\omega}$ with $k \simeq L(c)$. Hence d is a retract of $L(c)$, as desired. □

We obtain the following gluing of compact generators across localizations; a criterion for compact generators of right split Verdier sequences.

Proposition 5.2.6. Let

$$\mathcal{C} \xrightarrow{\ i\ } \mathcal{D} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}$$

be a right split Verdier sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Let $S \subseteq \mathcal{C}^{\omega}$ and $T \subseteq \mathcal{D}^{\omega}$ be full subcategories such that:

(1) It is $\mathcal{C} = \mathrm{Loc}_{\mathcal{C}}(S)$ and $i(S) \subseteq \mathcal{D}^{\omega}$.
(2) It is $\mathcal{E} = \mathrm{Loc}_{\mathcal{E}}(L(T))$ and $L(T) \subseteq \mathcal{E}^{\omega}$.

Then $\mathcal{D} = \mathrm{Loc}_{\mathcal{D}}(i(S) \cup T)$. In particular, $\mathcal{D}$ is weakly generated by a family of compact objects.

Proof. Let $G := i(S) \cup T \subseteq \mathcal{D}^{\omega}$. Then G weakly generates $\mathcal{D}$ by Remark 5.2.3 since S and T weakly generate $\mathcal{C}$ and $\mathcal{E}$ by Lemma 5.1.11. Consider now the localizing closure $\mathrm{Loc}_{\mathcal{D}}(G) \subseteq \mathcal{D}$. If $\mathrm{Loc}_{\mathcal{D}}(G) \neq \mathcal{D}$ then the corresponding Verdier quotient $\mathcal{D} \to \mathcal{D}/\mathrm{Loc}_{\mathcal{D}}(G)$ is nonzero, hence admits a nonzero object y. Its fully faithful right adjoint sends y to a nonzero $\mathrm{Loc}_{\mathcal{D}}(G)$-local object $x \in \mathcal{D}$ satisfying $\hom_{\mathcal{D}}(k, x) \simeq 0$ for all $k \in \mathrm{Loc}_{\mathcal{D}}(G)$, and in particular for all $g \in G$. This contradicts weak generation by G. Hence $\mathrm{Loc}_{\mathcal{D}}(G) = \mathcal{D}$, which is the claim. □

Corollary 5.2.7. In a right split Verdier sequence for which the outer terms are compactly generated, then also the middle term is compactly generated.

Remark 5.2.8. It is not known if there exists a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ in which $\mathcal{C}$ and $\mathcal{E}$ are compactly generated and $\mathcal{D}$ is not.

5.3. **Generators for schemes.** The goal of this section is to prove the existence of a single compact generator for quasi-compact quasi-separated scheme, a result due to Bondal and Van den Bergh, [BB02, Theorem 3.1.1]. We begin with the "induction principle" for schemes.

Lemma 5.3.1. Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X. Assume that

(1) The property P holds for every affine open of X.
(2) If U is quasi-compact open, V affine open and P holds for U, V, and $U \cap V$, then P holds for $U \cup V$.

Then P holds for every quasi-compact open of X and in particular for X.

Proof. Let $W \subseteq X$ be a quasi-compact open of X. Then W may be covered by a finite number of affine opens of X, say $W = U_1 \cup \dots U_n$. The proof goes by induction on n. If $n = 1$, then the claim follows by point (1). Assume therefore the claim for $n-1$. Let $U = U_1 \cup \dots U_{n-1}$ and $V = U_n$ so that $W = U \cup V$. Since V is affine open and P holds for V, by (2) it suffices to check that P holds for U (which is quasi-compact open, being a finite union of affine opens) and $U \cap V$. But P holds for U, since it is covered by $n-1$ affine opens. Consider therefore $U \cap V$. Since X is quasi-separated and $U, V \subseteq X$ are quasi-compact opens, the intersection $U \cap V$ is quasi-compact. Moreover $U \cap V$ is an open subscheme of the affine scheme V, hence it is separated. Choose a finite affine open cover of $U \cap V$. By the separated case (proved first by induction on the number of affines, using that intersections of affine opens in a separated scheme are affine), it follows that P holds for $U \cap V$, so the claim. □

Corollary 5.3.2. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ be an open immersion of a quasi-compact. Then $j^* : \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is essentially surjective up to direct summands.

Proof. Apply Lemma 5.2.5. □

To prove the main result of this section we will use Proposition 5.2.6 in the following situation. Let W be a quasi-compact quasi-separated scheme and assume that $W = U \cup V$ with U quasi-compact open and V affine open. If $j : U \hookrightarrow W$ is the inclusion, then Proposition 3.5.3 implies the existence of a right split Verdier sequence

$$\mathrm{QCoh}_{W \smallsetminus U}(W) \xrightarrow{\iota_*} \mathrm{QCoh}(W) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{QCoh}(U)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Notice now that $W \smallsetminus U \subseteq V$ is a closed in an affine, so that $W \smallsetminus U$ is affine. This observation, leaves us to prove that $\mathrm{QCoh}_{W \smallsetminus U}(W)$ is singly compactly generated.

Remark 5.3.3. Let $A \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring and let $f_1, \dots f_n \in A$ be elements. The Koszul complex of $(f_1, \dots, f_n)$ is defined to compute the *derived intersection of the section* $(f_1, \dots, f_n)$ *with zero in the affine space* $\mathbb{A}^n_A$. More precisely, the *Koszul complex* of $(f_1, \dots, f_n)$ is defined as

$$K^\bullet(f_1, \dots, f_n) = \bigotimes_{i=1}^{n} \mathrm{cofib}(A \xrightarrow{f_i} A)$$

Notice that $K^\bullet(f_1, \dots, f_n) \in \mathrm{Perf}_A$.

Lemma 5.3.4. Let A be a commutative ring, let $f_1, \dots, f_r \in A$, set $X = \mathrm{Spec}(A)$ and let $U = \cup_{i=1}^r D(f_i) \subseteq X$ be the union of the distinguished. Let $j : U \hookrightarrow X$ be the inclusion and let $Z := X \smallsetminus U = V(f_1, \dots, f_r)$. Write $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$ and $j^* : \mathrm{Mod}_A \to \mathrm{Mod}_{A[f_1^{-1}, \dots, f_r^{-1}]}$. Consider the (derived) Koszul object $K := K(f_1, \dots, f_r)$. Then $\ker(j^*) = \mathrm{QCoh}_Z(X) = \mathrm{Loc}_{\mathrm{Mod}_A}(K)$.

Proof. Indeed, since j^* is a localization it follows that

$$\ker(j^*) = \mathrm{Loc}\big(\ker(j^*) \cap \mathrm{Mod}_A^{\omega}\big) = \mathrm{Loc}\big(\mathrm{Perf}_Z(X)\big),$$

where $\mathrm{Perf}_Z(X)$ is the full subcategory on $P \in \mathrm{Perf}(X)$ such that $P|_U \simeq 0$. By Thomason's classification of thick tensor ideals in $\mathrm{Perf}(X)$ (see [Tho97, Theorem 3.15]), it suffices to show that the homological support of K is exactly Z. For a prime $\mathfrak{p} \in X$, if $\mathfrak{p} \in U$ then some $f_i \notin \mathfrak{p}$, so $(\mathrm{cofib}(A \xrightarrow{f_i} A))_{\mathfrak{p}} \simeq 0$ and hence $K_{\mathfrak{p}} \simeq 0$. If $\mathfrak{p} \in Z$, then all $f_i \in \mathfrak{p}$ and $\pi_0(K_{\mathfrak{p}}) \cong A_{\mathfrak{p}}/(f_1, \dots, f_r) \neq 0$, hence $K_{\mathfrak{p}} \not\simeq 0$. Thus $\mathrm{supp}(K) = Z$, so $\mathrm{Perf}_Z(X) = \mathrm{thick}^{\otimes}(K)$ and therefore $\mathrm{QCoh}_Z(X) = \mathrm{Loc}(K)$. □

We need the following technical result.

Lemma 5.3.5. Let W be a quasi-compact quasi-separated scheme and assume that $W = U \cup V$ with U quasi-compact open and V affine open, say $V = \mathrm{Spec}(A)$. Let $j : U \hookrightarrow W$ and $k : V \hookrightarrow W$ be the open immersions and set $Z := W \setminus U$. Choose $f_1, \dots, f_r \in A$ such that $U \cap V = D(f_1) \cup \dots \cup D(f_r) \subseteq V$, and let $K := K^{\bullet}(f_1, \dots, f_r) \in \mathrm{Perf}(V)$ be the Koszul complex. Then there exists an object $K_W \in \mathrm{Perf}(W) \subseteq \mathrm{QCoh}(W)$ such that:

(1) It is $K_W|_U = j^* K_W \simeq 0$ and $K_W|_V = k^* K_W \simeq K$.
(2) It is a single compact generator of $\mathrm{QCoh}_Z(W)$, that is $\mathrm{QCoh}_Z(W) = \mathrm{Loc}_{\mathrm{QCoh}(W)}(K_W)$.

Proof. By Proposition 3.5.3 there exists a right split Verdier sequence

$$\mathrm{QCoh}_Z(W) \to \mathrm{QCoh}(W) \xrightarrow{j^*} \mathrm{QCoh}(U),$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Using Zariski descent for QCoh along the cover $W = U \cup V$, an object of $\mathrm{QCoh}(W)$ is equivalent to a triple (E_U, E_V, α) with $E_U \in \mathrm{QCoh}(U)$, $E_V \in \mathrm{QCoh}(V)$ and $\alpha : E_U|_{U\cap V} \simeq E_V|_{U \cap V}$. Under this identification, an object $E \in \mathrm{QCoh}_Z(W)$ belongs to the kernel of j^* if and only if $E_U \simeq 0$. For these objects it is also $E_V|_{U\cap V} \simeq 0$. Therefore restriction along the open immersion $l : U \cap V \hookrightarrow V$. induces an equivalence

$$\mathrm{QCoh}_Z(W) \simeq \ker(\mathrm{QCoh}(V) \to \mathrm{QCoh}(U \cap V)).$$

Let $L := l_* \circ l^* : \mathrm{QCoh}(V) \to \mathrm{QCoh}(V)$ be the corresponding localization. Since l_* is fully-faithful, it follows that $\ker(L) = \ker(\mathrm{QCoh}(V) \to \mathrm{QCoh}(U \cap V))$. Use now Lemma 5.3.4 (with $X = V$ and $U = U \cap V$) to deduce that

$$\mathrm{QCoh}_Z(W) \simeq \ker(\mathrm{QCoh}(V) \to \mathrm{QCoh}(U \cap V)) \simeq \ker(L) = \mathrm{Loc}_{\mathrm{QCoh}(V)}(K),$$

is singly generated by the Koszul complex $K = K^{\bullet}(f_1, \dots, f_r)$ corresponding to $U \cap V = D(f_1) \cup \dots \cup D(f_r)$. Now, since $K|_{U\cap V} \simeq 0$ (because the f_i's become invertible on U), the triple $(0, K, 0)$ glues to an object $K_W \in \mathrm{QCoh}(W)$ with $K_W|_U \simeq 0$ and $K_W|_V \simeq K$ and since perfection is Zariski-local, it follows also that $K_W \in \mathrm{Perf}(W)$ and is compact. This proves (1) and (2) follows from the above chain of equivalences. □

We can finally prove [BB02, Theorem 3.1.1].

Theorem 5.3.6. Let X be a quasi-compact quasi-separated scheme. Then $\mathrm{QCoh}(X)$ can be generated by a single compact object.

Proof. We prove the statement by the induction principle on quasi-compact opens. Define, for a quasi-compact open $W \subseteq X$, the property $P(W)$ as "$\mathrm{QCoh}(W)$ admits a compact generator". Now, if $W = \mathrm{Spec}(A)$ is affine, then $\mathrm{QCoh}(W) \simeq \mathrm{Mod}_A$ and the unit $A \in \mathrm{Mod}_A$ is compact and generates. Hence $P(W)$ holds on affines. Let now W be quasi-compact open and write $W = U \cup V$ with U

quasi-compact open and V affine open and assume that the property P holds for U, V and $U \cap V$. Let $j : U \hookrightarrow W$ be the inclusion. Then there exists a right split Verdier sequence

$$\mathrm{QCoh}_{W\setminus U}(W) \xrightarrow{\iota_*} \mathrm{QCoh}(W) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{QCoh}(U)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. The goal is to apply Proposition 5.2.6, so it suffices to check its assumptions.

(1) An application of Lemma 5.3.5 produces a single compact generator $K \in \mathrm{QCoh}_Z(W)$ whose image $K_W \in \mathrm{Perf}(W)$ is compact.
(2) Let $G_U \in \mathrm{QCoh}(U)$ be the single compact generator (which exists by assumption). Notice that it does not suffice to use the object $j_*(G_U)$, since it is not necessarily compact in W. Consider therefore Corollary 5.3.2; it provides a compact object $P_U \in \mathrm{Perf}(W)$ such that G_U fits into a retract of $j^*(P_U)$. In particular, $\mathrm{Loc}_{\mathrm{QCoh}(U)}(j^*(P_U))$ contains G_U, so that $\mathrm{QCoh}(U) = \mathrm{Loc}_{\mathrm{QCoh}(U)}(j^*(P_U))$. Thus the object P_U satisfies the assumptions.

Thus Proposition 5.2.6 implies that $\mathrm{QCoh}(W)$ is the localizing closure of $K_W \oplus P_U$, that is, it is singly compactly generated. □

Corollary 5.3.7. Let X be a quasi-compact quasi-separated scheme. Then $\mathrm{Perf}(X)$ is classicaly generated.

5.4. **Resolutions of the diagonal.** We begin with a useful result: the resolution of diagonal.

Proposition 5.4.1. Let R be a noetherian ring. Let X be a separated finite type scheme over R with the resolution property. Set $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_X)$ where $\Delta : X \to X \times_R X$ is the diagonal of X/R. Then there exists a resolution

$$\cdots \to E_1 \boxtimes G_1 \to E_0 \boxtimes G_0 \to \mathcal{O}_\Delta \to 0$$

where each E_i and G_i is a finite locally free $\mathcal{O}_X$-module.

Proof. This is [Aut18, Lemma 0FZ2]. Since X is separated over R, the diagonal $\Delta : X \to X \times_R X$ is a closed immersion. Hence $\mathcal{O}_\Delta$ is a coherent $\mathcal{O}_{X\times_R X}$-module[Aut18, Tag 0FZ2]. Now, in general, if F is a coherent $\mathcal{O}_{X\times_R X}$-module, then there exists a surjection $E \boxtimes G \twoheadrightarrow F$ where E and G are finite locally free $\mathcal{O}_X$-modules, see [Aut18, Tag 0FZ0]. Applying this to $F = \mathcal{O}_\Delta$, one gets a surjection $E_0 \boxtimes G_0 \twoheadrightarrow \mathcal{O}_\Delta$. Let K_1 be its kernel. Since $X \times_R X$ is locally Noetherian, kernels of morphisms of coherent sheaves are coherent, so K_1 is coherent. Applying again the previous statement to K_1, one gets a surjection $E_1 \boxtimes G_1 \twoheadrightarrow K_1$. Iterating this construction for the successive kernels produces the required resolution. □

Lemma 5.4.2. Let k be a field and let X be a quasi-compact separated smooth scheme over k and let d be the dimension of X. There exist finite locally free $\mathcal{O}_X$-modules E and G such that $\mathcal{O}_\Delta \in \mathrm{thick}_{2d+1}(E \boxtimes G)$.

In particular, $\mathrm{Perf}(X) = \mathrm{Coh}(X)$ is smooth over $\mathrm{Perf}(k) = \mathrm{Coh}(k)$.

Proof. This is [Aut18, Lemma 0FZ5]. Since X is smooth over k, it is regular. Hence X has the resolution property. By Proposition 5.4.1, there exists a resolution $\cdots \to E_1 \boxtimes G_1 \to E_0 \boxtimes G_0 \to \mathcal{O}_\Delta \to 0$ where each E_i and G_i is finite locally free on X. Since $X \times_k X$ is smooth over k, it is regular, noetherian and of dimension $2d$.

Let K be the perfect complex on $X \times_k X$ given by $K = (E_{2d} \boxtimes G_{2d} \to \cdots \to E_0 \boxtimes G_0)$. Then $\pi_0(K) \cong \mathcal{O}_\Delta$, $\pi_{2d}(K) \cong \ker(E_{2d} \boxtimes G_{2d} \to E_{2d-1} \boxtimes G_{2d-1})$, and $\pi_i(K) = 0$ for $0 < i < 2d$. By the splitting lemma for regular noetherian schemes Exercise E.5.3, it follows that $\mathcal{O}_\Delta$ is a direct summand of K. On the other hand, K is obtained from the objects $E_i \boxtimes G_i$ for $0 \le i \le 2d$ by finitely many extensions, hence $K \in \mathrm{thick}\big(\bigoplus_{i=0}^{2d} E_i \boxtimes G_i\big)$. Set $E = \bigoplus_{i=0}^{2d} E_i$ and $G = \bigoplus_{i=0}^{2d} G_i$ so that each $E_i \boxtimes G_i$ is a direct summand of $E \boxtimes G$, so $K \in \mathrm{thick}(E \boxtimes G)$. Since $\mathcal{O}_\Delta$ is a direct summand of K, it follows that $\mathcal{O}_\Delta \in \mathrm{thick}(E \boxtimes G)$.

Under the equivalence $\mathrm{Perf}(X)^{\mathrm{op}} \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(X) \simeq \mathrm{Perf}(X \times_k X)$, the diagonal bimodule corresponds to $\mathcal{O}_\Delta$. Hence the diagonal bimodule is perfect, so $\mathrm{Perf}(X)$ is smooth over $\mathrm{Perf}(k)$. Finally, since X is regular noetherian, one has $\mathrm{Perf}(X) = \mathrm{Coh}(X)$. □

Corollary 5.4.3. Let k be a field and let X be a proper smooth scheme over k with dimension d. Then $\mathrm{Perf}(X)$ has a strong generator with generation time $2d$.

Proof. this is [Aut18, Lemma 0FZ6]. By Lemma 5.4.2, choose finite locally free $\mathcal{O}_X$-modules E and G such that $\mathcal{O}_\Delta \in \mathrm{thick}(E \boxtimes G)$. Choose $n \in \mathbb{N}$ such that $\mathcal{O}_\Delta \in \mathrm{thick}_n(E \boxtimes G)$, for example $2d+1$ for d the dimension of X. Then G is a strong generator of $\mathrm{Perf}(X)$.

Let $M \in \mathrm{Perf}(X)$. Consider the exact functor $\Phi_M(-) = p_{2,*}(p_1^* M \otimes -) : \mathrm{Perf}(X \times_k X) \to \mathrm{Perf}(X)$. Since exact functors preserve thickenings, from $\mathcal{O}_\Delta \in \mathrm{thick}_n(E \boxtimes G)$ one gets $M \cong \Phi_M(\mathcal{O}_\Delta) \in \mathrm{thick}_n(\Phi_M(E \boxtimes G))$. Now $\Phi_M(E \boxtimes G) \cong p_{2,*}(p_1^* M \otimes (E \boxtimes G)) \cong p_{2,*}((M \otimes E) \boxtimes G) \cong \Gamma(X, M \otimes E) \otimes_k G$. Since X is proper over k and $M \otimes E$ is perfect, the object $\Gamma(X, M \otimes E)$ is a perfect complex over k. As k is a field, every perfect complex over k is a finite direct sum of shifts of k (see Example 8.4.2). Therefore $\Gamma(X, M \otimes E) \otimes_k G \in \mathrm{thick}_1(G)$. It follows that $M \in \mathrm{thick}_n(G)$. Since this holds for every $M \in \mathrm{Perf}(X)$, it follows that $\mathrm{Perf}(X) = \mathrm{thick}_n(G)$, so G is a strong generator. □

For the applications of these notes, the following will be useful.

Proposition 5.4.4. Let X be a quasi-projective scheme over a field k, and let L be an ample invertible sheaf on X. Then there exists an exact complex

$$\cdots \to C^{-2} \to C^{-1} \to C^0 \to \mathcal{O}_\Delta \to 0$$

on $X \times_k X$ such that each C^{-i} is a finite direct sum of sheaves of the form $(L \boxtimes L)^{-n_i}$. Equivalently, each C^{-i} is a finite direct sum of sheaves of the form $L^{-n_i} \boxtimes L^{-n_i}$.

Proof. Since L is ample on X, the line bundle $L \boxtimes L$ is ample on $X \times_k X$. For every coherent sheaf F on $X \times_k X$, ampleness implies that there exist integers $n \geq 1$ and $N \geq 0$ together with a surjection $(L \boxtimes L)^{-n, \oplus N} \twoheadrightarrow F$. Apply this first to $F = O_\Delta$, and then inductively to the successive kernels. This produces the required exact complex. □

Remark 5.4.5. The resolution of Proposition 5.4.1 (and Proposition 5.4.4) can be viewed as a universal filtration of the identity functor. Let X be a separated finite type scheme over R with the resolution property and let $p_1, p_2 : X \times_R X \to X$ be the two projections. For every kernel $K \in \mathrm{QCoh}(X \times_R X)$, consider the *Fourier-Mukai* functor

$$\Phi_K(-) := p_{2*}(p_1^*(-) \otimes K) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X).$$

Since $\Delta : X \to X \times_R X$ is the diagonal, one has $\Phi_{\mathcal{O}_\Delta} \simeq \mathrm{id}$ on $\mathrm{QCoh}(X)$ (see Exercise E.5.2). Therefore a resolution of $\mathcal{O}_\Delta$ by objects of the form $E_i \boxtimes G_i$ induces, after applying $\Phi_{(-)}$, a filtration of the identity functor by simpler functors. If moreover $X \to \mathrm{Spec}(R)$ is proper and the kernels $E_i \boxtimes G_i$ have finite Tor-dimension over the first factor (for instance if X is smooth over R), then these transforms preserve $D\mathrm{Coh}(X)$, and for every $F \in \mathrm{Coh}(X)$ the above resolution yields a filtration of F with successive terms $p_{2*}(p_1^* F \otimes (E_i \boxtimes G_i))$.

Remark 5.4.6. In the case $X = \mathbb{P}^n_k$, where k is a field, the preceding remark applies without any further difficulty, since $\mathbb{P}^n_k$ is smooth and proper over k. Thus the resolution of the diagonal gives, for every $F \in \mathrm{Coh}(\mathbb{P}^n_k)$, a filtration of F whose successive terms are

$$p_{2*}(p_1^* F \otimes (E_i \boxtimes G_i)) \simeq \Gamma(\mathbb{P}^n_k, F \otimes E_i) \otimes_k G_i.$$

Hence a resolution of O_Δ by elementary kernels $E_i \boxtimes G_i$ produces a universal expression of every object of $\mathrm{Coh}(\mathbb{P}^n_k)$ in terms of the objects G_i.

In the case of projective space, one can do even better.

Proposition 5.4.7 (Beilinson's resolution of the diagonal)**.** Let k be a field, and set $X = \mathbb{P}^n_k$. Let $\Delta : X \to X \times_k X$ be the diagonal. Then there is an exact sequence

$$0 \to \Omega^n_X(n) \boxtimes \mathcal{O}_X(-n) \to \cdots \to \Omega^1_X(1) \boxtimes \mathcal{O}_X(-1) \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Proof. Let V be a $(n+1)$-dimensional k-vector space such that $X = \mathbb{P}(V)$. Consider the tautological exact sequence $0 \to \mathcal{O}_X(-1) \to V \otimes_k \mathcal{O}_X \to Q \to 0$ where Q is a vector bundle of rank n. By the Euler sequence, one has $Q^\vee \simeq \Omega^1_X(1)$.

Let $p_1, p_2 : X \times X \to X$ be the projections. Pulling back the tautological inclusion along p_2 and the tautological quotient along p_1, one obtains a morphism $\beta : p_2^*\mathcal{O}_X(-1) \to p_1^*Q$ which corresponds to a global section $s \in \Gamma(X \times X, p_1^*Q \otimes p_2^*\mathcal{O}_X(1))$. The zero locus of s is precisely the diagonal.

Indeed, a point of $X \times X$ is a pair $([\ell_1], [\ell_2])$ of lines in V, and the fibre of β at such a point is the composite $\ell_2 \hookrightarrow V \twoheadrightarrow V/\ell_1$ This map is zero if and only if $\ell_2 \subset \ell_1$, hence if and only if $\ell_1 = \ell_2$. Therefore the vanishing locus of s is set-theoretically equal to Δ. For the scheme-theoretically claim, let $U_i = \{x_i \neq 0\} \subset X$ be the standard affine open. On $U_i \times U_i$, write affine coordinates

$$u_j = \frac{x_j}{x_i}, \qquad v_j = \frac{y_j}{y_i}, \qquad j \neq i,$$

on the first and second factor respectively. Under the natural trivialization of $p_1^*Q \otimes p_2^*\mathcal{O}_X(1)$ on $U_i \times U_i$, the section s is given by $(v_j - u_j)_{j\neq i}$. Hence its zero scheme on $U_i \times U_i$ is cut out by the ideal $(v_j - u_j \mid j \neq i)$ which is exactly the ideal of $\Delta \cap (U_i \times U_i)$. Since these functions form a regular sequence, the section s is regular, proving the claim.

The vector bundle

$$E := p_1^*Q \otimes p_2^*\mathcal{O}_X(1)$$

has rank n, and $\Delta \subset X \times X$ has codimension n. Since Δ is the zero scheme of the regular section s, the Koszul complex of s yields an exact sequence

$$0 \to \Lambda^n(E^\vee) \to \cdots \to E^\vee \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Finally,

$$E^\vee \simeq p_1^*(Q^\vee) \otimes p_2^*\mathcal{O}_X(-1) \simeq p_1^*(\Omega^1_X(1)) \otimes p_2^*\mathcal{O}_X(-1) = \Omega^1_X(1) \boxtimes \mathcal{O}_X(-1),$$

and therefore $\Lambda^i(E^\vee) \simeq \Omega^i_X(i) \boxtimes \mathcal{O}_X(-i)$. Substituting into the Koszul complex gives the claimed resolution. □

Theorem 5.4.8 (Beilinson)**.** Over a field k, it is $\mathrm{Coh}(\mathbb{P}^n_k) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\rangle$.

Proof. First, the collection is semi-orthogonal. Indeed, for $0 \le j < i \le n$ one has

$$\pi_* \hom_{\mathbb{P}^n_k}(\mathcal{O}(i), \mathcal{O}(j)) \cong H^*(\mathbb{P}^n_k, \mathcal{O}(j-i)),$$

and these groups vanish because $-n \le j - i \le -1$. In particular, the mapping spectrum vanishes, proving semi-orthogonality. To prove generation, consider the diagonal $\Delta \subset \mathbb{P}^n_k \times \mathbb{P}^n_k$ and consider the resolution of Proposition 5.4.7

$$0 \to \Omega^n_X(n) \boxtimes \mathcal{O}_X(-n) \to \cdots \to \Omega^1_X(1) \boxtimes \mathcal{O}_X(-1) \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Let $p_1, p_2 : \mathbb{P}^n_k \times \mathbb{P}^n_k \to \mathbb{P}^n_k$ be the projections. Let $F \in \mathrm{Coh}(\mathbb{P}^n_k)$. Then Remark 5.4.6 implies that $F \simeq p_{2*}(p_1^*F \otimes \mathcal{O}_\Delta)$ and an application of the exact functor $p_{2*}(p_1^*F \otimes -)$ to the above resolution produces a filtration of F whose successive terms are of the form

$$p_{2*}(p_1^*(\Omega^i(i) \otimes F) \otimes p_2^*\mathcal{O}(-i)) \simeq \Gamma(\mathbb{P}^n_k, \Omega^i(i) \otimes F) \otimes_k \mathcal{O}(-i),$$

by the projection formula and flat base change. Therefore $F \in \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}\rangle$ and twisting by $\mathcal{O}(n)$ shows $F \in \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\rangle$ thus proving the claim. □

The semiorthogonal decompoistion appearing in the statement is "exceptional".

Definition 5.4.9. Let k be a field and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_k}(\mathrm{Cat}^{\mathrm{perf}})$ a k-linear category.

(1) An object $E \in \mathcal{C}$ is *exceptional* if $\pi_* \hom_{\mathcal{C}}(E, E) \cong k$ concentrated in degree 0.
(2) A sequence $(E_0, \dots, E_n)$ of objects of $\mathcal{D}$ is an *exceptional collection* if each E_i is exceptional and $\pi_* \hom_{\mathcal{C}}(E_i, E_j) = 0$ for all $i > j$.
(3) An exceptional collection $(E_0, \dots, E_n)$ is *full* $\mathcal{C} = \langle E_0, \dots, E_n \rangle$.

Corollary 5.4.10. The semi-orthogonal decomposition of Theorem 5.4.8 is a full exceptional collection.

Proof. For every i one has $\pi_* \hom_{\mathbb{P}^n_k}(\mathcal{O}(i), \mathcal{O}(i)) \cong H^*(\mathbb{P}^n_k, \mathcal{O}) \cong k$ concentrated in degree 0, so each $\mathcal{O}(i)$ is exceptional. The semi-orthogonality was proved in the proof of Theorem 5.4.8, and fullness is exactly the content of that theorem. □

6. Projective classes

In this chapter we discuss projective classes on stable categories, a useful tool to reduce the complexity of a category in two pieces: projective objects and ideal maps. We begin with the basic formalism of ideals and projective objects, and we explain how these two pieces of data determine each other. We then discuss examples and constructions, with particular attention to projective classes generated by a small collection of objects. This leads to the associated cellular and phantom towers, which will be used later as a technical tool. We conclude with the second representability result of these notes, a Brown representability statement.

6.1. The definition.

Since these two pieces of data determine each other, we start with introducing ideals.

Definition 6.1.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. An *ideal* is a a collection of morphisms $\mathcal{I} \subseteq \mathcal{C}^{\Delta^1}$ such that:

(1) The category $\mathcal{I}$ is closed under finite sums. Notice that the zero map belongs to $\mathcal{I}$.
(2) For every composable morphisms f, g, h in $\mathcal{C}$, if $g \in \mathcal{I}$ then $gf \in \mathcal{I}$ and $hg \in \mathcal{I}$.

We let $\mathrm{Ideals}(\mathcal{C})$ denote the subcategory of $\mathcal{C}^{\Delta^1}$, which we regard as a poset order by inclusion.

Notation 6.1.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{I}$ be an ideal. We will denote by $\mathcal{I}$-proj the full subcategory of $\mathcal{C}$ spanned by those objects $x \in \mathcal{C}$ such that $\pi_* \hom_{\mathcal{C}}(x, i) \simeq$ is the zero map for every $* \in \mathbb{Z}$ and every $i \in \mathcal{I}$. We will refer to $\mathcal{I}$-proj as the *$\mathcal{I}$-projectives*.

Lemma 6.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{I}$ be an ideal. Then $\mathcal{I}$-proj is closed under suspensions, direct sums and retracts.

Proof. Let $x \in \mathcal{C}$ and $i \in \mathcal{I}$. For suspensions, the claim follows since $\pi_* \hom_{\mathcal{C}}(\Sigma^n x, i) \simeq \pi_{*-n} \hom_{\mathcal{C}}(x, i)$, and for direct sums, the claim follows since $\hom_{\mathcal{C}}(-, i)$ is exact. For retracts, it follows since the retract of a zero map of abelian groups is a zero map. □

We now discuss projective objects.

Notation 6.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{P} \subseteq \mathcal{C}$ be a full subcategory. We let $\mathcal{P}$-null be the colletion of morphisms $i \in \mathcal{C}^{\Delta^1}$ such that $\pi_* \hom_{\mathcal{C}}(p, i) \simeq 0$ is the zero map for every $* \in \mathbb{Z}$ and $p \in \mathcal{P}$. We will refer to $\mathcal{P}$-null as the *ideal of null maps of $\mathcal{P}$*. It is immediate to check that this is indeed an ideal using the properties of the mapping spectrum functor

We have then the following "Galois connection".

Lemma 6.1.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then there exists an adjunction $(-)\text{-proj} : \mathrm{Ideals}(\mathcal{C}) \rightleftarrows \mathrm{Sub}(\mathcal{C})^{\mathrm{op}} : (-)\text{-null}$.

Proof. First of all, taking projective and null maps furnishes the required functors. Indeed:

(1) If $\mathcal{I} \subseteq \mathcal{J}$ are two ideals, and if $x \in \mathcal{J}$-proj, then $\pi_* \hom_{\mathcal{C}}(x, j) \simeq 0$ for all $j \in \mathcal{J}$, thus $\pi_* \hom_{\mathcal{C}}(x, i) \simeq 0$ for all $i \in \mathcal{I}$, making $x \in \mathcal{I}$-proj, so that $\mathcal{J}$-proj $\subseteq \mathcal{I}$-proj.
(2) If $\mathcal{P} \subseteq \mathcal{Q}$, then for every $i \in \mathcal{Q}$-null then for every $x \in \mathcal{Q}$ the maps $\pi_* \hom_{\mathcal{C}}(x, i) \simeq 0$ are zero, so that for every $y \in \mathcal{P}$ it is $\pi_* \hom_{\mathcal{C}}(y, i) \simeq 0$, making $\mathcal{Q}$-null $\subseteq \mathcal{P}$-null.

For the adjunction, let $\mathcal{I}$ be an ideal and $\mathcal{P} \subseteq \mathcal{C}$ a class of objects. Then $\mathcal{I}$-proj $\subseteq \mathcal{P}$ in $\mathrm{Sub}(\mathcal{C})^{\mathrm{op}}$ if and only if $\mathcal{P} \subseteq \mathcal{I}$-proj as subcategories of $\mathcal{C}$. By definition of $\mathcal{I}$-proj, this happens if and only if for every $p \in \mathcal{P}$ and for every $i \in \mathcal{I}$ the maps $\pi_* \hom_{\mathcal{C}}(p, i) \simeq 0$ vanish, which happens if and only if $\mathcal{I} \subseteq \mathcal{P}$-null, being $\mathcal{P}\text{-null} = \{j \in \mathcal{C}^{\Delta^1} \mid \pi_* \hom_{\mathcal{C}}(p, j) \simeq 0 \text{ for all } p \in \mathcal{P}\}$. □

With ideals and projectives objects we can finally give the following.

Definition 6.1.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. A *projective class* on $\mathcal{C}$ consists of a pair $(\mathcal{P}, \mathcal{I})$ of a subcategory and an ideal of $\mathcal{C}$ such that:

(1) The pair is *orthogonal*, in the sense that $\mathcal{P}$-null $= \mathcal{I}$ and $\mathcal{I}$-proj $= \mathcal{P}$.
(2) There are *enough projectives*, in the sense that for every $x \in \mathcal{C}$ there esists an exact sequence $p \to x \to y$ with $p \in \mathcal{P}$ and $(x \to y) \in \mathcal{I}$.

The moprhism $p \to x$ is often called a *$\mathcal{P}$-precover* of x.

We recall the following trivial (but extremely useful) result.

Lemma 6.1.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $(\mathcal{P}, \mathcal{I})$ be an orthogonal pair. If $f : p \to x$ is in $\mathcal{I}$ and has domain $p \in \mathcal{P}$, then f is null-homotopic.

Proof. Indeed, $f \in \mathcal{I}$ implies that $\pi_* \hom_{\mathcal{C}}(p, p) \xrightarrow{f_*} \pi_* \hom_{\mathcal{C}}(p, x)$ is the zero map, that is, $f \simeq f \circ \mathrm{id}_p$ is the zero element of $\pi_* \hom_{\mathcal{C}}(p, x)$, implying that f is null-homotopic. □

6.2. **Examples and constructions.** We now discuss basic examples and tools to construct projective classes.

Example 6.2.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then there are always two trivial projective classes, namely $(\mathcal{C}, 0)$ and $(0, \mathcal{C}^{\Delta^1})$. In the first projective class, 0 denotes the ideal of *null-homotopic maps*, and not of zero maps!

Remark 6.2.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. In practice, there are essentially two ways to to define a projective class on $\mathcal{C}$:

(1) Start with a collection of objects $S \subseteq \mathcal{C}$, produce the ideal S-null and define the subcategory (S-null)-proj and then check that the pair ((S-null)-proj, S-null) has enough projectives.
(2) Start with a collection of maps $M \subseteq \mathcal{C}^{\Delta^1}$, produce the subcategory M-proj and define the ideal (M-proj)-null and then check that the pair (M-proj, (M-proj)-null) has enough projectives.

The next result shows that for presentable stable categories the first construction produces indeed a projective class, which is moreover functorial.

Proposition 6.2.3. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category, and let $S \subseteq \mathcal{C}$ be a small subcategory. Let $\mathcal{P} = \mathrm{Thick}_1(S)$ be the smallest full subcategory of $\mathcal{C}$ closed under small coproducts, retracts and shifts that contains S. Then the pair $(\mathcal{P}, \mathcal{P}\text{-null})$ is a projective class on $\mathcal{C}$.

Proof. Assume, without loss of generality, that S is closed under shifts. Let $R : \mathcal{C} \to \mathrm{Fun}(S, \mathrm{Set})$ be the functor mapping $x \in \mathcal{C}$ to $\pi_0 \hom_{\mathcal{C}}(x, -) : S \to \mathrm{Set}$ and let $L : \mathrm{Fun}(S, \mathrm{Set}) \to \mathcal{C}$ be defined by mapping $F : S \to \mathrm{Set}$ to $L(F) = \bigoplus_{s \in S} \bigoplus_{a \in F(s)} s$. Notice that there is an adjunction $L : \mathrm{Fun}(S, \mathrm{Set}) \rightleftarrows \mathcal{C} : R$. Let $P = LR : \mathcal{C} \to \mathcal{C}$ be the induced comonad and denote by $\varepsilon : P \to \mathrm{id}_{\mathcal{C}}$ the counit of the adjunction. Given $x \in \mathcal{C}$, let y_x be the cofibre of the map $\varepsilon_x : P(x) \to x$.

Notice that, by construction $P(x) = L(Rx)$ is a coproduct of objects of S, hence lies in the smallest full subcategory closed under small coproducts and containing S. Since $\mathcal{P}$ is, by definition, thick and closed under small coproducts (and contains S), it follows that $Px \in \mathcal{P}$. Notice also that the map $x \to y_x$ is $\mathcal{P}$-null. Indeed, by applying $\pi_0 \hom_{\mathcal{C}}(s,-)$, for $s \in S$, at the cofibre sequence defining y_x, it follows the existence of an exact sequence

$$\pi_0 \hom_{\mathcal{C}}(s, P(x)) \to \pi_0 \hom_{\mathcal{C}}(s,x) \to \pi_0 \hom_{\mathcal{C}}(s, y_x).$$

in which the first map is surjective. Indeed, under the adjunction $L \dashv R$, the counit $\varepsilon_x : LR(x) \to x$ corresponds to the identity natural transformation $R(x) \to R(x)$; evaluating at s says exactly that every element of $\pi_0 \hom_{\mathcal{C}}(s,x) = R(x)(s)$ is hit by the summand $s \to Px$ indexed by that element, thus making the fist map zero. Since this holds for every $s \in S$ and since this property is stable under shifts, retracts and small coproducts, it follows that $\pi_0 \hom_{\mathcal{C}}(p, P(x)) \to \pi_0 \hom_{\mathcal{C}}(p,x)$ is surjective for every $p \in \mathcal{P}$, making $x \to y_x \in \mathcal{P}$-null. The claim the follows since $P(x) \to x \to y_x$ is the required $\mathcal{P}$-precover. □

Remark 6.2.4. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $\mathcal{D} \subseteq \mathcal{C}$ be a small full subcategory. Then a projective class $(\mathcal{P}, \mathcal{I})$ on $\mathcal{C}$ does not necessarily restrict to a projective class on $\mathcal{D}$; it does if and only if the relevant precovers exists in $\mathcal{D}$. If S is a small full subcategory of $\mathcal{C}$ and the projective class is generated via Proposition 6.2.3, then the restriction to $\mathcal{D}$ amounts to check that the intersection $\mathcal{D} \cap \mathcal{P}$ is is closed under the coproducts needed to build the canonical $\mathcal{P}$-precover of objects in $\mathcal{D}$.

The next result does not use the theory of projective classes, but only the thick calculus.

Lemma 6.2.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $S \subseteq \mathcal{C}$ be a small subcategory. If $x \in \mathrm{thick}_n(S)$, then the n-fold composites of $\mathrm{thick}_1(S)$-null maps out of x is null-homotopic.

Proof. The proof goes by induction on n. If $n = 1$ and $x \in \mathrm{thick}_1(S)$, then for every $f \in \mathrm{thick}_1(S)$-null it is $\pi_* \hom_{\mathcal{C}}(x, f) = 0$ by definition. Evaluating at id_x shows that f is null-homotopic. Assume now the claim for $n-1$, and let $x \in \mathrm{thick}_n(S)$. Since $\mathrm{thick}_n(S) = \mathrm{smd}(\mathrm{thick}_1(S) * \mathrm{thick}_{n-1}(S))$, it is enough to treat the case in which x fits into an exact triangle $p \to x \to z$ with $p \in \mathrm{thick}_1(S)$ and $z \in \mathrm{thick}_{n-1}(S)$ (indeed the case of retracts has a similar proof). Let $f : x \to y$ be a composite of n maps in $\mathrm{thick}_1(S)$-null, say $f = i_n \circ \cdots \circ i_1$ with each $i_j \in \mathrm{thick}_1(S)$-null. Since $i_1 \circ (p \to x)$ is again in $\mathrm{thick}_1(S)$-null and $p \in \mathrm{thick}_1(S)$, it follows $i_1 \circ (p \to x)$ is null-homotopic by Lemma 6.1.7. By exactness, i_1 therefore factors through $z \to \Sigma p$, so there exists a map $\alpha : z \to \mathrm{cod}(i_1)$ such that $i_1 \simeq \alpha \circ (x \to z)$. Hence $f \simeq i_n \circ \cdots \circ i_2 \circ \alpha \circ (x \to z)$. Set $g \coloneqq i_n \circ \cdots \circ i_2 \circ \alpha : z \to y$. Since $\mathrm{thick}_1(S)\text{-null}^{n-1}$ is an ideal and $i_n \circ \cdots \circ i_2 \in \mathrm{thick}_1(S)\text{-null}^{n-1}$, it follows that $g \in \mathrm{thick}_1(S)\text{-null}^{n-1}$. By the inductive hypothesis, since $z \in \mathrm{thick}_{n-1}(S)$, it follows that g is null-homotopic. Therefore $f \simeq g \circ (x \to z)$ is null-homotopic. □

We now introduce *ghosts*, ideals associated to particular projective classes.

Remark 6.2.6. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $G \in \mathcal{C}^\omega$ be an object. Then Proposition 6.2.3 implies the existence of a projective class whose subcategory of projectives is given by $\mathrm{Thick}_1(G)$. The ideal of null maps is then given by those maps $x \to y$ such that $\pi_* \hom_{\mathcal{C}}(\Sigma^n G, x) \to \pi_* \hom_{\mathcal{C}}(\Sigma^n G, y)$ is the zero map. We will refer to such maps as *G-ghosts* and we will denote the ideal of G-ghosts maps by $\mathcal{Gh}_G$. The projective class $(\mathrm{Thick}_1(G), \mathcal{Gh}_G)$ is called the *projective class of G-ghosts*.

We obtain the following extremely useful result.

Remark 6.2.7 (The ghost lemma)**.** Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $G \in \mathcal{C}^\omega$ be a compact object. Consider the projective class of G-ghosts $(\mathrm{Thick}_1(G), \mathcal{Gh}_G)$ on $\mathcal{C}$. An argument with compactness shows that $\mathrm{thick}_n(G) = \mathcal{C}^\omega \cap \mathrm{Thick}_n(G)$ for every $n \in \mathbb{N}$. In particular, if $x \in \mathrm{thick}_n(G)$, then Lemma 6.2.5 implies that the n-fold composite of G-ghosts out of x vanishes. Equivalently, if there exists a non null-homotopic n-fold composite of G-ghosts out of some compact object $x \in \mathcal{C}^\omega$, then $x \notin \mathrm{thick}_n(G)$. In particular, if G is a strong generator of $\mathcal{C}^\omega$, this gives a lower bound on the generation time of G: the existence of such a composite implies that ⏰$(G) > n$.

We now study projective classes closed under extensions. The next observation tells us that there are few projective classes closed under extenions.

Example 6.2.8. Let $\mathrm{Mod}_{\mathbb{Z}}$ be the derived category of $\mathbb{Z}$-modules. Let $S = \{\mathbb{Z}\}$ and apply Proposition 6.2.3 to deduce the existence of a projective class $(\mathcal{P}, \mathcal{P}\text{-null})$ on $\mathrm{Mod}_{\mathbb{Z}}$. Notice that $\mathcal{P}$-null may be identified with the ideal of $\mathbb{Z}$-ghosts maps. Let now p be a prime and consider the extension $\mathbb{Z} \to \mathbb{Z}/p \to \Sigma\mathbb{Z}$. Notice that the connective morphism $\mathbb{Z}/p \to \Sigma\mathbb{Z}$ is ghost, since the homotopy of $\mathbb{Z}/p$ exists only in degree zero and the homotopy of $\Sigma\mathbb{Z}$ only in degree -1. If now $\mathbb{Z}/p$ was projective, then the ghost morphism $\mathbb{Z}/p \to \Sigma\mathbb{Z}$ would have projective domain, and Lemma 6.1.7 would imply that $\mathbb{Z}/p \to \Sigma\mathbb{Z}$ is the zero morphism, making $\mathbb{Z}/p$ a retract of $\mathbb{Z}$, which is clearly false.

Fortunately, there is a way of making projective classes closed under extensions.

Lemma 6.2.9. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. For every $n \in \mathbb{N}$ let $\mathcal{P}_n = \mathrm{smd}(\mathcal{P}^{*n})$ be the retract closure of the n-extension of $\mathcal{P}$ and let $\mathcal{I}^n$ be the ideal of n-composites of $\mathcal{I}$. Then $(\mathcal{P}_n, \mathcal{I}^n)$ is a projective class.

Proof. The proof goes by induction on n, the cases $n = 0, 1$ being trivial. Assume therefore that $(\mathcal{P}_{n-1}, \mathcal{I}^{n-1})$ defines a projective class. For orthogonality:

(1) To show that $\mathcal{P}_n$-null $= \mathcal{I}^n$, notice first of all that $\mathcal{I}^n \subseteq \mathcal{I}$, being $\mathcal{I}$ an ideal, so that $\mathcal{P} = \mathcal{I}$-proj $\subseteq (\mathcal{I}^n)$-proj. Since $(\mathcal{I}^n)$-proj is thick (it is closed under shifts, retracts and extensions, by exactness of the mapping spectrum), it follows that $\mathcal{P}_n = \mathcal{P}^{*n} \subseteq (\mathcal{I}^n)$-proj. In particular, every $f \in \mathcal{I}^n$ is annihilated by every $p \in \mathcal{P}_n$, hence $\mathcal{I}^n \subseteq (\mathcal{P}_n)$-null. For the reverse inclusion, let $f : x \to y$ be a morphism in $(\mathcal{P}_n)$-null. Choose an exact sequence $p \to x \to x_1$ with $p \in \mathcal{P}$ and $i_1 \in \mathcal{I}$. Since $p \in \mathcal{P} \subseteq \mathcal{P}_n$, it follows that $f \circ (p \to x) = 0$, so f factors as $f = f_1 \circ (x \to x_1)$ for some $f_1 : x_1 \to y$. The inclusion will be proved if $f_1 \in (\mathcal{P}_{n-1})$-null. This follows from a trivial computantion (after having picked a $\mathcal{P}_{n-1}$-precover for x_1).

(2) To show that $(\mathcal{I}^n)$-proj $= \mathcal{P}_n$, since $\mathcal{P}_n \subseteq (\mathcal{I}^n)$-proj holds from the first part, it suffices to prove $(\mathcal{I}^n)$-proj $\subseteq \mathcal{P}_n$. Let $z \in (\mathcal{I}^n)$-proj and assume for a moment that there are enough projectives (see the paragraph below), so that there exists an exact sequence $p_n \to z \to z'$ with $p_n \in \mathcal{P}_n$ and $(z \to z') \in \mathcal{I}^n = (\mathcal{P}_n)$. Since $z \in (\mathcal{I}^n)$-proj, the induced map $\pi_0 \hom_{\mathcal{C}}(z, z) \to \pi_0 \hom_{\mathcal{C}}(z, z')$ is zero, hence id_z maps to 0. Therefore the cofibre sequence splits and z is a retract of p_n. Since $\mathcal{P}_n$ is closed under retracts, it follows that $z \in \mathcal{P}_n$.

To prove that there are enough projectives, let $x \in \mathcal{C}$ and choose an exact sequence $p_{n-1} \to x \to x_{n-1}$ with $p_{n-1} \in \mathcal{P}_{n-1}$ and $(x \to x_{n-1}) \in \mathcal{I}^{n-1}$. Pick then also an exact sequence $p_1 \to x_{n-1} \to x_1$ with $p_1 \in \mathcal{P}_1$ and $(x_{n-1} \to x_1) \in \mathcal{I}^1$. Since the composite $x \to x_{n-1} \to x_1$ lies in $\mathcal{I}^n$ and since the octahedral axiom furnishes an exact sequence $p_n \to x \to x_1$, it suffices to notice that the extension $p_{n-1} \to p_n \to p_1$ implies $p_n \in \mathcal{P}_n$. □

There is another interesting operation on projective classes.

Construction 6.2.10. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with A-coproducts and let $((\mathcal{P}_\alpha, \mathcal{I}_\alpha))_{\alpha \in A}$ be a family of projective classes. Define $\bigcap_{\alpha \in A} (\mathcal{P}_\alpha, \mathcal{I}_\alpha)$ to be the pair $(\mathcal{P}, \mathcal{I})$ with the full subcategory of $\mathcal{C}$ spanned by retracts of coproducts $\bigoplus_{\alpha \in A} p_\alpha$ with $p_\alpha \in \mathcal{P}_\alpha$ and $\mathcal{I} = \mathcal{P}$-null.

Lemma 6.2.11. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $((\mathcal{P}_\alpha, \mathcal{I}_\alpha))_{\alpha \in A}$ be a family of projective classes. Assume that $\mathcal{C}$ admits A-indexed coproducts. Then the pair $\bigcap_{\alpha \in A} (\mathcal{P}_\alpha, \mathcal{I}_\alpha)$ is a projective class.

Proof. Denote by $(\mathcal{P}, \mathcal{I})$ the pair $\bigcap_{\alpha \in A} (\mathcal{P}_\alpha, \mathcal{I}_\alpha)$. Then:

(1) First, one has $\mathcal{I} = \bigcap_{\alpha\in A}\mathcal{I}_\alpha$. Indeed, let $f \in \mathcal{I} = \mathcal{P}$-null. For every fixed α and every $p_\alpha \in \mathcal{P}_\alpha$, the object p_α lies in $\mathcal{P}$ (take $p_\beta = 0 \in \mathcal{P}_\beta$ for $\beta \neq \alpha$), hence $\pi_* \hom_{\mathcal{C}}(p_\alpha, f) \simeq 0$, so $f \in \mathcal{P}_\alpha$-null $= \mathcal{I}_\alpha$. Thus $\mathcal{I} \subseteq \bigcap_\alpha \mathcal{I}_\alpha$. Conversely, if $f \in \bigcap_\alpha \mathcal{I}_\alpha$ and $p \in \mathcal{P}$ is a retract of $\bigoplus_\alpha p_\alpha$ with $p_\alpha \in \mathcal{P}_\alpha$, then $\pi_* \hom_{\mathcal{C}}(p_\alpha, f) \simeq 0$ for all α, hence $\pi_* \hom_{\mathcal{C}}(\bigoplus_\alpha p_\alpha, f) \simeq 0$, and therefore $\pi_* \hom_{\mathcal{C}}(p, f) \simeq 0$. Hence $f \in \mathcal{P}$-null $= \mathcal{I}$, proving $\bigcap_\alpha \mathcal{I}_\alpha \subseteq \mathcal{I}$.
(2) To check orthogonality, notice that if $f \in \mathcal{I}$ and $p \in \mathcal{P}$, then by writing p as a retract of $\bigoplus_\alpha p_\alpha$ with $p_\alpha \in \mathcal{P}_\alpha$. and using $f \in \mathcal{I}_\alpha = \mathcal{P}_\alpha$-null, it follows that $\pi_* \hom_{\mathcal{C}}(p_\alpha, f) \simeq 0$ for all α, so that $\pi_* \hom_{\mathcal{C}}(p, f) \simeq 0$ for all $p \in \mathcal{P}$.
(3) To show that there are enough projectives, fix $x \in \mathcal{C}$ and choose exact sequences $p_\alpha \to x \to x_\alpha$ with $p_\alpha \in \mathcal{P}_\alpha$ and $x \to x_\alpha \in \mathcal{I}_\alpha$. Let $\bigoplus_\alpha p_\alpha \to x \to z$ be the cofiber triangle. Then each composite $p_\alpha \to x \to z$ is 0, so $x \to z \in \mathcal{P}_\alpha$-null $= \mathcal{I}_\alpha$ for all α, hence $x \to z \in \mathcal{I}$, and $\bigoplus_\alpha p_\alpha \in \mathcal{P}$.

The claim follows. □

6.3. **Towers.**

Construction 6.3.1. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category, and let $(\mathcal{P}, \mathcal{I})$ be the projective class generated by a small subcategory $S \subseteq \mathcal{C}$ as in Proposition 6.2.3. Let $P : \mathcal{C} \to \mathcal{C}$ be the associated comonad, and let $B_\bullet(P, -) : \Delta^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ be the associated augmented simplicial bar construction. Fix an object $x \in \mathcal{C}$.

(1) Define the cellular tower of x by setting $C_0(x) := 0$ and
$$C_n(x) := |\mathrm{sk}_{n-1} B_\bullet(P, x)|$$
for $n \geq 1$. Equivalently, with the convention $\mathrm{sk}_{-1} = 0$, one has $C_n(x) = |\mathrm{sk}_{n-1} B_\bullet(P, x)|$ for all $n \in \mathbb{N}$. The natural inclusions $\mathrm{sk}_{n-1} B_\bullet(P, x) \hookrightarrow \mathrm{sk}_n B_\bullet(P, x)$ induce maps
$$C_0(x) \to C_1(x) \to C_2(x) \to \cdots \to |B_\bullet(P, x)|.$$
Composing with the augmentation $|B_\bullet(P, x)| \to x$ yields compatible maps $C_n(x) \to x$
(2) Define the phantom tower of x by
$$P_n(x) := \mathrm{fib}(C_n(x) \to x)$$
for $n \in \mathbb{N}$. Thus there are induced maps
$$P_0(x) \to P_1(x) \to P_2(x) \to \cdots.$$
Finally, if $Q = \mathrm{cofib}(P \to \mathrm{id}_{\mathcal{C}})$, then $C_1(x) \simeq P(x)$, $P_0(x) \simeq \Omega x$ and $\Sigma P_1(x) \simeq Q(x)$.

Remark 6.3.2. One can construct phantom and cellular towers for an arbitrary projective class $(\mathcal{P}, \mathcal{I})$, not necessarily generated by a small full subcategory S, by iterating chosen $\mathcal{P}$-precovers; however, without small generation one should not expect the resulting towers to be functorial.

Lemma 6.3.3. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category, and let $(\mathcal{P}, \mathcal{I})$ be the projective class generated by a small subcategory $S \subseteq \mathcal{C}$. For every $n \in \mathbb{N}$ there exists an object $N_n(x) \in \mathcal{P}$ and an exact sequencq
$$C_n(x) \to C_{n+1}(x) \to \Sigma^n N_n(x)$$
in $\mathcal{C}$. In particular, the fibre and cofibre of $C_n(x) \to C_{n+1}(x)$ lie in $\mathcal{P}$ for every $n \in \mathbb{N}$.

Proof. Let $L_n B_\bullet(P, x) \to B_n(P, x)$ denote the n-th latching map, and let $N_n(x) := \mathrm{cofib}(L_n B_\bullet(P, x) \to B_n(P, x))$. Since $\mathcal{C}$ is presentable and stable, it is tensored over spaces and geometric realization is exact. The skeletal filtration of realization yields a pushout square
$$\begin{array}{ccc} L_n B_\bullet(P, x) \otimes \partial\Delta^n & \longrightarrow & C_n(x) \\ \downarrow & & \downarrow \\ B_n(P, x) \otimes \Delta^n & \longrightarrow & C_{n+1}(x). \end{array}$$

Passing to cofibres gives

$$\mathrm{cofib}(C_n(x) \to C_{n+1}(x)) \simeq \mathrm{cofib}(L_n B_\bullet(P,x) \to B_n(P,x)) \otimes (\Delta^n/\partial\Delta^n) \simeq \Sigma^n N_n(x).$$

It remains to prove that $N_n(x) \in \mathcal{P}$. Since $\mathcal{C}$ is stable, it is additive. For a simplicial object in an additive category, one has the usual decomposition into normalized and degenerate parts; in particular, the latching map is split mono and $B_n(P,x) \simeq L_n B_\bullet(P,x) \oplus N_n(x)$. Hence $N_n(x)$ is a retract of $B_n(P,x)$. On the other hand, $B_n(P,x) \simeq P^{n+1}(x)$ and by construction $P(y)$ is always a coproduct of objects of S. Therefore every iterate $P^{n+1}(x)$ belongs to $\mathcal{P}$. Since $\mathcal{P}$ is closed under retracts, it follows that $N_n(x) \in \mathcal{P}$. As $\mathcal{P}$ is also closed under shifts, this proves that $\mathrm{cofib}(C_n(x) \to C_{n+1}(x)) \in \mathcal{P}$. □

Corollary 6.3.4. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category, and let $(\mathcal{P},\mathcal{I})$ be the projective class generated by a small subcategory $S \subseteq \mathcal{C}$. For every $n \in \mathbb{N}$ one has $C_n(x) \in \mathcal{P}^{*n}$.

Proof. The proof goes by induction on n. For $n = 0$ one has $C_0(x) = 0 \in \mathcal{P}^{*0}$ by definition. Assume now $C_n(x) \in \mathcal{P}^{*n}$. By Lemma 6.3.3, the object $\mathrm{cofib}(C_n(x) \to C_{n+1}(x))$ belongs to $\mathcal{P}$. Hence $C_{n+1}(x)$ fits into an exact sequence $C_n(x) \to C_{n+1}(x) \to p$ with $p \in \mathcal{P}$. Therefore $C_{n+1}(x) \in \mathcal{P} * \mathcal{P}^{*n} = \mathcal{P}^{*(n+1)}$. □

The next result motivates the nomenclature "phantom tower".

Lemma 6.3.5. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P},\mathcal{I})$ be a projective class generated by a small subcategory. Let $x \in \mathcal{C}$. Then the maps $P_n(x) \to P_{n+1}(x)$ in the phantom tower are in $\mathcal{I}$ for every $n \in \mathbb{N}$.

Proof. By definition, $f \in \mathcal{I}$ if and only if for every $p \in \mathcal{P}$ the induced morphism $\pi_0 \hom_{\mathcal{C}}(p,f)$ is zero. With the notation of Proposition 6.2.3, it suffices to check this when $p = P(y)$ for $y \in \mathcal{C}$, since the claim then immediately extends to retracts.

Consider the augmented simplicial spectrum $X_\bullet \to X_{-1}$ defined by

$$X_m := \hom_{\mathcal{C}}(P(y), B_m(P,x)) \qquad \text{and} \qquad X_{-1} := \hom_{\mathcal{C}}(P(y), x).$$

Since $\hom_{\mathcal{C}}(P(y),-) : \mathcal{C} \to \mathrm{Sp}$ is exact, there are natural equivalences $\hom_{\mathcal{C}}(P(y), C_n(x)) \simeq |\mathrm{sk}_{n-1} X_\bullet|$ and hence $\hom_{\mathcal{C}}(P(y), P_n(x)) \simeq \mathrm{fib}(|\mathrm{sk}_{n-1}X_\bullet| \to X_{-1})$. Write $F_n := \mathrm{fib}(|\mathrm{sk}_{n-1}X_\bullet| \to X_{-1})$. Now the augmented simplicial spectrum $X_\bullet \to X_{-1}$ is split: the assignment induced by the comultiplication of the comonad defines an extra degeneracy $s_{-1} : X_m \to X_{m+1}$. Therefore, after applying π_0, one gets a split augmented simplicial abelian group. Its Moore complex is contractible, so the induced maps on the fibres of the skeletal truncations vanish, in that $\pi_0(F_n) \to \pi_0(F_{n+1}) = 0$ for all $n \in \mathbb{N}$. Equivalently, $\pi_0 \hom_{\mathcal{C}}(P(y), P_n(x)) \to \pi_0 \hom_{\mathcal{C}}(P(y), P_{n+1}(x))$ is zero for every y. Hence also $\pi_0 \hom_{\mathcal{C}}(p, P_n(x)) \to \pi_0 \hom_{\mathcal{C}}(p, P_{n+1}(x))$ is zero for every retract $p \in \mathcal{P}$. Therefore $P_n(x) \to P_{n+1}(x)$ belongs to $\mathcal{I}$. □

Lemma 6.3.6. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P},\mathcal{I})$ be a projective class generated by a small subcategory. Let $x \in \mathcal{C}$ be an object. Then there exists an exact sequence

$$\mathrm{colim}_{n\in\mathbb{N}} P_n(x) \to \mathrm{colim}_{n\in\mathbb{N}} C_n(x) \to x.$$

Proof. By construction there exists an exact sequence $P_n(x) \to C_n(x) \to x$ for every $n \in \mathbb{N}$. The maps $\mathrm{sk}_n \hookrightarrow \mathrm{sk}_{n+1}$ induce maps $C_n(x) \to C_{n+1}(x)$, and the morphisms $C_n(x) \to x$ are compatible with these, hence the fibre sequences above form a filtered diagram of fibre sequences. Since $\mathcal{C}$ is presentable and stable, filtered colimits are exact, hence they preserve fibres. Taking $\mathrm{colim}_{n\in\mathbb{N}}$ of the diagram of fibre sequences yields the required exact sequence. □

6.4. **A Brown representability result.** Projective classes can be used to prove Brown representability results.

Construction 6.4.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ be an additive category. Let $\mathrm{Ab}[\mathcal{C}]$ be the *abelianization of* $\mathcal{C}$, and recall that

$$\mathrm{Ab}[\mathcal{C}] \simeq \mathrm{PSh}(\mathcal{C}) \otimes \mathrm{Ab} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$$

identifies the abelianization with Ab-presheaves. Let $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ be the full subcategory of $\mathrm{Ab}[\mathcal{C}]$ spanned by the *additive presheaves*, that is, by those presheaves that preserves the zero object and biproducts. Then there exists a left Bousfield localization $L : \mathrm{Ab}[\mathcal{C}] \to \mathrm{Ab}^{\oplus}[\mathcal{C}]$ and that the additive Yoneda factors as $よ_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ab}^{\oplus}[\mathcal{C}]$ and is given by $x \mapsto \pi_0 \hom(-, x)$. When $\mathcal{C} \in \mathrm{Pr}^{\mathrm{add}}$ is furthermore presentable, we let $\mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$ denote the full subcategory spanned by the compact objects.

Warning 6.4.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ be an additive category. In general the additive Yoneda $よ_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathrm{Ab}^{\oplus}[\mathcal{C}]$ is not exact; it only preserves biproducts.

Remark 6.4.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and consider the additive Yoneda embedding $よ_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathrm{Ab}^{\oplus}[\mathcal{C}]$. If $x \to y \to z$ is an exact sequence in $\mathcal{C}$, then there exists a long exact sequence

$$\dots \to \pi_1 \hom_{\mathcal{C}}(-, z) \to \pi_0 \hom_{\mathcal{C}}(-, x) \to \pi_0 \hom_{\mathcal{C}}(-, y) \to \pi_0 \hom_{\mathcal{C}}(-, z) \to \pi_{-1} \hom_{\mathcal{C}}(-, x) \to \dots$$

of abelian groups. Since $\pi_n \simeq \pi_0 \Sigma^{-n}$, and since $\hom_{\mathcal{C}}(-, -)$ is exact, it follows that the long exact sequence depicted above is the evaluation of a long exact sequence

$$\dots \to よ_{\mathcal{C}}(\Sigma z) \to よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y) \to よ_{\mathcal{C}}(z) \to よ_{\mathcal{C}}(\Sigma^{-1} z) \to \dots$$

of additive presheaves.

Lemma 6.4.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then $\mathrm{Ab}^{\oplus}[\mathcal{C}] \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ab}}$ is a Grothendieck abelian 1-category.

Proof. Since $\mathrm{Ab}[\mathcal{C}] \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, it follows that $\mathrm{Ab}[\mathcal{C}]$ is a Grothendieck abelian 1-category. In particular, $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ is a left Bousfield localization of a Grothendieck abelian 1-category. □

Remark 6.4.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$ be an additive category. Let $x \in \mathcal{C}$. Then the additive Yoneda functor $よ_{\mathcal{C}}(x)$ is projective, since $\mathrm{Hom}_{\mathrm{Ab}^{\oplus}[\mathcal{C}]}(よ_{\mathcal{C}}(x), -) \simeq (-)(x)$ and evaluation is exact.

The next result identifies the compact objects of $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ with the classical finitely presented functors of Auslander.

Lemma 6.4.6. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Then an additive presheaf $H : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is compact if and only if it is finitely presented, in the sense that there exists an exact sequence $よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y) \to H \to 0$.

Proof. First of all, notice that representable additive presheaves are compact as a consequence of the Yoneda lemma (plus the fact that the colimits are computed pointwise). For the actual proof, assume ($\Leftarrow$) that H fits into an exact sequence $よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y) \to H \to 0$. Then for every additive presheaf F the set of natural transformations $H \to F$ can be computed as

$$\hom(H, F) \simeq \ker(\hom(よ_{\mathcal{C}}(y), F) \to \hom(よ_{\mathcal{C}}(x), F)).$$

Let $(F_i)_{i \in I}$ be a filtered diagram in $\mathrm{Ab}^{\oplus}[C]$. Using compactness of $よ_{\mathcal{C}}(x)$ and $よ_{\mathcal{C}}(y)$ and exactness of filtered colimits in Ab, it follows that

$$\begin{aligned} \hom(H, \mathrm{colim}_{i \in I} F_i) &\simeq \ker(\mathrm{colim}_i \hom(よ_{\mathcal{C}}(y), F_i) \to \mathrm{colim}_i \hom(よ_{\mathcal{C}}(x), F_i)) \\ &\simeq \mathrm{colim}_i(\ker(\hom(よ_{\mathcal{C}}(y), F_i) \to \hom(よ_{\mathcal{C}}(x), F_i))) \\ &\simeq \mathrm{colim}_i \hom(H, F_i), \end{aligned}$$

so that $\hom(H, -)$ preserves filtered colimits, proving the compactness of H.

Conversely ($\Rightarrow$), assume that H is compact. First construct a surjection from a (possibly large) coproduct of representables onto H as follows: for each object $c \in \mathcal{C}$ and each element $a \in H(c)$, let

$\varphi_{c,a} : よ_{\mathcal{C}}(c) \to H$ be the corresponding natural transformation under Yoneda. Taking the induced map $\bigoplus_{(c,a)} よ_{\mathcal{C}}(c) \to H$ gives an epimorphism (since it is pointwise surjective by construction). Write the domain as a filtered colimit of its finite subsums

$$\bigoplus_{(c,a)} よ_{\mathcal{C}}(c) \simeq \operatorname{colim}_J \bigoplus_{(c,a)\in J} よ_{\mathcal{C}}(c),$$

where J runs over the filtered poset of finite subsets of the indexing set. Let $I_J \subseteq H$ be the image of the composite $\bigoplus_{(c,a)\in J} よ_{\mathcal{C}}(c) \to H$. Then $H \simeq \operatorname{colim}_J I_J$. Since H is compact, the identity map $\mathrm{id}_H : H \to H \simeq \operatorname{colim}_J I_J$ factors through some stage I_{J_0}. The composite $H \to I_{J_0} \hookrightarrow H$ is then id_H, hence $I_{J_0} = H$. Therefore the above epimorphism restricts to an epimorphism $よ_{\mathcal{C}}(y) \to H$ for $y = \bigoplus_{(c,a)\in J_0} c$.

Let $K = \ker(よ_{\mathcal{C}}(y) \to H)$. Write K as the filtered colimit of its finitely generated subobjects K_α (equivalently, of subobjects generated by finitely many elements in finitely many values) and notice that, since filtered colimits are exact, one has

$$\operatorname{colim}_\alpha(よ_{\mathcal{C}}(y)/K_\alpha) \simeq よ_{\mathcal{C}}(y)/K \simeq H.$$

Again by compactness of H, the identity $\mathrm{id}_H : H \to \operatorname{colim}_\alpha H_C(y)/K_\alpha$ factors through some $H_C(y)/K_{\alpha_0}$, forcing the canonical epimorphism $よ_{\mathcal{C}}(y)/K_{\alpha_0} \to H$ to be an isomorphism. Hence $K_{\alpha_0} = K$, so K is finitely generated. Finally, since representables generate, there exists an epimorphism $よ_{\mathcal{C}}(x) \to K$ with x a finite direct sum of objects of C. Composing $よ_{\mathcal{C}}(x) \to K \hookrightarrow よ_{\mathcal{C}}(y)$ yields a morphism $よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y)$ whose cokernel is H, giving the required exact sequence. □

Corollary 6.4.7. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Then $\mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$ is an abelian 1-category. Furthermore, the additive Yoneda emebdding preserves small coproducts.

Proof. Since compact objects are always closed under cokernel, it suffices to prove the closure under kernels. First of all, it is closed under kernels of representables. Indeed, let $f : x \to Y$ be a map in $\mathcal{C}$ and let $k = \mathrm{fib}(f)$, so there is a fibre sequence $k \to x \to y$ in $\mathcal{C}$. By Remark 6.4.3 there exists an exact sequence in $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ of the form $よ_{\mathcal{C}}(\Sigma^{-1}y) \to よ_{\mathcal{C}}(k) \to よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y)$. Therefore $\ker(よ_{\mathcal{C}}(x) \to よ_{\mathcal{C}}(y)) \simeq \mathrm{coker}(よ_{\mathcal{C}}(\Sigma^{-1}y) \to よ_{\mathcal{C}}(k))$ is finitely presented, hence belongs to $\mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$.

Finally, for a general morphism $\alpha : H \to H'$ in $\mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$, choose presentations $よ_{\mathcal{C}}(x) \to v_{\mathcal{C}}(y) \to H \to 0$ and $よ_{\mathcal{C}}(x') \to よ_{\mathcal{C}}(y') \to H' \to 0$. Since $よ_{\mathcal{C}}(y)$ is projective, α lifts to a map $よ_{\mathcal{C}}(y) \to よ_{\mathcal{C}}(y')$, and using projectivity again one may represent α by a morphism between the two presentations. Taking kernels in the abelian category $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ expresses $\ker(\alpha)$ as a cokernel of a morphism between finite direct sums of kernels of maps between representables; by the previous paragraph all those kernels are finitely presented, hence so is $\ker(\alpha)$.

For the "furthermore" part, notice that, since every $F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$ is finitely presented, evaluating the exact sequence $よ_{\mathcal{C}}(a) \to よ_{\mathcal{C}}(b) \to F \to 0$ at a coproduct $\bigoplus_{i\in I} x_i$ and using that $\hom_{\mathcal{C}}(-,b)$ carries coproducts to products, shows that $F(\bigoplus_{i\in I} x_i) \simeq \prod_{i\in I} F(x_i)$. Therefore, for every $F \in \mathrm{Ab}^{\oplus}[C]^{\omega}$,

$$\mathrm{Hom}(よ_{\mathcal{C}}(\bigoplus_i x_i), F) \cong F(\bigoplus_i x_i) \cong \prod_i F(x_i) \cong \prod_i \mathrm{Hom}(よ_{\mathcal{C}}(x_i), F),$$

so $よ_{\mathcal{C}}(\bigoplus_i x_i)$ represents the coproduct of the $よ_{\mathcal{C}}(x_i)$ in $\mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$. □

Remark 6.4.8. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Then the inclusion $i : \mathcal{P} \hookrightarrow \mathcal{C}$ induces an adjunction $\mathrm{Lan}_i : \mathrm{Ab}^{\oplus}[\mathcal{P}] \rightleftarrows \mathrm{Ab}^{\oplus}[\mathcal{C}] : \mathrm{res}_i$ via the usual yoga with Kan extenions (plus the observation that both the left Kan extension and the restriction along i preserves additive presheaves). Since limits and colimits of additive presheaves are computed pointwise (that is, the inclusion $\mathrm{Ab}^{\oplus}[\mathcal{C}] \hookrightarrow \mathrm{Ab}[\mathcal{C}]$ creates them), it follows that the restriction $\mathrm{res}_i : \mathrm{Ab}^{\oplus}[\mathcal{C}] \to \mathrm{Ab}^{\oplus}[\mathcal{P}]$ preserves all limits and colimits (limits since it is a right adjoint, and colimits since the restriction at the level of abelian presheaves is also a left adjoint, by the usual yoga of Kan extensions).

The next result shows that this adjunction restricts to compact objects.

Lemma 6.4.9. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Then ther exists an adjunction $\mathrm{Lan}_i : \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega} \rightleftarrows \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} : \mathrm{res}_i$.

Proof. The left adjoint clearly restricts, since it preserves colimits and sends representable to representables. For the right adjoint, it suffices to show that it preserves cokernels (which follows from Remark 6.4.8) sends representables to compact additive presheaves. Let $x \in \mathcal{C}$ and consider the additive abelian presheaf $\text{よ}_{\mathcal{C}}(x) = \pi_0 \hom_{\mathcal{C}}(-, x)$ and restrict it to $\mathcal{P}$. Pick a $\mathcal{P}$-precover $p_0 \to x \to x_1$ and notice that $\mathrm{res}_i \text{よ}_{\mathcal{C}}(x) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(x_1)$ vanishes, since $x \to x_1$ is in $\mathcal{I}$. Thus the long exact sequence of Remark 6.4.3 reduces to

$$\cdots \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(\Sigma x_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(p) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(x) \to 0$$

an exact sequence. On the other side, let $p_1 \to \Sigma x_1 \to x_2$ be a $\mathcal{P}$-precover of Σx_1. As before, the map $\mathrm{res}_i \text{よ}_{\mathcal{C}}(\Sigma x_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(x_2)$ vanishes, so that making $\mathrm{res}_i \text{よ}_{\mathcal{C}}(p_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(\Sigma x_1)$ an epimorphism. In particular, since the image of $\mathrm{res}_i \text{よ}_{\mathcal{C}}(\Sigma x_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(p_0)$ equals the image of the composite $\mathrm{res}_i \text{よ}_{\mathcal{C}}(p_1) \mathrm{res}_i \text{よ}_{\mathcal{C}}(\Sigma x_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(p_0)$, it follows the existence of an exact sequence

$$\mathrm{res}_i \text{よ}_{\mathcal{C}}(p_1) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(p_0) \to \mathrm{res}_i \text{よ}_{\mathcal{C}}(x) \to 0$$

which exhibits $\mathrm{res}_i \text{よ}_{\mathcal{C}}(x)$ as finitely presented, hence compact by Lemma 6.4.6. □

Definition 6.4.10. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Let κ be a regular cardinal. We will say that:

(1) The projective class *generates* if $\mathcal{P}$ weakly generates.
(2) The projective class is *κ-perfect* if $\mathcal{I}$ is closed under κ-coproducts.

We will furthermore say that the projective class is *perfect* if κ-perfect for every regular cardinal κ.

Remark 6.4.11. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Then the projective class generates if and only if the only identity contained in $\mathcal{I}$ is the zero identity.

Remark 6.4.12. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Then the projective class generates if and only if $\mathrm{res}_i \circ \text{よ}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ab}[\mathcal{P}]^{\omega}$ reflects equivalences.

Lemma 6.4.13. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Then there exists a left split localization sequence

$$\ker(\mathrm{res}_i) \to \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} \xrightarrow{\mathrm{res}_i} \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega}$$

of abelian 1-categories.

Proof. It follows from the adjunction $\mathrm{Lan}_i : \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega} \rightleftarrows \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} : \mathrm{res}_i$, since the left adjoint is fully-faithful, being the left Kan extension of a fully-faithful functor. □

The following result is fundamental.

Proposition 6.4.14. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a projective class. Let κ be a regular cardinal. Then the projective class is κ-perfect if and only the restriction $\mathrm{res}_i : \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} \to \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega}$ preserves κ-coproducts.

Proof. By Remark 6.4.8 the functor $\mathrm{res}_i : \mathrm{Ab}^{\oplus}[\mathcal{C}] \to \mathrm{Ab}^{\oplus}[\mathcal{P}]$ preserves all colimits, and by Lemma 6.4.9 it restricts to $\mathrm{res}_i : \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} \to \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega}$. Together with Lemma 6.4.13 and the general criterion for left split localization sequences of abelian categories proved earlier, it is therefore enough to show that $\ker(\mathrm{res}_i) \subseteq \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$ is closed under κ-coproducts if and only if $\mathcal{I}$ is closed under κ-coproducts.

First of all, the kernel is $\ker(\mathrm{res}_i) = \{F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} \mid F \subseteq \mathrm{im}(\text{よ}(f)) \text{ for some } f \in \mathcal{I}\}$. Indeed, if $F \subseteq \mathrm{im}(\text{よ}(f))$ for some $f \in \mathcal{I}$, then $\mathrm{res}_i(F) \subseteq \mathrm{res}_i(\mathrm{im}(\text{よ}(f))) = \mathrm{im}(\mathrm{res}_i \text{よ}(f)) = 0$, since $f \in \mathcal{I}$

means precisely that $\operatorname{res}_i \text{よ}(f) = 0$. Conversely, let $F \in \ker(\operatorname{res}_i)$. Since F is finitely presented, by Lemma 6.4.6 there exists an exact sequence $\text{よ}(x) \xrightarrow{\text{よ}(u)} \text{よ}(y) \to F \to 0$. Extend $u : x \to y$ to an exact sequence $x \xrightarrow{u} y \xrightarrow{v} z$. Since $\operatorname{res}_i(F) = 0$, the morphism $\operatorname{res}_i \text{よ}(u) : \operatorname{res}_i \text{よ}(x) \to \operatorname{res}_i \text{よ}(y)$ is an epimorphism. Evaluating at every $p \in \mathcal{P}$, it follows that $\pi_0 \hom_{\mathcal{C}}(p, x) \to \pi_0 \hom_{\mathcal{C}}(p, y)$ is surjective. Hence $v \in \mathcal{I}$. On the other hand, Remark 6.4.3 applied to the exact sequence $x \to y \to z$ produces an exact sequence

$$\text{よ}(x) \xrightarrow{\text{よ}(u)} \text{よ}(y) \xrightarrow{\text{よ}(v)} \text{よ}(z),$$

and therefore $F \simeq \operatorname{coker}(\text{よ}(u)) \simeq \operatorname{im}(\text{よ}(v))$. This proves the identification of $\ker(\operatorname{res}_i)$.

Assume now that $\mathcal{I}$ is closed under κ-coproducts, and let $\{F_\alpha\}_{\alpha \in A}$ be a family of objects of $\ker(\operatorname{res}_i)$ with $|A| < \kappa$. By what was just shown, for every α there exists $f_\alpha \in \mathcal{I}$ such that $F_\alpha \subseteq \operatorname{im}(\text{よ}(f_\alpha))$. Since $\mathrm{Ab}^{\oplus}[\mathcal{C}]$ is a Grothendieck abelian category, coproducts are exact, and hence

$$\amalg_{\alpha \in A} F_\alpha \subseteq \amalg_{\alpha \in A} \operatorname{im}(\text{よ}(f_\alpha)) \simeq \operatorname{im}(\amalg_{\alpha \in A} \text{よ}(f_\alpha)).$$

By Corollary 6.4.7 the functor よ preserves coproducts, so

$$\amalg_{\alpha \in A} \text{よ}(f_\alpha) \simeq \text{よ}(\amalg_{\alpha \in A} f_\alpha).$$

Since $\mathcal{I}$ is closed under κ-coproducts, one has $\amalg_{\alpha \in A} f_\alpha \in \mathcal{I}$. It follows that $\operatorname{im}(\text{よ}(\amalg_{\alpha \in A} f_\alpha))$ belongs to $\ker(\operatorname{res}_i)$, and since $\ker(\operatorname{res}_i)$ is closed under subobjects, it follows that $\amalg_{\alpha \in A} F_\alpha \in \ker(\operatorname{res}_i)$. Thus $\ker(\operatorname{res}_i)$ is closed under κ-coproducts.

Conversely, assume that $\ker(\operatorname{res}_i)$ is closed under κ-coproducts, and let $\{f_\alpha\}_{\alpha \in A}$ be a family of morphisms in $\mathcal{I}$ with $|A| < \kappa$. Then each object $\operatorname{im}(\text{よ}(f_\alpha))$ belongs to $\ker(\operatorname{res}_i)$, and hence so does $\amalg_{\alpha \in A} \operatorname{im}(\text{よ}(f_\alpha))$. As above, exactness of coproducts and preservation of coproducts by よ imply that

$$\amalg_{\alpha \in A} \operatorname{im}(\text{よ}(f_\alpha)) \simeq \operatorname{im}(\text{よ}(\amalg_{\alpha \in A} f_\alpha)).$$

Applying res_i and using exactness, one obtains $\operatorname{im}(\operatorname{res}_i \text{よ}(\amalg_{\alpha \in A} f_\alpha)) = 0$, hence $\operatorname{res}_i \text{よ}(\amalg_{\alpha \in A} f_\alpha) = 0$. Equivalently, $\amalg_{\alpha \in A} f_\alpha \in \mathcal{I}$. Therefore $\mathcal{I}$ is closed under κ-coproducts. ☐

Lemma 6.4.15. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a generating ω_1-perfect projective class. Let $x_0 \to x_1 \to \dots$ be a diagram with morphisms in $\mathcal{I}$. Then $\operatorname{colim}_{n \in \mathbb{N}} x_n \simeq 0$.

Proof. Since Proposition 6.4.14 implies that the restriction functor $\operatorname{res}_i : \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega} \to \mathrm{Ab}^{\oplus}[\mathcal{P}]^{\omega}$ preserves ω_1-coproducts and since Corollary 6.4.7 implies that $\text{よ}_{\mathcal{C}}$ preserves all coproducts, it follows that $\operatorname{res}_i \circ \text{よ}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ab}^{\oplus}[\mathcal{C}]^{\omega}$ preserves ω_1-coproducts. The claim then follows by the Milnor exact sequence for colimits[3] plus the fact that $\operatorname{res}_i \circ \text{よ}_{\mathcal{C}}(x_n) \to \operatorname{res}_i \circ \text{よ}_{\mathcal{C}}(x_{n+1})$ vanishes, being $x_n \to x_{n+1}$ in $\mathcal{I}$. ☐

Proposition 6.4.16. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a generating ω_1-perfect projective class, and assume that $\mathcal{P}$ is generated by a small subcategory $S \subseteq \mathcal{C}$. Let $x \in \mathcal{C}$ be an object. Then $\operatorname{colim}_{n \in \mathbb{N}} C_n(x) \to x$ is an equivalence.

Proof. Consider the exact sequence $\operatorname{colim}_n P_n(x) \to \operatorname{colim}_n C_n(x) \to x$ of Lemma 6.3.6. It suffices to prove $\operatorname{colim}_n P_n(x) \simeq 0$. By Lemma 6.3.5, the tower $P_0(x) \to P_1(x) \to \cdots$ has all structure maps in $\mathcal{I}$. Since $(\mathcal{P}, \mathcal{I})$ is generating and ω_1-perfect, Lemma 6.4.15 applies and yields $\operatorname{colim}_n P_n(x) \simeq 0$. ☐

We can finally discuss Brown representability.

[3] Say something!

Remark 6.4.17. Let $\mathcal{C} \in \Pr^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Let $F \in \mathrm{Ab}^{\oplus}(\mathcal{C})$ and for $\mathcal{D} \subseteq \mathcal{C}$ a full subcategory define the *category of elements* via the cartesian unstraightening

$$\begin{array}{ccc} \mathrm{El}(F,\mathcal{D}) & \longrightarrow & \mathrm{Cat}_* \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{op}} & \xrightarrow[F|_{\mathcal{D}}]{} & \mathrm{Cat}. \end{array}$$

Recall that an element of $\mathrm{El}(F,\mathcal{D})$ is a pair $(d \in \mathcal{D}, x \in F(d))$ and a map $(d,x) \to (d',x')$ in $\mathrm{El}(F,\mathcal{D})$ is a map $f : d' \to d$ in $\mathcal{D}$ such that $F(f)(x') = x$.

Brown representability amounts to showing that, under certain assumptions, the category of elements $\mathrm{El}(F,\mathcal{D})$ admits a weak terminal object, that is, a pair $(c,x) \in \mathrm{El}(F,\mathcal{C})$ for which every element $(d,y) \in \mathrm{El}(F,\mathcal{D})$ admits a map $(d,y) \to (c,x)$. In other terms, the natural transformation $よ_{\mathcal{C}}(t) \to F$ which corresponds under the Yoneda isomorphism to $b \in F(t)$ is an epimorphism.

Lemma 6.4.18. Let $\mathcal{C} \in \Pr^{\mathrm{L}}_{\mathrm{st}}$ and let $(\mathcal{P},\mathcal{I})$ be a generating ω_1-perfect projective class. Let $F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]$ and assume that F is cohomological and sends coproducts into products. Assume also that $\mathrm{El}(F,\mathcal{P}^{*n})$ has a weak terminal object for every $n \in \mathbb{N}$. Then $\mathrm{El}(F,\mathcal{C})$ has a weak terminal object.

Proof. Let $(t_n \in \mathcal{P}^{*n}, b_n \in F(t_n))$ be the weak terminal object of $\mathrm{El}(F,\mathcal{P}^{*n})$ and let I be the set of towers $(0 = t_0 \xrightarrow{\tau_0} t_1 \xrightarrow{\tau_1} \dots)$ satisfying $F(\tau_i)(b_{i+1}) = b_i$. Then $I \neq \varnothing$. Indeed, since $t_n \in \mathcal{P}^{*n} \subseteq \mathcal{P}^{*(n+1)}$ and (t_{n+1}, b_{n+1}) is weak terminal in $\mathrm{El}(F,\mathcal{P}^{*(n+1)})$, the system of maps α_i exists. Given $i = (t_0^i \to t_1^i \to \dots) \in I$, let $t_i^{\infty} := \mathrm{colim}_n\, t_n^i$ be the colimit of the tower, and note that the exact sequence

$$0 \to \lim^1_{n\in\mathbb{N}} F(t_n^i) \to F(\mathrm{colim}\, t_n^i) \to \lim_{n\in\mathbb{N}} F(t_n^i) \to 0$$

shows that the sequence $(b_n)_{n\in\mathbb{N}}$ may be lifted to an element $b_i \in F(\mathrm{colim}\, t^i)$. The claim is that $(t,b) := (\amalg_{i\in I} t_i^{\infty}, (b_i)_{i\in I}) \in \mathrm{El}(F,\mathcal{C})$ is a weak terminal object.

Let $x \in \mathcal{C}$ and use Proposition 6.4.16 to produce an equivalence $\mathrm{colim}_{n\in\mathbb{N}} C_n(x) \to x$ with the colimit of the cellular tower. Consider then the diagram

$$\begin{array}{ccccccccc} 0 & \to & \lim^1 \pi_0 \hom_{\mathcal{C}}(\Sigma C_n(x), t) & \to & \pi_0 \hom_{\mathcal{C}}(x,t) & \to & \lim \pi_0 \hom_{\mathcal{C}}(C_n(x), t) & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\ 0 & \to & \lim^1 F(\Sigma C_n(x)) & \to & F(x) & \to & \lim F(C_n(x)) & \to & 0. \end{array}$$

and notice that if the two extreme vertical maps are surjective, then so is the middle one, by the snake lemma. Consider therefore α. Since $\Sigma C_n(x) \in \mathcal{P}^{*n}$ by Corollary 6.3.4 and since (t_n, b_n) is weak terminal in $\mathrm{El}(F,\mathcal{P}^{*n})$, there is a map $(\Sigma C_n(x), a_n) \to (t_n, b_n)$ in $\mathrm{El}(F,\mathcal{P}^{*n})$ for every $a_n \in F(\Sigma C_n(x))$. Since $I \neq \varnothing$, there exists a map

$$(\Sigma C_n(x), a_n) \to (t_n, b_n) \to (t_i, b_i) \to (t,b),$$

and this shows that $\hom_{\mathcal{C}}(\Sigma C_n(x), t) \to F(\Sigma C_n(x))$ is surjective. Since this property is preserved by $\lim^1$ (see Exercise E.6.3), the map α is surjective. Consider now β. Let $(a_n)_{n\in\mathbb{N}} \in \lim_{n\in\mathbb{N}} F(C_n(x))$, so that $a_n \in F(C_n(x))$ and $F(\alpha_n)(a_{n+1}) = a_n$ for $\alpha_n : C_n(x) \to C_{n+1}(x)$. The goal is to construct a diagram

$$\begin{array}{ccccc} C_0(x) & \to & C_1(x) & \to & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ t_0 & \to & t_1 & \to & \cdots \end{array}$$

such that the bottom row belongs to I and $F(f_n)(b_n) = a_n$ for every $n \in \mathbb{N}$. The construction is by induction. Set $f_0 = 0$ and pick f_1 from the fact that (t_1, b_1) is weak terminal in $\mathrm{El}(F,\mathcal{P})$. Assume the construction has been made for the first $n-1$ steps. Consider now

$$\begin{array}{ccc} C_{n-1}(x) & \longrightarrow & C_n(x) \\ {\scriptstyle f_{n-1}}\downarrow & & \downarrow \\ t_n & \longrightarrow & y_n \end{array}$$

Since pushouts are pullbacks, it follows that the fibre of $t_{n-1} \to y_n$ is the same of $C_{n-1}(x) \to C_n(x)$, hence it is in $\mathcal{P}$ by Lemma 6.3.3. In particular $y_n \in \mathcal{P}^{*(n-1)} * \mathcal{P} \simeq \mathcal{P}^{*n}$. Apply F to the exact sequence defining y_n to get

$$\cdots \to F(y_n) \to F(C_{n-1}(x)) \times F(t_{n-1}) \to F(C_n(x)) \to \cdots$$

Since $F(\alpha_{n-1})(a_n) - F(f_{n-1})(b_{n-1}) = a_{n-1} - a_{n-1} = 0$ in $F(C_{n-1}(x))$, there is an element $b'_n \in F(y_n)$ lifting (a_n, b_{n-1}). It follows that the bottom horizontal and right vertical maps in the above pushout are maps $(C_n(x), a_n) \to (y_n, b'_n)$ and $(t_{n-1}, b_{n-1}) \to (y_n, b'_n)$ in $\mathrm{El}(F, \mathcal{P}^{*n})$. Since (t_n, b_n) is weak terminal in $\mathrm{El}(F, \mathcal{P}^{*n})$, there is a map $(y_n, b'_n) \to (t_n, b_n)$ in $\mathrm{El}(F, \mathcal{P}^{*n})$ that does the required job, thus completing the construction process. The proof of surjectivity of β follows by considering the composite $F(t) \to F(t_i^\infty) \to \lim F(t_n) \to \lim F(C_n(x))$, since it sends $b \in F(t)$ to $(a_n)_{n\in\mathbb{N}}$. □

Theorem 6.4.19. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ be a generating ω_1-perfect projective class generated by a small category. Assume also that every category of elements $\mathcal{P}^{*n}/F$ has a weak terminal object for every $n \in \mathbb{N}$ and every cohomological functor $F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]$ which sends coproducts into products. Then $\mathcal{C}$ satisfies Brown representability.

Proof. By [Nee09, Theorem 1.3] $\mathcal{C}$ satisfies Brown representability if and only if every cohomological functor $F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]$ which sends coproducts into products has a category of elements with a weak terminal object. Thus the claim follows from Lemma 6.4.18. □

Corollary 6.4.20. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category which is ω_1-compactly generated. Then satisfies the Brown representability theorem.

Proof. It follows from the previous theorem together with Exercise E.6.4. □

7. Strong generation and descent

In this chapter we discuss how strong generation behaves with respect to descent. We begin by recalling some basic facts on totalizations, pro-objects, and the general theory of descent. We then introduce descendable algebra objects and explain how descendability may be interpreted as a finite form of descent. After discussing Aoki's result on strong generation of quasi-excellent schemes, we use this formalism to prove descent statements for strong generation in geometric situations.

7.1. **Pro-objects.** We recall the construction of the pro-completion, the dual of the ind-completion.

Remark 7.1.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{lex}}$ be a category with finite limits. Then the *pro-completion of* $\mathcal{C}$ is the datum of a category $\mathrm{Pro}(\mathcal{C}) \in \mathrm{Cat}^{\mathrm{lex}}$ with all limits together with a functor $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ such that:

(1) The functor $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is left exact.
(2) Given a category $\mathcal{D}$ with all limits, restriction induces an equivalence
$$j^* : \mathrm{Fun}^{\mathrm{R}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$
of categories.

Explicitly, the pro-completion may be constructed as $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and the required functor is given by the Yoneda. In particular, every object in $\mathrm{Pro}(\mathcal{C})$ may be written as a "formal" filtered inverse limit of objects in $\mathcal{C}$: that is, $\mathcal{C}$ generates $\mathrm{Pro}(\mathcal{C})$ under cofiltered limits. Moreover, $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ identifies a full sucbategory, and, if $\mathcal{C}$ is idempotent complete, then it exhibits the cocompact objects.

Definition 7.1.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{lex}}$ be a category with finite limits. An object in $\mathrm{Pro}(\mathcal{C})$ is called *constant* if it is equivalent to an object in the image of $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$.

Split cosimplicial objects furnish a great numver of constant pro-objects.

Example 7.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{lex}}$ be a category with finite limits and let $X^\bullet : \Delta \to \mathcal{C}$ be a cosimplicial object. If $X^\bullet$ extends to a split augmented cosimplicial object $X^\bullet_+ : \Delta_+ \to \mathcal{C}$ then the pro-object $\{\mathrm{Tot}_{\leq n} X^\bullet\}$ is constant.

Remark 7.1.4. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring. Let $F : I \to \mathcal{C}$ be a cofiltered diagram for which the induced pro-object is constant. Then for any $x \in \mathcal{C}$, the natural map $(\lim_I F) \otimes x \to \lim_I (f \otimes x)$ is an equivalence.

We no show that in a finite diagram of categories, a pro-object is constant if and only if it is constant at each stage.

Remark 7.1.5. Let K be a finite category and let $p : K \to \mathrm{Cat}^{\mathrm{lex}}$ be a functor. Then there is a a natural functor $\mathrm{Pro}(\lim_K p(k)) \to \lim_K(\mathrm{Pro}(p(k)))$ which respects all limits. The claim is that this functor is is fully faithful.

Indeed, the functor $p(k) \to \mathrm{Pro}(p(k))$ are fully faithful for every $k \in K$ so that $\lim_{k\in K} p(k) \to \lim_{k\in K} \mathrm{Pro}(p(k))$ is fully-faithful and preserves finite limits. In order for the right Kan extension $\mathrm{Pro}(\lim_{k\in K} p(k)) \to \lim_{k\in K}(\mathrm{Pro}(p(k)))$ to be fully faithful, it follows by [Lur09, Section 5.3] that it suffices for the embedding $\lim_{k\in K} p(k) \to \lim_{k\in K} \mathrm{Pro}(p(k))$ to land in the cocompact objects. However, over a finite diagram of categories, an object is cocompact if and only if it is cocompact pointwise, because finite limits commute with filtered colimits in spaces.

Corollary 7.1.6. Let K be a finite category and let $p : K \to \mathrm{Cat}^{\mathrm{lex}}$ be a functor. Then a pro-object in $\lim_{k\in K} p(k)$ is constant if and only if its evaluation in $\mathrm{Pro}(p(k))$ is constant for each vertex $k \in K$.

Proof. Consider

$$\begin{array}{ccc} \lim_{k\in K} p(k) & \xrightarrow{\simeq} & \lim_{k\in K} p(k) \\ \downarrow & & \downarrow \\ \mathrm{Pro}(\lim_{k\in K} p(k)) & \longrightarrow & \lim_{k\in K} \mathrm{Pro}(p(k)) \end{array}$$

where the bottom horizontal map is fully-faithful. It follows that an object of $\mathrm{Pro}(\lim_{k\in K} p(k))$ is pro-constant if and only if its image in $\lim_{k\in K} \mathrm{Pro}(p(k))$ belongs to $\lim_{k\in K} p(k)$. Since each $p(k) \to \mathrm{Pro}(p(k))$ is fully-faithful, the claim follows. □

7.2. **Descendable algebra objects.** We now study the theory of descendable algebras, a finite truncation of the more general theory of descent due to Mathew [Mat16, Definition 3.18]. It already appeared in Balmer [Bal16] under the name *nil-faithfulness*.

Definition 7.2.1. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring, and let $A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C})$. We will say that $A \to B$ is *descendable* if $\mathrm{thick}_\otimes(B) = \mathrm{Mod}_A(\mathcal{C})$.

Recall that $\mathrm{thick}_\otimes(B) \subseteq \mathrm{Mod}_A(\mathcal{C})$ denotes the smallest thick subcategory closed under $\otimes$ containing B, regarded as an A-module.

The main statement in descent theory is the Barr-Beck-Lurie theorem. In order to state it, let us recall the construction of totalizations.

Remark 7.2.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{lex}}$ be a category with finite limits and let $X^\bullet : \Delta \to \mathcal{C}$ be a cosimplicial object. Let $\mathrm{Tot}_{\leq n} X^\bullet$ be the limit $\lim_{\Delta_{\leq n}} X^\bullet$. Then there exists a diagram

$$\cdots \to \mathrm{Tot}_{\leq n} X^\bullet \to \mathrm{Tot}_{\leq n-1} X^\bullet \cdots \to \mathrm{Tot}_{\leq 1} X^\bullet \to \mathrm{Tot}_{\leq 0} X^\bullet$$

of *partial n-totalizations* whose limit (if it exists) is the *totalization* $\mathrm{Tot}(X^\bullet)$.

Theorem 7.2.3 (The Barr-Beck-Lurie theorem)**.** Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Then the adjunction is comonadic if and only if:

(1) The left adjoint F is conservative.
(2) Given a cosimplicial object $X^\bullet : \Delta \to \mathcal{C}$ such that $F(X^\bullet)$ admits a splitting, then $\mathrm{Tot}(X^\bullet)$ exists in $\mathcal{C}$ and the map $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot} F(X^\bullet)$ is an equivalence.

Our next goal is to show that descendable morphisms of commutative algebras do admit descent.

Lemma 7.2.4. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \in \mathrm{CAlg}(\mathcal{C})$ be descendable. Then $- \otimes A : \mathcal{C} \to \mathcal{C}$ is faithful.

Proof. Let $M \in \mathcal{C}$ be such that $M \otimes A \simeq 0$. To show that $M \simeq 0$, consider $\mathcal{C}_M = \{x \in \mathcal{C} \mid M \otimes x \simeq 0\}$ Then $\mathcal{C}_M$ is a thick $\otimes$-ideal and $A \in \mathcal{C}_M$, so that $\mathcal{C}_M = \mathcal{C}$. Thus $\mathbb{1} \in \mathcal{C}_M$ and $M \simeq M \otimes \mathbb{1} \simeq 0$. □

Remark 7.2.5. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \in \mathrm{CAlg}(\mathcal{C})$. Then the multiplication maps of A construct a cosimplicial object $A^\bullet : \Delta \to \mathrm{CAlg}(\mathcal{C})$, depicted as $A \rightrightarrows A \otimes A \overset{\to}{\underset{\to}{\to}} A \otimes A \otimes A \cdots$, called the *cobar resolution of* A. The unit map furnishes an augmentation, thus providing an augmented cosimplicial object $A^\bullet_+ : \Delta_+ \to \mathrm{CAlg}(\mathcal{C})$.

Proposition 7.2.6. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and $A \in \mathrm{CAlg}(\mathcal{C})$. Then A is descendable if and only if $A^\bullet_+ : \Delta_+ \to \mathrm{CAlg}(\mathcal{C})$ is a limit diagram and $A^\bullet$ defines a constant pro-object.

In other words, A admits descent if and only if $\mathbb{1}$ is a retract of a finite colimit of a diagram consisting of A-modules.

Proof. Assume first A admits descent. Let $\mathcal{C}_{\mathrm{good}}$ be the full subcategory of $\mathcal{C}$ spanned by $M \in \mathcal{C}$ such that $A^\bullet_+ \otimes M$ is a limit diagram and such that the induced Tot tower converging to M defines a constant pro-object. The claim will be proved if $\mathbb{1} \in \mathcal{C}_{\mathrm{good}}$. Notice that $A \in \mathcal{C}_{\mathrm{good}}$ since $A^\bullet_+ \otimes A$ is split (via the multiplication) and so it is a limit diagram; by Example 7.1.3 the associated pro-object is constant. Since $\mathcal{C}_{\mathrm{good}}$ is clearly a thick $\otimes$-ideal (here it is used that the limit commutes with $\otimes$, being the Tot tower a constant pro-object), the claim follows. Conversely, it follows that $\mathbb{1}$ is a retract of $\mathrm{Tot}_{\le n} A^\bullet$ for some $n \in \mathbb{N}$, and since $\mathrm{Tot}_{\le n} A^\bullet \in \mathrm{thick}_\otimes(A)$, it follows that $\mathbb{1} \in \mathrm{thick}_\otimes(A)$, hence $\mathcal{C} = \mathrm{thick}_\otimes(A)$, making A descendable. □

Now to get the version for a morphism of algebras $A \to B$, it suffices to notice that we have worked with slices all along.

Remark 7.2.7. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and $A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C})$. Then Proposition 7.2.6 implies that $A \to B$ is descendable if and only if $\{A\} \to \{\mathrm{Tot}_{\le n} B^\bullet\}$ is a pro-isomorphism of A-modules.

We note the connection with the theory of descent.

Proposition 7.2.8. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \in \mathrm{CAlg}(\mathcal{C})$ be descendable. Then the adjunction $- \otimes A : \mathcal{C} \rightleftarrows \mathrm{Mod}_A(\mathcal{C}) : \mathrm{res}_A$ is comonadic.

Proof. It suffices to check the assumptions of the Barr-Beck-Lurie theorem Theorem 7.2.3. Now Lemma 7.2.4 implies that $- \otimes A$ is conservative, so fix a cosimplicial object $X^\bullet : \Delta \to \mathcal{C}$ such that $X^\bullet \otimes A$ is split. To show that the map $A \otimes \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(A \otimes X^\bullet)$ is an equivalence, notice that the collection of $M \in \mathcal{C}$ such that $M \otimes X^\bullet$ defines a constant pro-object contains A (by Example 7.1.3) and is a thick $\otimes$-ideal, so that the above map is an equivalence (being $\mathrm{Tot}(X^\bullet)$ a finite limit, hence a finite colimit). □

Lemma 7.2.9. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring, and let $A \to B \to C$ be in $\mathrm{CAlg}(\mathcal{C})$.

(1) If $A \to B$ and $B \to C$ are descendable, then so is $A \to C$.
(2) If $A \to C$ is descendable, then so is $A \to B$.

Proof. For (1), since $A \to B$ is descendable, there exists a finite diagram $D : I \to \mathrm{Mod}_B(\mathcal{C})$ such that $\mathbb{1}$ is a retract of $\lim_I D$ in $\mathcal{C}$. Since $B \to C$ is descendable, there exists a finite diagram $E : J \to \mathrm{Mod}_C(\mathcal{C})$ such that B is a retract of $\lim_J E$ in $\mathrm{Mod}_B(\mathcal{C})$. Now fix $i \in I$. Since D_i is a B-module, tensoring the latter retract diagram with D_i over B yields $D_i \simeq D_i \otimes_B B$ as a retract of $D_i \otimes_B \lim_J E \simeq \lim_J (D_i \otimes_B E_j)$. Each object $D_i \otimes_B E_j$ carries a canonical C-module structure. Hence every D_i is a retract of a finite limit of C-modules. It follows that $\lim_I D$, and therefore also its retract $\mathbb{1}$, is a retract of a finite colimit of C-modules. By Proposition 7.2.6, this shows that $A \to C$ is descendable.

For (2), assume that $A \to C$ is descendable. Then by Proposition 7.2.6, $\mathbb{1}$ is a retract of a finite colimit of a diagram consisting of C-modules. Restricting scalars along $B \to C$, the same diagram may be viewed as a diagram consisting of B-modules. Therefore $\mathbb{1}$ is also a retract of a finite colimit of B-modules, and Proposition 7.2.6 implies that $A \to B$ is descendable. □

Descendability may be interpreted in terms of projective classes.

Lemma 7.2.10. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C})$. Consider the adjunction $- \otimes B : \mathrm{Mod}_A(\mathcal{C}) \rightleftarrows \mathrm{Mod}_B(\mathcal{C}) : \mathrm{res}$ and define:

(1) The ideal $\mathcal{I}_B$ of $\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$ to be given by those maps $f : M \to N$ of A-modules such that $f \otimes_A B$ is null-homotopic, equivalently, $\pi_0(f \otimes_A B)$ is zero. Maps in $\mathcal{I}_B$ are called *B-zero maps.*

(2) The full subcategory $\mathcal{P}_B$ of $\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$ to be given by the retract closure of $\mathrm{res}(P)$ for $P \in \mathrm{Mod}_B(\mathcal{C})$.

Then the pair $(\mathcal{P}_B, \mathcal{I}_B)$ defines a projective class on $\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$.

Proof. The proof seems much more complicated than what actually is since there is an opposite category everywhere (and in fact it would be much easier if the notion of *injective classes* was available). In any case, consider first the orthogonality problems:

(1) First of all, $\mathcal{P}_B$-null $= \mathcal{I}_B$. Indeed if $(f : M \to N) \in \mathcal{P}$-null then $\pi_* \hom_{\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}}(P, f) \simeq 0$ for every $P \in \mathcal{P}_B$. In particular, for every $\mathrm{res}(Q)$ with $Q \in \mathrm{Mod}_B(\mathcal{C})$. By exchaing the opposite and by adjunction, it follows that $\pi_* \hom(f \otimes_A B, Q) \simeq 0$. Choose now $Q = M \otimes_A B$ so that $\pi_0(f \otimes_A B)(\mathrm{id}_{M \otimes_A B}) \cong 0$, which implies $\pi_0(f \otimes_A B)$ and thus $\mathcal{P}_B$-null $\subseteq \mathcal{I}_B$. Conversely, let $f : M \to N$ be in $\mathcal{I}_B$ so that $f \otimes_A B \simeq 0$. Then for every $P \in \mathrm{Mod}_B(\mathcal{C})$ it is
$$\pi_* \hom_{\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}}(\mathrm{res}(P), f) = \pi_* \hom_{\mathrm{Mod}_A(\mathcal{C})}(f, \mathrm{res}(P)) \simeq \hom_{\mathrm{Mod}_B(\mathcal{C})}(f \otimes_A B, P) \simeq 0$$
and since this equivalence propagates to retracts, it follows that $\mathcal{I}_B \subseteq \mathcal{P}_B$-null.

(2) Secondly, $\mathcal{P}_B = \mathcal{I}_B$-proj. Indeed, if $M \in \mathcal{P}_B$ is of the form $\mathrm{res}(P)$ for $P \in \mathrm{Mod}_B(\mathcal{C})$, then the usual computation with adjunction shows that $M \in \mathcal{I}_B$-proj, and since this claim is stable under retracts it follows that $\mathcal{P}_B \subseteq \mathcal{I}_B$-proj. The other inclusion is analogous.

Finally, for existence of enough projectives, fix $M \in \mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$ and consider the exact sequence $\mathrm{fib}(\eta_M) \to M \to \mathrm{res}(M \otimes_A B)$ in $\mathrm{Mod}_A(\mathcal{C})$, that is, the exact sequence $\mathrm{res}(M \otimes_A B) \to M \to \mathrm{fib}(\eta_M)$ in $\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$. Since $\mathrm{res}(M \otimes_A B) \in \mathcal{P}_B$ and since the map $M \to \mathrm{fib}(\eta_M)$ is in $\mathcal{I}_B$ (indeed, it suffices to tensor with B to get that $\mathrm{fib}(\eta_M) \otimes_A B \simeq 0$), the claim follows. □

The following observation already appears in [Mat16, Proposition 3.27] and in [BS17, Lemma 11.20]. We observe the connection with projective classes.

Proposition 7.2.11. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C})$. Then $A \to B$ is descendable if and only if there exists $n \in \mathbb{N}$ such that $(\mathcal{P}_B, \mathcal{I}_B)^{*n}$ is the trivial projective class $(\mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}, 0)$.

In this case, we will say that $A \to B$ is *descendable of index* $\le n$.

Proof. Let F be the fibre of $A \to B$ and consider the Adams tower $F^{\otimes n} \to \cdots \to F \to A$. First of all notice that $\mathcal{I}_B^{*n} = 0$ if and only if $F^{\otimes n} \to A$ is null-homotopic. Consider indeed the exact sequence $F \to A \to B$ and tensor it with $M \in \mathrm{Mod}_A(\mathcal{C})$ to get $M \otimes_A F \to M \to M \otimes_A B$. Then the map

$M \otimes_A F \to M$ is the universal “injective map”, in the sense that a map $N \to M$ is null-homotopic if and only if it factors trough[4] $M \otimes_A F \to M$.

Therefore the claim will be proved once $A \to B$ is descendable if and only if $F^{\otimes n} \to A$ is null-homotopic. Assume first that $A \to B$ is descendable. Given $M \in \mathrm{Mod}_A(\mathcal{C})^{\mathrm{op}}$ let $\mathcal{I}_M$ denote the ideal of maps f such that $f \otimes_A M$ is null-homotopic. Then a long but trivial computation shows that $\mathcal{I}_M \subseteq \mathcal{I}_{M\otimes N}$, that if N is a retract of M then $\mathcal{I}_M \subseteq \mathcal{I}_N$ and if $M_0 \to M \to M_1$ is exact then $\mathcal{I}_{M_0} \circ \mathcal{I}_{M_1} \subseteq \mathcal{I}_M$. In particular, since $A \in \mathrm{thick}_\otimes(B)$ it follows that $\mathcal{I}_B^n \subseteq \mathcal{I}_A$ for some $n \in \mathbb{N}$. Since $F^{\otimes n} \to A$, being a composition of elements of $\mathcal{I}_B$ belongs to $\mathcal{I}_A$, and hence is null-homotopic.

Conversely, assume that $F^{\otimes_A n} \to A$ is null-homotopic for some $n \geq 1$. Fix $M \in \mathrm{Mod}_A(\mathcal{C})$ and tensor the Adams tower with M to get

$$M \otimes_A F^{\otimes_A n} \to \cdots \to M \otimes_A F \to M.$$

For each $i \geq 1$, tensoring the fibre sequence $F \to A \to B$ with $M \otimes_A F^{\otimes_A (i-1)}$ gives a cofiber sequence

$$M \otimes_A F^{\otimes_A i} \to M \otimes_A F^{\otimes_A (i-1)} \to M \otimes_A B \otimes_A F^{\otimes_A (i-1)}.$$

The third term is naturally a B-module, hence belongs to the thick tensor ideal generated by B. By induction on i, it follows that the cofiber of $M \otimes_A F^{\otimes_A i} \to M$ belongs to the thick tensor ideal generated by B for every $i \geq 1$. Now for $i = n$, the map $M \otimes_A F^{\otimes_A n} \to M$ is precisely $\mathrm{id}_M \otimes \alpha_n$, hence is null-homotopic by assumption. Therefore its cofiber is $M \oplus \Sigma(M \otimes_A F^{\otimes_A n})$ and it belongs to the thick tensor ideal generated by B. As this thick tensor ideal is closed under retracts, it follows that M itself belongs to the thick tensor ideal generated by B. Since M was arbitrary, the thick tensor ideal generated by B is all of $\mathrm{Mod}_A(\mathcal{C})$. Equivalently, $A \to B$ is descendable. □

Corollary 7.2.12. Lax monoidal left adjoints preserve the property of being descendable of index $\leq n$.

Proof. Let $f : \mathcal{C} \to \mathcal{D}$ be a lax-monoidal left adjoint between objects in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$, and assume that $A \to B$ in $\mathrm{CAlg}(\mathcal{C})$ is descendable of index $\leq n$; so, if F is the fibre, then $F^{\otimes_A n} \to A$ is null-homotopic in $\mathrm{Mod}_A(\mathcal{C})$. As f is exact, $f(F)$ is the fibre of $f(A) \to f(B)$. As f is lax-monoidal and colimit preserving, one checks, using the bar-resolution construction of the tensor product of A-modules, that the induced functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_{f(A)}(\mathcal{D})$ is also lax-monoidal. This gives, for each i   N, natural maps $f(F)^{\otimes_{f(A)} i} \to f(F^{\otimes_A i})$ whose composition with the canonical map $f(F^{\otimes_A i}) \to f(A)$ is the canonical map $f(F)^{\otimes_{f(A)} i} \to f(A)$. Specializing to $i = n$ then shows that the canonical map $f(F)^{\otimes_{f(A)} n} \to f(A)$ is null-homotopic, proving the claim. □

We conclude this section with some examples.

Example 7.2.13. Let $A \to B$ be a faithfully flat map in $\mathrm{CAlg}(\mathrm{Sp})$ such that $\pi_*(A)$ is countable. Then $A \to B$ is descendable. Indeed, by Proposition 7.2.11 it suffices to show that the composition of n B-zero maps is null-homotopic. The claim is that $n = 2$ works. Let $M \to M' \to M''$ be a sequence of B-zero maps and consider a perfct A-module P. If $P \to M \to M'$ is a composite, then $P \to M'$ is a B-zero and to show that it is actually null-hmotopic, consider the unit $M' \to \mathrm{res}_A(M' \otimes_A B)$, which is B-zero, tensor it with the dual $P^\vee$ and take homotopy groups to get

$$\pi_*(P^\vee \otimes_A M') \to \pi_*(P^\vee \otimes_A \mathrm{res}_A(M' \otimes_A B)).$$

This map is injective by faithfully-flatness (see Exercise E.7.1). Since the class of $P \to M'$ belongs to $\pi_*(P^\vee \otimes_A M')$ and since it is sent to zero in $\pi_*(P^\vee \otimes_A \mathrm{res}_A(M' \otimes_A B))$, the injectivity shows that $P \to M'$ is null-homotopic and so that $M \to M'$ is phantom. But now $\pi_*(A)$ is countable, so that [HPS97, Theorem 4.1.5] implies that Mod_A satisfies Brown representability and then [HPS97, Theorem 4.1.8] implies that the composite of two phantom maps in Mod_A is zero, thus showing the claim.

[4]Excercise!

Example 7.2.14. Let $A \to B$ be a faithfully flat map in CAlg(Sp). On can improve the previous result by asking $\pi_*(A)$ to have cardinality $\aleph_k$ for some $k \in \mathbb{N}$, see [Mat16, Proposition 3.32]. Even more [Mat16, Corollary 3.33] shows that every faithfully flat map $A \to B$ in CAlg(Sp) such that $\pi_0(B)$ has a presentation $\pi_0(A)$-algebra with at most $\aleph_k$ generators and relations for some $k \in \mathbb{N}$ admits descent. This covers finitely presented faithfully flat map of discrete rings. However, it is known [Aok24] and [Zel24] that there are faithfully flat extensions of classical rings which are not descendable.

Example 7.2.15. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ be connective and such that $\pi_i A = 0$ for $i >> 0$. Then $A \to \pi_0(A)$ is descendable. In this case is actually possible to check the definition, that is, that $A \in \mathrm{thick}_\otimes(\pi_0(A))$. Indeed, given an A-module M with $\pi_*(M)$ is concentrated in one degree, it canonically admits the structure of a $\pi_0 A$-module and thus it belongs to the $\mathrm{thick}_\otimes(\pi_0(A))$. However, the (finite!) Postnikov decomposition of A in Mod_A has successive cofibres with π_* concentrated in one degree, so that an argument by induction shows that A belongs to the thick $\otimes$-ideal generated by $\pi_0 A$.

Example 7.2.16. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^\heartsuit$ be a classical ring and let $I \subseteq R$ be a nilpotent ideal. Then the quotient map $R \to R/I$ of classical rings is descendable.

Again this can be checked via the definition. Indeed let $\mathrm{thick}_\otimes(R/I) \subseteq \mathrm{Mod}_R$ be the thick $\otimes$-ideal generated by R/I and notice that $\mathrm{Mod}_{R/I} \subseteq \mathrm{thick}_\otimes(R/I)$. If n is the index of nilpotence, there exists a filtration $0 = I^n \subseteq \cdots \subseteq I \subseteq R$ whose successive quotients are R/I-modules. In particular, I^j/I^{j+1} belongs to $\mathrm{thick}_\otimes(R/I)$ and an argument by induction on the exact sequences $I^{j+1} \to I^j \to I^j/I^{j+1}$ allows to conclude that $R \in \mathrm{thick}_\otimes(R/I)$, thus showing that $R \to R/I$ is descendable.

We also have the following examples coming from algebraic geometry.

Lemma 7.2.17. Let K be a finite category and let $p : K \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a diagram. Let $A \in \mathrm{CAlg}(\lim_K p)$. Then A is descendable if and only if the projection $A \in \mathrm{CAlg}(p(k))$ is descendable for every $k \in K$.

Proof. One implication is obvious as being descendable is preserved by lax monoidal left adjoints by Corollary 7.2.12. Conversely, consider the cobar construction $A^\bullet : \Delta \to \mathrm{CAlg}(\lim_K p)$. Since it becomes pro-constant after evaluation at each vertex $k \in K$, it is constant by Corollary 7.1.6 and since the limit is exactly $\mathbb{1}$ at each vertex, it follows that $\mathbb{1} \to A$ must be a limit diagram. □

Lemma 7.2.18. Let $f : Y \to X$ be a map of quasi-compact quasi-separated schemes.

(1) Let $\{j_i : U_i \hookrightarrow X\}_{i \in I}$ be a finite cover by quasi-compact opens. Then $\mathcal{O}_X \to \prod_{i \in I}(j_i)_*\mathcal{O}_{U_i}$ is descendable.
(2) If f is fppf, then $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is descendable.

Proof. Consider first point (1). Assume first that X is affine, and let $u : \amalg_{i \in I} U_i \to X$ be the induced morphism. Since the cover is finite by quasi-compact opens, each U_i is affine (since X is affine), hence $\amalg_i U_i$ is affine. Moreover, u is fppf: each j_i is an open immersion, hence flat and locally of finite presentation, and the family is jointly surjective. Therefore the corresponding map of discrete rings is faithfully flat and finitely presented. Thus example: finitely presented faithfully flat map of discrete rings is descendable implies that $\mathcal{O}_X \to u_*\mathcal{O}_{\amalg_i U_i}$ is descendable. Since pushforward along a finite disjoint union is the product, one has $u_*\mathcal{O}_{\amalg_i U_i} \simeq \prod_{i \in I}(j_i)_*\mathcal{O}_{U_i}$ which proves point (1) when X is affine. One now passes to a general quasi-compact quasi-separated scheme X. Choose a finite affine open cover $\{X_\alpha\}$ of X, and let K be the corresponding finite Čech diagram consisting of all finite intersections $X_{\alpha_0 \cdots \alpha_n}$. By Zariski descent,

$$\mathrm{QCoh}(X) \simeq \lim_{\sigma \in K} \mathrm{QCoh}(X_\sigma).$$

Under this identification, the algebra object

$$\mathcal{O}_X \to \prod_{i \in I}(j_i)_*\mathcal{O}_{U_i}$$

is descendable if and only if its restriction to each vertex X_σ is descendable by Lemma 7.2.17. Since each X_σ is affine, the restriction of the above map to X_σ is descendable by the affine case already proved. This proves point (1).

The proof of point (2) is similar. Assume first that X is affine. Choose a finite affine open cover $\{V_j\}_{j\in J}$ of Y, and let $g : \amalg_{j\in J} V_j \to Y$ be the induced morphism. By point (1), the map $\mathcal{O}_Y \to g_*\mathcal{O}_{\amalg_j V_j}$ is descendable in $\mathrm{QCoh}(Y)$. Consider now the composite $fg : \amalg_{j\in J} V_j \to X$. Since both f and g are fppf, so is fg. As both source and target are affine, the corresponding ring map is faithfully flat and finitely presented, hence

$$\mathcal{O}_X \to (fg)_*\mathcal{O}_{\amalg_j V_j} \simeq f_*g_*\mathcal{O}_{\amalg_j V_j}$$

is descendable by example: finitely presented faithfully flat map of discrete rings is descendable. Applying point (2) of Lemma 7.2.9 to the composite of algebra maps $\mathcal{O}_X \to f_*\mathcal{O}_Y \to f_*g_*\mathcal{O}_{\amalg_j V_j}$ shows that $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is descendable, thus proving the claim in the case where X is affine. Finally, let X be arbitrary. Choose again a finite affine open cover $\{X_\alpha\}$ and let $f_\sigma : Y \times_X X_\sigma \to X_\sigma$ be the base change of f to each vertex of the finite Čech diagram. Each f_σ is again fppf, and since X_σ is affine, the affine-target case already proved shows that $\mathcal{O}_{X_\sigma} \to (f_\sigma)_*\mathcal{O}_{Y\times_X X_\sigma}$ is descendable. Using again Lemma 7.2.17, it follows that $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is descendable. □

7.3. **Aoki's result.** We now present a huge result due to Aoki [Aok21].

Definition 7.3.1. Let $f : Y \to X$ be a morphism between noetherian schemes. We will say that f is *coherently bounded* if for every $G \in \mathrm{Coh}(Y)$, there exists an object $F \in \mathrm{Coh}(X)$ and an integer $n \in \mathbb{N}$ such that $f_*G \in \mathrm{Thick}_n(F)$.

Lemma 7.3.2. The following holds:

(1) Composition of two coherently bounded morphisms is coherently bounded.
(2) Proper morphisms between noetherian schemes are coherently bounded.
(3) Open immersions between separated noetherian schemes are coherently bounded.
(4) Morphism of finite type between separated noetherian schemes is coherently bounded.

Proof. Point (1) is a computation. Point (2) is trivial since quasi-proper morphisms send coherent complexes to coherent ones (and every proper morphism between noetherian schemes is an example of such). Point (3) requires more work. Let $j : U \to X$ be an open immersion between separated noetherian schemes, and let $G \in \mathrm{Coh}(U)$. It is enough to find an object $F \in \mathrm{Coh}(X)$ and an integer $n \in \mathbb{N}$ such that $j_*G \in \mathrm{Coprod}_n(F)$. Since every object of $\mathrm{Coh}(U)$ is a direct summand of an object of the form j^*F' for some $F' \in \mathrm{Coh}(X)$, one may assume that $G \simeq j^*F'$ for some $F' \in \mathrm{Coh}(X)$. By [Nee21, Theorem 6.2], there exist an object $F'' \in \mathrm{Perf}(X)$ and an integer $n \in \mathbb{N}$ such that $j_*\mathcal{O}_U \in \mathrm{Coprod}_n(\mathbb{Z}F'')$. Setting $F := F' \otimes F'' \in \mathrm{Coh}(X)$, one gets

$$j_*G \simeq j_*j^*F' \simeq F' \otimes j_*\mathcal{O}_U \in F' \otimes \mathrm{Coprod}_n(F'') \subseteq \mathrm{Coprod}_n(F' \otimes F'') = \mathrm{Coprod}_n(F)$$

as required. Finally, point (4) follows from Nagata's compactification theorem: every morphism of finite type between separated noetherian schemes is factored into an open immersion followed by a proper morphism. □

We now recall a consequence of Proposition 7.2.11.

Corollary 7.3.3. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a big 2-ring and let $A \in \mathrm{CAlg}(\mathcal{C})$ be descendable. Then there exists an integer $n \in \mathbb{N}$ such that $\mathcal{C} = \mathrm{thick}_n(\{A \otimes C \mid C \in \mathcal{C}\})$.

Proof. Saying that A is descendable means that the unit map $\mathbb{1} \to A$ is descendable. In particular, the above mentioned proposition states that $\mathcal{C}^{\mathrm{op}} \simeq \mathrm{Mod}_{\mathbb{1}}(\mathcal{C})^{\mathrm{op}} = \mathcal{P}_A^{*n}$ where $\mathcal{P}_A$ is the retract closure of $\mathrm{res}(P)$ for $P \in \mathrm{Mod}_A(\mathcal{C})$. Since passing to opposites is an inert operation (being the conditions only involving finite coproducts, self-extensions and retracts) it follows that $\mathcal{C} = \mathcal{P}_A^{*n}$. Thus the claim will be proved if $\mathrm{thick}_1(\{A \otimes C \mid C \in \mathcal{C}\}) = \mathcal{P}_A$. The inclusion $(\subseteq)$ is clear (since $A \otimes C = \mathrm{res}(A \otimes C)$ and since every projective class is closed under finite coproducts, retracts and shifts by Lemma 6.1.3); for ($\supseteq$), if $P \in \mathrm{Mod}_A(\mathcal{C})$, then the triangle identity $\mathrm{res}(P) \to \mathrm{res}(A \otimes \mathrm{res}(P)) \to \mathrm{res}(P)$ exhibits $\mathrm{res}(P)$ as a retract of $A \otimes \mathrm{res}(P) = \mathrm{res}(A \otimes \mathrm{res}(P))$, as claimed. □

Lemma 7.3.4. Let $L : \mathcal{C} \to \mathcal{D}$ be in $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ with right adjoint R. Let $S \subseteq \mathcal{D}$ is such that $\mathcal{D} = \mathrm{Thick}_n(S)$ for some integer $n \in \mathbb{N}$ and that $R(\mathbb{1}_{\mathcal{D}})$ is descendable. Then $\mathcal{C} = \mathrm{Thick}_m(R(S))$ for some $m \in \mathbb{N}$.

Proof. Since $R(\mathbb{1}_{\mathcal{D}})$ is descendable, Corollary 7.3.3 implies that there exists $k \in \mathbb{N}$ such that $\mathcal{C} = \mathrm{thick}_k(\{R(\mathbb{1}_{\mathcal{D}}) \otimes_{\mathcal{C}} c \mid c \in \mathcal{C}\})$. Now it is not hard to see that $L \dashv R$ satisfies the projection formula [BDS16, Cororllary 2.14], that is $c \otimes R(d) \simeq R(L(c) \otimes d)$ for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$. In particular, $R(\mathbb{1}_{\mathcal{D}}) \otimes c \simeq RL(c)$ for every $c \in \mathcal{C}$, so that $\mathcal{C} = \mathrm{thick}_k(\{R(L(c)) \mid c \in \mathcal{C}\})$. But now $L(\mathcal{C}) \subseteq \mathcal{D} = \mathrm{Thick}_n(S)$, so that

$$\mathcal{C} = \mathrm{thick}_k(\{R(L(c)) \mid c \in \mathcal{C}\}) \subseteq \mathrm{thick}_k(\mathrm{Thick}_n(R(S))) \subseteq \mathrm{Thick}_{kn}(R(S)) \subseteq \mathcal{C}$$

thus showing the claim. □

To continue we need a new notion of morphism of schemes.

Definition 7.3.5. Let $f : Y \to X$ be a morphism between noetherian schemes. We will say that f is an *h-cover* if it is of finite type and universally topologically submersive.

Bhatt and Scholze [BS17, Proposition 11.25 and Theorem 11.12] showed the following.

Proposition 7.3.6. Let $f : Y \to X$ be an h-cover of noetherian schemes. Then:

(1) The morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is descendable.

(2) There is an equivalence of categories $\mathrm{QCoh}(X) \simeq \lim \mathrm{QCoh}(Y^{\bullet/X})$ which specializes to a similar descent result for perfect and pseudo-coherent complexes.

Proposition 7.3.7. Let $f : Y \to X$ be an h-cover between quasicompact separated noetherian schemes. Suppose that there is an object $G \in \mathrm{Coh}(Y)$ satisfying $\mathrm{QCoh}(Y) = \mathrm{Thick}_n(G)$ for some integer $n \in \mathbb{N}$. Then there exists an object $F \in \mathrm{Coh}(X)$ and an integer $m \in \mathbb{N}$ such that $\mathrm{QCoh}(X) = \mathrm{Thick}_n(F)$.

Proof. Since f is of finite type between noetherian separated schemes, Lemma 7.3.2 shows that f is coherently bounded. In particular, there exists $F \in \mathrm{Coh}(X)$ and an integer $k \in \mathbb{N}$ such that $f_*G \in \mathrm{Thick}_k(F)$. On the other hand, the morphism f is an h-cover, and so Proposition 7.3.6 and Lemma 7.3.4 applied to G imply that $\mathrm{QCoh}(X) = \mathrm{Thick}_m(f_*(G))$ Putting all together implies that $\mathrm{QCoh}(X) = \mathrm{Thick}_{km}(F)$. □

We recall the following result due to Neeman [Nee21, Lemma 2.7].

Theorem 7.3.8. Let X be a noetherian scheme and let $F \in \mathrm{Coh}(X)$ be such that $\mathrm{QCoh}(X) = \mathrm{Thick}_n(F)$ for some integer $n \in \mathbb{N}$. Then F is a strong generator of $\mathrm{Coh}(X)$.

Proof. Since the proof of this result would occupy the all section, we refer the reader to Neeman's article (which is also very pleasent to read). We only observe that Neeman works with the Coprod_n operators instead of the Thick_n ones. This is ininfluent for the statemente here presented, since Lemma 2.6 in loc. requires to closu under retracts. In any case, $\mathrm{Coh}(X) = \mathrm{thick}_{2n}(F)$. □

We deduce the following result.

Theorem 7.3.9 ([Aok21, Main theorem]). Let X be a quasi-compact separated quasi-excellent scheme of finite dimension. Then $\mathrm{Coh}(X)$ has a strong generator.

Proof. By Gabber's weak local uniformization theorem, there exists an h-cover $\mathrm{Spec}(R) \to X$ with R a regular ring, which automatically has finite (global) dimension n. Since Proposition 8.4.5 says that $\mathrm{Mod}_R = \mathrm{Thick}_{n+1}(R)$ is big $(n+1)$-thickly generated by R, Proposition 7.3.7 implies that there exists an object $F \in \mathrm{Coh}(X)$ such that $\mathrm{QCoh}(X) = \mathrm{Thick}_m(F)$ for some $m \in \mathbb{N}$ Apply now Theorem 7.3.8. □

7.4. **Descending strong generation.** We now discuss how to descend strong generation. The ideas of this section belong to [DLR25].

Notation 7.4.1. Let $\mathcal{C}$ be a category with finite limits, and let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a functor. We will refer to $\mathcal{D}$ as a *presheaf of stable categories*. Given a morphism $j : U \to X$ in $\mathcal{C}$, we write $\ker(j^*) \subseteq \mathcal{D}(X)$ for the kernel of the pullback functor $j^* = \mathcal{D}(j) : \mathcal{D}(X) \to \mathcal{D}(U)$ so that there is a fiber sequence of stable categories $\ker(j^*) \to \mathcal{D}(X) \to \mathcal{D}(U)$. We will furthermore say that $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ *has adjoints* if every j^* has a right adjoint j_*. We allow change of universes and decorations on the codomain of $\mathcal{D}$.

Example 7.4.2. Let Sch be the 1-category of quasi-compact quasi-separated schemes. The assignment $\mathrm{QCoh}(-) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ is an example of a presheaf with values in rigidly-compactly generated stable categories, and for an open immersion $j : U \hookrightarrow X$, the category $\ker(j^*)$ is exactly the subcategory of objects supported on the complement of U.

Example 7.4.3. Let $X \in \mathrm{Sch}$ be a quasi-compact quasi-separated scheme, and let $\mathrm{Open}(X)$ be the category of Zariski open subsets of X. Then QCoh, Perf, and Coh define stable presheaves on $\mathrm{Open}(X)$, with values in $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$, $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ and $\mathrm{Mod}_{\mathrm{Perf}(X)}(\mathrm{Cat}^{\mathrm{perf}})$ respectively.

We now introduce a Mayer-Vietoris type notion of flatness.

Definition 7.4.4. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a stable presheaf with adjoints. Let $f : U \to V$ be a morphism in $\mathcal{C}$.

(1) We will say that f is $\mathcal{D}$-*preflat* if for every pullback square

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow[f]{} & V \end{array}$$

the base-change transformation $f^* g_* \to g'_*(f')^*$ is an equivalence, that is, the induced square is *horizontally right adjointable*.

(2) We will say that f is $\mathcal{D}$-*flat* if for every morphism $g : V' \to V$, the pullback $f' : U \times_V V' \to V'$ is $\mathcal{D}$-preflat.

Example 7.4.5. For the presheaf $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$, a morphism $U \to V$ is QCoh-preflat if and only if $U \to V$ is flat. The same holds for algebraic spaces.

Lemma 7.4.6. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a stable presheaf with adjoints, and let $j : U \to X$ be a $\mathcal{D}$-preflat monomorphism. Then the restriction functor $j^* : \mathcal{D}(X) \to \mathcal{D}(U)$ is a localization.

Proof. It is enough to show that the right adjoint $j_* : \mathcal{D}(U) \to \mathcal{D}(X)$ is fully faithful, or equivalently that the counit $j^* j_* \to \mathrm{id}_{\mathcal{D}(U)}$ is an equivalence. Since j is a monomorphism, the square

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{id}_U} & U \\ {\scriptstyle \mathrm{id}_U}\downarrow & & \downarrow{\scriptstyle j} \\ U & \xrightarrow[j]{} & X \end{array}$$

is a pullback square. By $\mathcal{D}$-preflatness, the associated base-change morphism identifies $j^* j_*$ with $\mathrm{id}_{\mathcal{D}(U)}$. Hence j_* is fully faithful, and therefore j^* is a localization. □

Now the good definition.

Definition 7.4.7. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a stable presheaf with adjoints. A *Mayer–Vietoris square* is a pullback square

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array}$$

in $\mathcal{C}$ such that:

(1) The morphism j is a $\mathcal{D}$-flat monomorphism.
(2) The base-change morphism $f^* j_* \to j'_*(f')^*$ is an equivalence.
(3) The induced functor $f^* : \ker(j^*) \to \ker((j')^*)$ is an equivalence.

Let us discuss better these conditions.

Remark 7.4.8. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a stable presheaf with adjoints and consider a Mayer-Vietoris square as in Definition 7.4.7. Notice first of all that if f is $\mathcal{D}$-preflat, then condition (2) holds (and the converse does not hold, since here one is only testing base change against the single morphism j). Consider condition (3). Since j and j' are monomorphisms, Lemma 7.4.6 implies that there is a morphism of localization sequences

$$\begin{array}{ccccc} \ker((j')^*) & \longrightarrow & \mathcal{D}(X') & \xrightarrow{(j')^*} & \mathcal{D}(U') \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \ker(j^*) & \longrightarrow & \mathcal{D}(X) & \xrightarrow[j^*]{} & \mathcal{D}(U). \end{array}$$

in which the two horizontal sequences are fibre sequences in $\mathrm{Cat}^{\mathrm{st}}$ and the functor between the kernels is an equivalence. Finally, notice that the right square is also horizontally right adjointable.

Example 7.4.9. Let

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array}$$

be a Nisnevich distinguished square of quasi-compact quasi-separated schemes, that is, j is an open immersion, f is étale, and if $Z = X \smallsetminus U$, then the induced morphism $Z' := X' \times_X Z \to Z$ is an isomorphism. Then this is a Mayer-Vietoris square for the presheaf $\mathrm{QCoh}(-) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$. Indeed, since j is an open immersion, it is a flat monomorphism, and therefore condition (1) of Definition 7.4.7 holds by Example 7.4.5. Since f is étale, hence flat, condition (2) follows from flat base change for quasi-coherent sheaves. Finally, $\ker(j^*)$ identifies with the full subcategory $\mathrm{QCoh}_Z(X) \subseteq \mathrm{QCoh}(X)$ of objects supported on Z, and similarly $\ker((j')^*) \simeq \mathrm{QCoh}_{Z'}(X')$. Since $Z' \to Z$ is an isomorphism, pullback along f induces an equivalence $f^* : \mathrm{QCoh}_Z(X) \to \mathrm{QCoh}_{Z'}(X')$, so that condition (3) also holds.

Mayer-Vietoris squares produce, well, a Mayer-Vietoris sequence.

Lemma 7.4.10. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ be a stable presheaf with adjoints, and let

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array}$$

be a Mayer-Vietoris square in $\mathcal{C}$. Let $k = j \circ f' = f \circ j' : U' \to X$ be the composition. Then there is an exact sequence $\mathrm{id}_{\mathcal{D}(X)} \to j_* j^* \oplus f_* f^* \to k_* k^*$ in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}(X), \mathcal{D}(X))$.

Proof. This is [HR17, Lemma 5.9]. □

Notation 7.4.11. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a stable presheaf (with adjoints) and let $\mathcal{P}$ be a property of objects. Given an object $X \in \mathcal{C}$ we will denote by $\mathcal{P}(X)$ the full subcategory of $\mathcal{D}(X)$ spanned by the objects satisfiying $\mathcal{P}$. We will always assume that $\mathcal{P}(X)$ is a stable subcategory of $\mathcal{D}(X)$ and that the assignment $X \mapsto \mathcal{P}(X)$ extends to a subpresheaf of $\mathcal{D}$ in the sense that $\mathcal{P}$ extends to a functor $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ and there is a fully-faithful exact functor $\mathcal{P}(X) \hookrightarrow \mathcal{D}(X)$. Furthermore, we assume that for every $X \in \mathcal{C}$ there exists a function $\alpha_X : \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ such that $\alpha_X(n) < \infty$ for every finite n and $\alpha_X(\infty) = \infty$, and for every $x \in \mathcal{P}(X)$ and every $n \in \mathbb{N} \cup \{\infty\}$ one has

$$\mathcal{P}(X) \cap \mathrm{Thick}_n^{\mathcal{D}(X)}(x) \subseteq \mathrm{thick}_{\alpha_X(n)}^{\mathcal{P}(X)}(x).$$

Notice that the requirement is that the function α_X is independent from $x \in \mathcal{P}(X)$. Since the reverse inclusion $\mathrm{thick}_n^{\mathcal{P}(X)}(x) \subseteq \mathcal{P}(X) \cap \mathrm{Thick}_n^{\mathcal{D}(X)}(x)$ is automatic, the above assumption is providing a (finite) control on thickening for the subpresheaf $\mathcal{P}$.

We now need an improvement of Definition 7.3.1.

Definition 7.4.12. Let $F : \mathcal{E} \to \mathcal{D}$ be in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ and let $\mathcal{P}$ be a property of objects. We will say that:

1. The functor F is $\mathcal{P}$-*bounded* if for every $x \in \mathcal{P}(\mathcal{E})$ there exist $d \in \mathcal{P}(\mathcal{D})$ and $n \in \mathbb{N}$ such that $F(x) \in \mathrm{Thick}_n(d)$.
2. The functor F is *strongly* $\mathcal{P}$-*bounded* if there exists $n \in \mathbb{N}$ such that for every $x \in \mathcal{P}(\mathcal{E})$ there exists $d \in \mathcal{P}(\mathcal{D})$ with $F(x) \in \mathrm{Thick}_n(d)$.

Notice that the composite of $\mathcal{P}$-bounded functors is again $\mathcal{P}$-bounded. The same holds for strongly $\mathcal{P}$-bounded functors.

Example 7.4.13. In the context of QCoh of quasi-compact quasi-separated schemes, consider the property Perf of being a perfect complex. Let X be quasi-compact separated scheme. Then the pushforward of every open immersion $j : U \hookrightarrow X$ with U quasi-compact is strongly Perf-bounded. Indeed, the Perf-boundness follows from [Nee21, Theorem 6.2]. For the strong version, notice that Perf-boundness implies the existence of a perfect complex $P \in \mathrm{Perf}(X)$ and an integer $n \in \mathbb{N}$ such that $j_* \mathcal{O}_U \in \mathrm{Thick}_n(P)$. Now, by using Corollary 5.3.2, for every $Q \in \mathrm{Perf}(U)$ there exists $P' \in \mathrm{Perf}(X)$ such that Q is a retract of $j^* P'$. Now the projection formula implies that $P' \otimes j_* \mathcal{O}_U \simeq j_*(j^*(P') \otimes \mathcal{O}_U) \simeq j_*(j^*(P'))$, so that $j_*(Q)$ is a retract of $P' \otimes j_* \mathcal{O}_U$. Thus $j_* Q \in \mathrm{smd}(P' \otimes j_*(\mathcal{O}_U)) \subseteq \mathrm{Thick}_n(P' \otimes P)$, and since $P' \otimes P$ is compact, the claim follows.

Example 7.4.14. In the context of QCoh of noetherian separated schemes, consider the property Coh of being a coherent complex. Then Lemma 7.3.2 produces several examples of Coh-bounded morphism: the pushforward of an open immersion, of a proper morphism and of a morphism of finite type is Coh-bounded.

Example 7.4.15. Let $f : X \to Y$ be an étale morphism of quasi-compact separated schemes. Then:

1. By [DLR25, Proposition 4.9] every smooth morphism of quasi-compact separated schemes has pushforward which is Perf-bounded, In particular, since every étale morphism is smooth, it follows that f_* is Perf-bounded.
2. Assume that X and Y are noetherian. Since every étale quasi-compact morphism is of finite type, it follows by Example 7.4.14 that the pushforward f_* is Coh-bounded.

As a consequence of the definitions we deduce the following.

Theorem 7.4.16. Let $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a stable presheaf (with adjoints) and let

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array}$$

be a Mayer-Vietoris square. Let $\mathcal{P} \subseteq \mathcal{D}$ be a subpresheaf as in Notation 7.4.11. Assume that j_*, f_*, and k_* are $\mathcal{P}$-bounded. If $\mathcal{P}(U)$, $\mathcal{P}(U')$, and $\mathcal{P}(X')$ admit (strong) generators, then also $\mathcal{P}(X)$ admits a (strong) generator.

Proof. We prove the claim with strong generators; the one with generators follows by allowing the indexes to be infinite. Let $G_U \in \mathcal{P}(U)$, $G_{U'} \in \mathcal{P}(U')$, and $G_{X'} \in \mathcal{P}(X')$ be the strong generators and choose integers $m_U, m_{U'}, m_{X'} \in \mathbb{N}$ such that

$$\mathcal{P}(U) = \mathrm{thick}_{m_U}(G_U), \qquad \mathcal{P}(U') = \mathrm{thick}_{m_{U'}}(G_{U'}), \qquad \mathcal{P}(X') = \mathrm{thick}_{m_{X'}}(G_{X'}).$$

Since j_*, f_* and k_* are P-bounded, there exist objects $G, G', G'' \in \mathcal{P}(X)$ and integers $N_U, N_{X'}, N_{U'} \in \mathbb{N}$ such that

$$j_*(G_U) \in \mathrm{Thick}_{N_U}(G), \qquad f_*(G_{X'}) \in \mathrm{Thick}_{N_{X'}}(\mathbb{Z}G'), \qquad k_*(G_{U'}) \in \mathrm{Thick}_{N_{U'}}(\mathbb{Z}G'').$$

Let $x \in \mathcal{P}(X)$. Since $\mathcal{P}$ is a subpresheaf, the pullback functor restricts to $\mathcal{P}$-categories in the sense that it is $j^*(x) \in \mathcal{P}(U)$, $f^*(x) \in \mathcal{P}(X')$, and $k^*(x) \in \mathcal{P}(U')$. Hence $j^*(x) \in \mathrm{thick}_{m_U}(G_U)$, $f^*(x) \in \mathrm{thick}_{m_{X'}}(G_{X'})$, and $k^*(x) \in \mathrm{thick}_{m_{U'}}(G_{U'})$. Applying the exact functors j_* it follows that

$$j_*j^*(x) \in j_*\,\mathrm{thick}_{m_U}(G_U) \subseteq \mathrm{thick}_{m_U}(j_*(G_U)) \subseteq \mathrm{thick}_{m_U}(\mathrm{Thick}_{N_U}(G)) \subseteq \mathrm{Thick}_{m_U N_U}(G).$$

Here the first inclusion is the functoriality of Lemma 4.2.12, the second one is from above, and the third one follows by $\mathrm{thick}_n \subseteq \mathrm{Thick}_n$ and by multiplication of indexes (see Lemma 4.2.9 for the small case; the big one is analoge). Similarly, $f_*f^*(x) \in \mathrm{Thick}_{m_{X'}N_{X'}}(G')$ and $k_*k^*(x) \in \mathrm{Thick}_{m_{U'}N_{U'}}(G'')$. Denote by $m = \max\{m_U N_U,\ m_{X'}N_{X'},\ m_{U'}N_{U'}\}$ the maximum, so that $j_*j^*(x)$, $f_*f^*(x)$ and $k_*k^*(x)$ belong $\mathrm{Thick}_m(G \oplus G' \oplus G'')$. Use now the Mayer-Vietoris triangle

$$\Sigma^{-1}k_*k^*(x) \to x \to j_*j^*(x) \oplus f_*f^*(x) \to k_*k^*(x)$$

of Lemma 7.4.10 shows that x belongs to $\mathrm{Thick}_{2m}(G \oplus G' \oplus G'')$. Since $x \in \mathcal{P}(X)$, the comparison hypothesis of Notation 7.4.11 yields $x \in \mathrm{thick}_{\alpha_X(2m)}(G \oplus G' \oplus G'')$. Since the integer $N := \alpha_X(2m)$ is independent of x, it follows that

$$\mathrm{thick}_N(G \oplus G' \oplus G'') \subseteq \mathcal{P}(X) \subseteq \mathrm{thick}_N(G \oplus G' \oplus G'').$$

Here the first inclusion is by construction. In any case, this shows that $G \oplus G' \oplus G''$ is a strong generator of $P(X)$. ☐

Example 7.4.17. Let X be a quasi-compact quasi-separated scheme and consider the presheaf $\mathrm{QCoh} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ regarded as a presheaf with adjoints. The properties of being perfect and coherent define subpresheaves Perf, Coh of QCoh: for perfect complexes because pullback preserves compact objects, and for coherent complexes because pullback along an open immersion is t-exact. Consider now the convention on thickenings of Notation 7.4.11.

(1) For perfect complexes one has $\mathrm{thick}_n^{\mathrm{Perf}(U)}(F) = \mathrm{Perf}(U) \cap \mathrm{Thick}_n^{\mathrm{QCoh}(U)}(F)$ for every quasi-compact open $U \subseteq X$, every $F \in \mathrm{Perf}(U)$, and every $n \in \mathbb{N}$, since $\mathrm{Perf}(U) = \mathrm{QCoh}(U)^\omega$ and compactness reduces arbitrary coproducts to finite coproducts. See Exercise E.4.1,

(2) Likewise, if X is noetherian, then for coherent complexes one has $\mathrm{thick}_n^{\mathrm{Coh}(U)}(F) \subseteq \mathrm{Coh}(U) \cap \mathrm{Thick}_{2n}^{\mathrm{QCoh}(U)}(F)$for every quasi-compact open $U \subseteq X$, every $F \in \mathrm{Coh}(U)$, and every $n \in \mathbb{N}$. See Exercise E.4.2.

Now let

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & U \cup V \end{array}$$

be a Zariski distinguished square of quasi-compact opens. This is a Mayer–Vietoris square for QCoh. Moreover, the pushforwards of the three open immersions are strongly Perf-bounded by Example 7.4.13, and are Coh-bounded in the noetherian separated case by Example 7.4.14. Therefore Theorem 7.4.16 implies that the property of admitting a (strong) generator satisfies Zariski descent for Perf and also for Coh on noetherian separated schemes.

Example 7.4.18. Let $\mathrm{Et}^{\mathrm{qcqs}}(X)$ be the category of quasi-compact quasi-separated schemes étale over a quasi-compact quasi-separated scheme X. Consider the presheaf $\mathrm{QCoh} : \mathrm{Et}^{\mathrm{qcqs}}(X)^{op} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ regarded as a presheaf with adjoints. The property of being perfect defines a subpresheaf $\mathrm{Perf} \subseteq \mathrm{QCoh}$, and if X is noetherian then the property of being coherent defines likewise a subpresheaf $\mathrm{Coh} \subseteq \mathrm{QCoh}$. Again the convetion of Notation 7.4.11 are satisfied (since they are pointwise and the étale topology does not play a role). Now let

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array}$$

be a Nisnevich distinguished square of quasi-compact quasi-separated schemes. By Example 7.4.9 this is a Mayer–Vietoris square for QCoh. The morphism j is an open immersion, hence j_* is strongly Perf-bounded by Example 7.4.13; moreover f and $k = j \circ f' = f \circ j'$ are étale, hence Perf-bounded by Example 7.4.15. If all schemes involved are noetherian and separated, then j_*, f_*, and k_* are also Coh-bounded by Example 7.4.14 and Example 7.4.15. Therefore Theorem 7.4.16 implies that the property of admitting a (strong) generator satisfies Nisnevich descent for Perf, and also for Coh on noetherian separated schemes.

8. The Rouquier dimension

In this chapter we introduce the Rouquier dimension of a stable category. We begin with the definition and with its first formal properties. We then discuss a number of examples and counterexamples, in order to clarify the behaviour of this invariant under basic categorical constructions. We conclude with a representability result and with computations of the Rouquier dimension in geometric and algebraic examples.

8.1. The definition.

Definition 8.1.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. The *Rouquier dimension* of $\mathcal{C}$ is

$$\dim_{\mathrm{R}}(\mathcal{C}) = \inf\{⏱_G(\mathcal{C}) - 1 \text{ for } G \in \mathcal{C}, ⏱_G(\mathcal{C}) < \infty\}.$$

Notice that $\dim_{\mathrm{R}}(\mathcal{C}) \in \mathbb{N} \cup \{+\infty\}$.

Remark 8.1.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. There are a couple of immediate observations.

(1) The Rouquier dimension is computed as

$$\dim_{\mathrm{R}}(\mathcal{C}) = \inf\{n \in \mathbb{N} \text{ for which there exists } G \in \mathcal{C} \text{ such that } \mathrm{thick}_{n+1}(G) = \mathcal{C}\}.$$

Indeed, $\dim_{\mathrm{R}}(\mathcal{C}) \leq n$ if and only if there exists $G \in \mathcal{C}$ such that $\mathcal{C} \subseteq \mathrm{thick}_{n+1}(G)$.

(2) It is clear that exact quivalences of stable categories preserve the Rouquier dimension.

We now show that the Rouquier dimension depends only on the idempotent completion.

Lemma 8.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $\mathcal{D} \subseteq \mathcal{C}$ be a dense full stable subcategory. Then $\dim_{\mathrm{R}}(\mathcal{D}) = \dim_{\mathrm{R}}(\mathcal{C})$.

Proof. Assume $\dim_R(D) \leq n$. Then there exists $M \in \mathcal{D}$ such that $\text{⏰}_M(\mathcal{D}) \leq n+1$, that is $\mathcal{D} \subseteq \mathrm{thick}_{n+1}(M)$. Let $x \in \mathcal{C}$. By density there exist $y \in \mathcal{D}$ and $x' \in \mathcal{C}$ such that $x \oplus x' \simeq y$. Since $y \in \mathcal{D} \subseteq \mathrm{thick}_{n+1}(M)$ and $\mathrm{thick}_{n+1}(M)$ is closed under retracts, it follows that $x \in \mathrm{thick}_{n+1}(M)$. Hence $\mathcal{C} \subseteq \mathrm{thick}_{n+1}(M)$, so $\text{⏰}_M(\mathcal{C}) \leq n+1$, and therefore $\dim_R(\mathcal{C}) \leq n$.

Assume $\dim_R(\mathcal{C}) \leq n$. Then there exists $N \in \mathcal{C}$ such that $\text{⏰}_N(\mathcal{C}) \leq n+1$, that is $\mathcal{C} \subseteq \mathrm{thick}_{n+1}(N)$. By density, N is a retract of some object $N_0 \in \mathcal{D}$, hence $N \in \mathrm{thick}_1(N_0)$. By monotonicity of the thickenings, it follows $\mathrm{thick}_{n+1}(N) \subseteq \mathrm{thick}_{n+1}(N_0)$. Therefore $\mathcal{C} \subseteq \mathrm{thick}_{n+1}(N_0)$, and since $\mathcal{D} \subseteq \mathcal{C}$ it follows that $\mathcal{D} \subseteq \mathrm{thick}_{n+1}(N_0)$, that is $\text{⏰}_{N_0}(\mathcal{D}) \leq n+1$. Hence $\dim_R(\mathcal{D}) \leq n$. □

Corollary 8.1.4. The Rouquier dimension is invariant under idempotent-completion.

Proof. By construction, the canonical exact fully faithful functor $\mathcal{C} \hookrightarrow \mathcal{C}^\natural$ has dense image: every object of $\mathcal{C}^\natural$ is, by definition, a retract of an object in the essential image. Applying Lemma 8.1.3 to the dense embedding $\mathcal{C} \subseteq \mathcal{C}^\natural$ gives $\dim(\mathcal{C}) = \dim(\mathcal{C}^\natural)$. □

From now on, we will consider the Rouquier dimension of small idempontent-complete stable categories.

8.2. **Formal properties of Rouquier dimension.** We now collect a list of formal properties of the Rouquier dimension (and counterexamples to apparently innocent statements).

Example 8.2.1. The Rouquier dimension is not preserved by fully-faithful exact functors: there are fully faithful exact functor $\mathcal{C} \hookrightarrow \mathcal{D}$ with $\dim_R(\mathcal{C}) > \dim_R(\mathcal{D})$. For an explicit example, let k be a field and consider the ring $k[x]$. Then $\mathrm{Perf}_{k[x]}$ has dimension 1 (see Theorem 8.4.21). Consider now the subset $S = \{(x-n) \text{ for } n \in \mathbb{N}\} \subseteq k(x)$. Let $\mathcal{C} \coloneqq \{ P \in \mathrm{Perf}_{k[x]} \mid \mathrm{supp}(P) \subseteq S \} \subseteq \mathrm{Perf}_{k[x]}$. Then $\mathcal{C}$ is a thick subcategory of $\mathrm{Perf}_{k[x]}$, hence the inclusion $\mathcal{C} \hookrightarrow \mathrm{Perf}_{k[x]}$ is exact and fully faithful. However, $\mathcal{C}$ has no (strong) generator: for any $G \in \mathcal{C}$ the support $\mathrm{supp}(G)$ is a closed subset of $\mathrm{Spec}(k[x])$ contained in S, hence finite. Moreover, if $X \in \mathrm{thick}(G)$ then $\mathrm{supp}(X) \subseteq \mathrm{supp}(G)$. Choose $m \in \mathbb{N}$ outside this finite set. The perfect complex $R/(x-m)$ lies in $\mathcal{C}$ but cannot lie in $\mathrm{thick}(G)$. Thus no object generates $\mathcal{C}$, so $\dim_R(\mathcal{C}) = +\infty$.

The situation is different for essentially surjective functors.

Lemma 8.2.2. Let $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}^{\mathrm{st}}$ be essential surjective up to direct summands. Then $\dim_{\mathrm{R}}(\mathcal{C}) \geq \dim_{\mathrm{R}}(\mathcal{D})$.

Proof. If the Rouquier dimension of $\mathcal{C}$ is infinite then there is nothing to prove. Assume therefore $\dim_{\mathrm{R}}(\mathcal{C})$ is finite, say equal to n. Then there exists an object $G \in \mathcal{C}$ such that $\mathrm{thick}_n(G) = \mathcal{C}$. Then $f(G)$ is a strong generator of $\mathcal{D}$. Indeed, let $d \in \mathcal{D}$. Since f is essentially surjective up to direct summands, there exists $c \in \mathcal{C}$ such that d is a direct summand of $f(c)$. Since $c \in \mathrm{thick}_n(G)$ and f is exact, Lemma 4.2.12 implies that $f(c) \in \mathrm{thick}_n(f(G))$. As $\mathrm{thick}_n(f(G))$ is thick, it is closed under direct summands. Hence $d \in \mathrm{thick}_n(f(G))$. Therefore $\mathcal{D} = \mathrm{thick}_n(f(G))$, so $\mathcal{D}$ is strongly generated by $f(G)$ and the bound on dimensions follows. □

Corollary 8.2.3. Let $q : \mathcal{D} \to \mathcal{E}$ be a Verdier quotient in $\mathrm{Cat}^{\mathrm{st}}$. Then $\dim_{\mathrm{R}}(\mathcal{D}) \geq \dim_{\mathrm{R}}(\mathcal{E})$.

Proof. Follows from Lemma 8.2.2 since Verdier projections are essentially surjective. □

Lemma 8.2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories. Then

$$\dim_R(\mathcal{C} \oplus \mathcal{D}) = \max\{\dim_R(\mathcal{C}), \dim_R(\mathcal{D})\}.$$

Proof. If one of the Rouquier dimensions appearing on the right side is infinite then there is nothing to prove. Assume therefore that both $\dim_R(\mathcal{C})$ and $\dim_R(\mathcal{D})$ are finite, say equal to n and m, and assume $n \geq m$. Then there are objects $G \in \mathcal{C}$ and $H \in \mathcal{D}$ such that $\mathrm{thick}_{n+1}(G) = \mathcal{C}$ and $\mathrm{thick}_{m+1}(H) = \mathcal{D}$. Then $(G, H) \in \mathcal{C} \oplus \mathcal{D}$ is such that $\mathrm{thick}_n((G, H)) = \mathrm{thick}_n(G) \oplus \mathrm{thick}_n(H) = \mathrm{thick}_n(G) \oplus \mathrm{thick}_m(H) = \mathcal{C} \oplus \mathcal{D}$, so that

$$\dim_R(\mathcal{C} \oplus \mathcal{D}) \leq \dim_R(\mathcal{C}) = \max\{\dim_R(\mathcal{C}), \dim_R(\mathcal{D})\}.$$

On the other side, the Rouquier dimension of $\mathcal{C} \oplus \mathcal{D}$ cannot exceed the ones of $\mathcal{C}$ and $\mathcal{D}$, as an argument with the projections shows (together with Lemma 4.2.12). □

We now prove a bound for the Rouquier dimension of the middle term for right split Verdier sequence.

Lemma 8.2.5. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a right split Verdier sequence in $\mathrm{Cat}^{\mathrm{st}}$. Suppose that $\mathcal{C}$ and $\mathcal{E}$ admit strong generators. Then $\mathcal{D}$ admits a strong generator and one has

$$\dim_R(\mathcal{D}) \leq \dim_R(\mathcal{C}) + \dim_R(\mathcal{E}) + 1.$$

Proof. Set $n \coloneqq \dim_R(\mathcal{C})$ and $m \coloneqq \dim_R(\mathcal{E})$. Since $\mathcal{C}$ and $\mathcal{E}$ admit strong generators, one may choose objects $G \in \mathcal{C}$ and $H \in \mathcal{E}$ such that $\mathcal{C} = \mathrm{thick}_{n+1}(G)$ and $\mathcal{E} = \mathrm{thick}_{m+1}(H)$. Let $p^{\mathrm{R}} : \mathcal{E} \to \mathcal{D}$ be the fully faithful right adjoint of p, and set $F \coloneqq f(G) \oplus p^{\mathrm{R}}(H) \in \mathcal{D}$. Then

$$\mathcal{D} = \mathrm{thick}_{n+m+2}(F).$$

Indeed, let $d \in \mathcal{D}$. Since $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a right split Verdier sequence, there exists an exact sequence $k_d \to d \to p^{\mathrm{R}}p(d)$ with $k_d \in \ker(p) = \mathrm{im}(f)$. Hence there exists $c_d \in \mathcal{C}$ such that $k_d \simeq f(c_d)$. Since $c_d \in \mathcal{C} = \mathrm{thick}_{n+1}(G)$ and f is exact, it follows that

$$k_d \simeq f(c_d) \in \mathrm{thick}_{n+1}(f(G)) \subseteq \mathrm{thick}_{n+1}(F).$$

Similarly, since $p(d) \in \mathcal{E} = \mathrm{thick}_{m+1}(H)$ and p^{R} is exact, one has

$$p^{\mathrm{R}}p(d) \in \mathrm{thick}_{m+1}(p^{\mathrm{R}}(H)) \subseteq \mathrm{thick}_{m+1}(F).$$

The above fibre sequence therefore shows that $d \in \mathrm{thick}_{n+1}(F) \star \mathrm{thick}_{m+1}(F) \subseteq \mathrm{thick}_{n+m+2}(F)$, and since d was arbitrary, this proves that $\mathcal{D} = \mathrm{thick}_{n+m+2}(F)$. In particular, F is a strong generator of $\mathcal{D}$, and therefore $\dim_R(\mathcal{D}) \leq n + m + 1 = \dim_R(\mathcal{C}) + \dim_R(\mathcal{E}) + 1$. □

Example 8.2.6. The bound in the previous lemma is sharp. Let k be a field and consider the Beilinson semiorthogonal decomposition $\mathrm{Perf}(\mathbb{P}^1_k) = \langle \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \rangle$ proved in Theorem 5.4.8. Set $\mathcal{C} \coloneqq \mathrm{thick}(\mathcal{O}_{\mathbb{P}^1})$ and $\mathcal{E} \coloneqq \mathrm{thick}(\mathcal{O}_{\mathbb{P}^1}(1))$. Since both $\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1}$ are exceptional, there are equivalences $\mathcal{C} \simeq \mathcal{E} \simeq \mathrm{Perf}(k)$. In particular, $\dim_R(\mathcal{C}) = \dim_R(\mathcal{E}) = 0$ as we will see in Example 8.4.2. Since the above semiorthogonal decomposition is admissible, by Lemma 3.3.6 it gives a right split Verdier sequence $\mathcal{C} \to \mathrm{Perf}(\mathbb{P}^1_k) \to \mathcal{E}$. The previous lemma therefore yields

$$\dim_R(\mathrm{Perf}(\mathbb{P}^1_k)) \leq 0 + 0 + 1 = 1.$$

On the other hand, since $\mathbb{P}^1_k$ is a reduced separated scheme of finite type over k, Rouquier's lower bound (which we will see in Theorem 10.2.2) gives

$$\dim_R(\mathrm{Perf}(\mathbb{P}^1_k)) = \dim_R(\mathrm{Coh}(\mathbb{P}^1_k)) \geq \dim(\mathbb{P}^1_k) = 1$$

and hence a sharp bound.

Example 8.2.7. The author does not know if it is possible to drop the right splitting assumption in the previous result. It seems unlikely that such dropping is possible...

8.3. **The ghost criteria.** Due to its importance, we recall the ghost lemma Remark 6.2.7.

Lemma 8.3.1 (Ghost lemma)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G \in \mathcal{C}$ be a strong generator. If there exists an object $x \in \mathcal{C}$ such that the n-fold composite of G-ghost maps is not null-homotopic, then $x \notin \mathrm{thick}_n(G)$ and therefore ⏰$_G(\mathcal{C}) > n$.

In particular, if for every strong generator G there is n-fold composite of G-ghost maps which not null-homotopic, then $\dim_{\mathrm{R}}(\mathcal{C}) \geq n$.

Proof. It is Remark 6.2.7. □

There is also a partial converse to this result (partial because it requires the existence of $\mathrm{thick}_1(G)$-precovers).

Lemma 8.3.2 (Converse ghost lemma)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G \in \mathcal{C}$. Assume that for every object $x \in \mathcal{C}$ there exists a map $\nu_x : x_G \to x$ with $x_G \in \mathrm{thick}_1(G)$ such that every map $y \to x$ with $y \in \mathrm{thick}_1(G)$ factors through ν_x. If $x \notin \mathrm{thick}_n(G)$, then there exist objects $x_1, \ldots, x_n \in \mathcal{C}$ and G-ghost maps

$$x = x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

such that the composite $f_n \circ \cdots \circ f_1$ is not null-homotopic.

Proof. Set $x_0 := x$. For every $i \geq 0$, complete $\nu_{x_i} : (x_i)_G \to x_i$ to an exact sequence

$$(x_i)_G \xrightarrow{\nu_{x_i}} x_i \xrightarrow{f_{i+1}} x_{i+1}.$$

By the universal property of ν_{x_i}, every map $y \to x_i$ with $y \in \mathrm{thick}_1(G)$ factors through ν_{x_i}. Hence f_{i+1} is a G-ghost. Suppose now that the composite $f_n \circ \cdots \circ f_1$ were null-homotopic. A repeated application of the octahedral axiom shows that x can then be obtained from the objects $(x_0)_G, \ldots, (x_{n-1})_G$ by at most n extensions. Since each $(x_i)_G$ belongs to $\mathrm{thick}_1(G)$, it follows that $x \in \mathrm{thick}_n(G)$, a contradiction and therefore $f_n \circ \cdots \circ f_1$ is not null-homotopic. □

We can also tell the dual of the 👻-story.

Definition 8.3.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G \in \mathcal{C}$. A map $x \to y$ in $\mathcal{C}$ is called a *G-coghost* if the map $\mathrm{hom}_{\mathcal{C}}(y, g) \to \mathrm{hom}_{\mathcal{C}}(x, g)$ is null-homotopic for every $g \in \mathrm{thick}_1(G)$.

Lemma 8.3.4 (Co-ghost lemma and converse co-ghost lemma)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G \in \mathcal{C}$.

(1) If there exist objects $x_0, \ldots, x_n \in \mathcal{C}$ and G-co-ghost maps

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x_0$$

such that the composite $f_1 \circ \cdots \circ f_n$ is not null-homotopic, then $x_0 \notin \mathrm{thick}_n(G)$.

(2) Assume that for every object $x \in \mathcal{C}$ there exists a map $\nu_x : x \to x_G$ with $x_G \in \mathrm{thick}_1(G)$ such that every map $x \to y$ with $y \in \mathrm{thick}_1(G)$ factors through ν_x. If $x \notin \mathrm{thick}_n(G)$, then there exist objects $x_1, \ldots, x_n \in \mathcal{C}$ and G-co-ghost maps

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x_0 = x$$

such that the composite $f_1 \circ \cdots \circ f_n$ is not null-homotopic.

Proof. Both statements follow by applying the corresponding ghost statements to $\mathcal{C}^{\mathrm{op}}$. □

Remark 8.3.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G \in \mathcal{C}$.

(1) The ghost lemma gives lower bounds on the Rouquier dimension. More precisely, if for every strong generator G there exists an object $x \in \mathcal{C}$ and a non-nullhomotopic composite of d many G-ghosts starting at x, then $\dim_{\mathrm{R}}(\mathcal{C}) \geq d$. Equivalently, if $\dim_{\mathrm{R}}(\mathcal{C}) \leq d - 1$, then there exists a strong generator G for which every composite of d many G-ghosts is null-homotopic.

(2) Assume now the hypothesis of the converse ghost lemma, namely that every object $x \in \mathcal{C}$ admits a right $\mathrm{thick}_1(G)$-precover $x_G \to x$. Then for every $n \geq 1$ and every object $x \in \mathcal{C}$ one has

$$x \in \mathrm{thick}_n(G) \iff \text{every composite of } n \text{ many } G\text{-ghosts starting at } x \text{ is null-homotopic.}$$

In particular,

$$\mathcal{C} = \mathrm{thick}_n(G) \iff \text{every composite of } n \text{ many } G\text{-ghosts in } \mathcal{C} \text{ is null-homotopic.}$$

Hence vanishing of n-fold ghost composites gives an upper bound $\dim_{\mathrm{R}}(\mathcal{C}) \leq n-1$, while the existence of a nonzero n-fold ghost composite gives the opposite lower bound.

Therefore the ghost lemma and its converse combine into the following criterion:

$$x \in \mathrm{thick}_n(G) \setminus \mathrm{thick}_{n-1}(G)$$

if and only if there exists a nonzero composite of $n-1$ many G-ghosts starting at x, and equivalently a nonzero composite of $n-1$ many G-coghosts ending at x. In particular, for such categories the generation time of G is exactly one plus the maximal length of a nonzero composite of G-ghosts, and the Rouquier dimension is the minimum of these maximal lengths over all strong generators.

Example 8.3.6 ([BFK11, Lemma 2.17])**.** If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ is k-linear and Ext-finite over a field k, in the sense that for every $x, y \in \mathcal{C}$ the total graded vector space $\bigoplus_{i\in\mathbb{Z}} \pi_i \hom_{\mathcal{C}}(x,y)$ is finite-dimensional, then the evaluation and coevaluation morphisms

$$\bigoplus_{i\in\mathbb{Z}} \pi_0 \hom_{\mathcal{C}}(G, \Sigma^i x) \otimes_k \Sigma^{-i} G \to x. \qquad x \to \bigoplus_{i\in\mathbb{Z}} \pi_0 \hom_{\mathcal{C}}(\Sigma^i x, G)^\vee \otimes_k \Sigma^{-i} G.$$

produce the require precovers for Lemma 8.3.2 and Lemma 8.3.4. Notice that being ext-finite is implied but not equivalent[5] to $\mathcal{C}$ being proper over Mod_k.

Ext-finite categories are very “rare” (as proper categories are).

Example 8.3.7. Being quasi-projective over k does not imply being ext-finite. For example, $\mathbb{A}^n_k$ is quasi-projective but not ext finite since the global sections of the monoidal unit are given by $k[t]$ which is not finite dimensional over k.

8.4. Examples.

We now compute the Rouquier dimension in concrete examples.

Example 8.4.1. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Then:

(1) The Rouquier dimension $\dim_R(\mathcal{C}) = -1$ is negative if and only if $\mathcal{C} = 0$.
(2) The Rouquier dimension $\dim_R(\mathcal{C}) = 0$ is zero if and only if there exists an object $G \in \mathcal{C}$ such that every object of $\mathcal{C}$ is a retract of a finite sum of shifts of G.

This exhausts the trivial cases.

Example 8.4.2. Let $k \in \mathrm{CAlg}(\mathrm{Sp})^\heartsuit$ be a field and consider the category of perfect complexes Perf_k. Since every perfect complex is a bounded complex of finite rank projective k-modules, and since for k-modules being projective is the same of being free, it follows that every perfect complex is a bounded complex of finite generated k-modules. Thus $\mathrm{Perf}_k = \mathrm{thick}_1(k)$, making $\dim_{\mathrm{R}}(\mathrm{Perf}_k) = 0$.

Example 8.4.3. The category of compact spectra Sp^ω is not strongly generated. If it was, then Lemma 5.1.4 implies that the sphere spectrum $\mathbb{S}$, being a classical generator, should be a strong generator. In particular, there should exist an integer $n \in \mathbb{N}$ such that $\mathrm{Sp}^\omega = \mathrm{thick}_n(\mathbb{S})$. Christensen has showed in [Chr13] that there are n-fold composite of $\mathbb{S}$-ghosts of arbitrary lenght, so that $⏰_{\mathbb{S}}(\mathrm{Sp}) = \infty$ and the sphere spectrum cannot strongly generate. See Exercise E.6.2.

[5] Consider $\mathrm{QCoh}(\mathbb{B}\mathbb{G}_m)$. The stack $\mathbb{B}\mathbb{G}_m$ is not proper over k, but $\mathrm{QCoh}(\mathbb{B}\mathbb{G}_m) \simeq \mathrm{Fun}(\mathbb{Z}^{\mathrm{discr}}, \mathrm{Mod}_k)$ is clearly ext-finite.

The next goal is to provide a bound on Rouquier dimension on Perf_R for a classical ring R via the global dimension. We recall the definition and some properties, following Section 00O2.

Remark 8.4.4. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring and let $M \in \mathrm{Mod}_R^{\heartsuit}$ be a classical R-module. We say that M has *finite projective dimension* if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the *projective dimension* of M. It is clear that the projective dimension of M is 0 if and only if M is a projective module. The ring R is said to have *finite global dimension* if there exists an integer $n \in \mathbb{N}$ such that every classical R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the *global dimension of* R, and it is denoted by $\mathrm{gl.dim}(R)$. For a noetherian ring $R \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$, Lemma 00OE states that R has finite global dimension n if and only if it is regular of Krull dimension n.

Proposition 8.4.5. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring of global dimension n. Then $\mathrm{Mod}_R = \mathrm{Thick}_{n+1}(R)$.

Proof. Consider the big 1-thickening $\mathrm{Thick}_1(R)$. Then $M \in \mathrm{Thick}_1(R)$ if and only if $\pi_* M$ is a classical graded projective R-module. For $(\Rightarrow)$, if $M \in \mathrm{Thick}_1(R)$ then M is a retract of a coproduct of shifts of R, so that by applying π_* it follows that $\pi_* M$ is a retract of a free graded R-module, hence classical graded projective. Conversely $(\Leftarrow)$, assume that $\pi_* M$ is a classical graded projective R-module. Write

$$\pi_* M = \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} P_i$$

with $P_i \in \mathrm{Mod}_R^{\heartsuit}$ classical projective R-module so that $\pi_i M \simeq P_i$. Let HP_i be the Eilenberg-MacLane R-module associated to P_i and define

$$N = \bigoplus_{i \in \mathbb{Z}} \Sigma^i HP_i.$$

Then $\pi_* N \cong \pi_* M$ trivially. Notice that $N \in \mathrm{Thick}_1(R)$, being it a coproduct of shifts of HP_i, that is, of retracts of free R-modules. Thus the claim will be proved if there exists a map $N \to M$ inducing isomorphism on homotopy groups. Now notice that for $P \in \mathrm{Mod}_R^{\heartsuit}$ be a classical R-module there are equivalences

$$\begin{aligned}\pi_0 \hom_{\mathrm{Mod}_R}(\Sigma^i HP, M) &\simeq \pi_0 \hom_{\mathrm{Mod}_R}(P, \pi_i M) \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_R^{\heartsuit}}(P, \pi_i M)\end{aligned}$$

so that a map $N \to M$ is determined by maps $P_i \to \pi_i M$. Picking the isomorphisms $P_i \to \pi_i M$ implies the claim.

Given $M \in \mathrm{Mod}_R$, consider $\pi_* M = \bigoplus_{i \in \mathbb{Z}} \pi_i M$. Since the abelian 1-category of graded R-modules has enough projectives (and since the free objects are sums of shifts of R), there exists a surjection $P_{0,*} \to \pi_* M$ with $P_{0,*} = \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} P_{0,i}$ and $P_{0,i} \in \mathrm{Mod}_R^{\heartsuit}$ projective. By defining $Q_0 = \bigoplus_{i \in \mathbb{Z}} \Sigma^i HP_{0,i}$, it follows that $\pi_* Q_0 \simeq P_{0,*}$ and that $Q_0 \in \mathrm{Thick}_1(R)$. Now the surjection $P_{0,*} \to \pi_* M$ may be realized as a map $Q_0 \to M$ in Mod_R, so that every graded surjection may be realized by a map in Mod_R. Consider now the fibre sequence $M_1 \to Q_0 \to M$. By the long exact sequence in homotopy

$$\cdots \to \pi_i M_1 \to \pi_i Q_0 \to \pi_i M \to \pi_{i-1} M_1 \to \cdots$$

it follows that $\pi_i M \to \pi_{i-1} M_1$ is zero, because $\pi_i Q_0 \to \pi_i M$ is surjective. Thus splicing the sequence and using arbitrariness of i produces an exact sequence $0 \to \pi_* M_1 \to \pi_* Q_0 \to \pi_* M \to 0$ of graded R-modules. Iterate now: pick a surjection $P_{i,*} \twoheadrightarrow \pi_* M_i$, realize it via some $Q_i \in \mathrm{Thick}_1(R)$ and a map $Q_i \to M_i$, and deduce exact sequences $0 \to \pi_* M_{i+1} \to \pi_* Q_i \to \pi_* M_i \to 0$. This produces a resolution

$$\cdots \to \pi_* Q_1 \to \pi_* Q_0 \to \pi_* M \to 0$$

of $\pi_* M$ in classical graded R-modules.

Since the global dimension of R is n, every graded R-module has projective dimension $\leq n$. Hence $\pi_* M_n$ is projective, so $M_n \in \mathrm{Thick}_1(R)$. By induction on the fibre sequences $M_{i+1} \to Q_i \to M_i$, equivalently on $Q_i \to M_i \to \Sigma M_{i+1}$, it follows that $M_i \in \mathrm{Thick}_1(R) \star \mathrm{Thick}_{n-i+1}(R)$, so that $M \in \mathrm{Thick}_{n+1}(R)$. □

Corollary 8.4.6. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring of global dimension n. Then $\mathrm{Perf}_R = \mathrm{thick}_{n+1}(R)$ and $\dim_{\mathrm{R}}(\mathrm{Perf}_R) \leq n$.

Proof. Follows from Proposition 8.4.5 and Exercise E.4.1 since R has generation time ⏱$_R(\mathrm{Perf}_R) \leq n+1$. □

Example 8.4.7. Stevenson proved in [Ste24, Proposition 1, Proposition 2 and Example 3] that there is a commutative coherent ring of infinite global dimension for which the category of perfect complexes has finite Rouquier dimension: given a field k, take $\prod_{\aleph_\omega} k$. in particular, the converse of the statement on perfect complexes of Proposition 8.4.5 does not hold.

Our next goal is to compute the Rouquier dimension of the category of perfect complexes for a noetherian regular ring using Corollary 8.4.6 as an upper bound. To do that, we need a bound in the other direction and we use the argument of [Let25], which we recast in the language of A-linear stable categories. The idea is to use the ghost lemma.

Remark 8.4.8. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}})$ be a small idempotent-complete A-linear category. As seen in Construction 2.3.7, the action functor admits an ind-right adjoint. In the case where the base is a module category over a ring spectrum, this ind-right adjoint produces an A-module action on the mapping spectrum $\hom_{\mathcal{C}}(x,y)$ for every $x, y \in \mathcal{C}$. See Exercise E.8.1

Proposition 8.4.9. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}})$. Then the homotopy category $h\mathcal{C}$ is naturally π_*A-linear. Equivalently, for every homogeneous element $x \in \pi_d A$ and every object $M \in \mathcal{C}$, there is a natural morphism

$$x_M : M \simeq M \otimes_A A \to M \otimes_A \Sigma^d A \simeq \Sigma^d M$$

in $h\mathcal{C}$, and the graded groups $\pi_* \hom_{\mathcal{C}}(x,y)$ are naturally graded π_*A-modules.

Proof. First of all, a class $x \in \pi_d A$ is represented by a map $x : A \to \Sigma^d A$ in $\mathrm{Perf}(A)$; indeed, $\pi_d A \cong \pi_d \hom(A,A) \cong \pi_0 \hom(A, \Sigma^d A)$ as $\pi_0 A$-modules. Now tensoring with M produces a morphism $x_M : M \otimes_A A \to M \otimes_A \Sigma^d A$, hence a natural transformation $\mathrm{id}_{h\mathcal{C}} \Rightarrow \Sigma^d$. This yields a morphism of graded rings $\pi_* A \to Z^*(h\mathcal{C})$ from $\pi_* A$ to the graded center of $h\mathcal{C}$. In particular, every $\pi_* \hom_{\mathcal{C}}(x,y)$ is naturally a graded π_*A-module. □

We now need refinement of the notion of Koszul complex Remark 5.3.3.

Notation 8.4.10. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathcal{C}^{\mathrm{perf}})$ be an A-linear category. Let $x \in \pi_d A$ be homogeneous. The *Koszul object of* x in $\mathrm{Perf}(A)$ is defined to be the cofibre $K_A(x) \coloneqq \mathrm{cofib}(A \xrightarrow{x} \Sigma^d A) \in \mathrm{Perf}(A)$. If $M \in \mathcal{C}$, define the *Koszul object of* x *on* M by

$$M /\!\!/ x \coloneqq \mathrm{cofib}(x_M : M \to \Sigma^d M) \in \mathcal{C}.$$

More generally, for a finite sequence $(x_1, \ldots, x_t)$ of homogeneous elements of $\pi_* A$, define recursively

$$M /\!\!/ (x_1, \ldots, x_t) \coloneqq (M /\!\!/ (x_1, \ldots, x_{t-1})) /\!\!/ x_t.$$

Lemma 8.4.11. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathcal{C}^{\mathrm{perf}})$ be an A-linear category. Let $(x_1, \ldots, x_n) \in \pi_*(A)$ be a finite sequence and $M \in \mathcal{C}$ be an object. Then there is a canonical equivalence $M /\!\!/ (x_1, \ldots, x_n) \simeq M \otimes_A K_A(x_1) \otimes_A \cdots \otimes_A K_A(x_n)$.

Proof. The action functor $- \otimes_A - : \mathcal{C} \times \mathrm{Perf}(A) \to \mathcal{C}$ is exact in each variable, hence carries the exact sequence $A \xrightarrow{x} \Sigma^{|x|} A \to K_A(x)$ to the exact sequence $M \xrightarrow{x_M} \Sigma^{|x|} M \to M \otimes_A K_A(x)$. This identifies $M /\!\!/ x$ with $M \otimes_A K_A(x)$. The iterated statement follows by induction. □

Remark 8.4.12. Let $A \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring and regard Perf_A as a module over itself. Given $f_1, \dots, f_r \in A$, then Lemma 8.4.11 implies that the Koszul complex $A /\!\!/ (f_1, \dots, f_r)$ of Notation 8.4.10 coincides with the Koszul complex $K^\bullet(f_1, \dots, f_r)$ of Remark 5.3.3.

Definition 8.4.13. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathrm{Cat}^{\mathrm{perf}})$ be a A-linear category. Let $M \in \mathcal{C}$ be an object and regard $\pi_* \mathrm{End}_{\mathcal{C}}(M)$ as a graded $\pi_* A$-module.

(1) A sequence of homogeneous elements $x_1, \dots, x_t \in \pi_* A$ is said to be *M-regular* if for every $1 \le i \le t$, multiplication by x_i induces an injective morphism
$$x_i : \pi_* \mathrm{End}_{\mathcal{C}}(M)/(x_1, \dots, x_{i-1})\pi_* \mathrm{End}_{\mathcal{C}}(M) \to \Sigma^{|x_i|}(\pi_* \mathrm{End}_{\mathcal{C}}(M)/(x_1, \dots, x_{i-1})\pi_* \mathrm{End}_{\mathcal{C}}(M)),$$
and moreover
$$\pi_* \mathrm{End}_{\mathcal{C}}(M)/(x_1, \dots, x_t)\pi_* \mathrm{End}_{\mathcal{C}}(M) \neq 0.$$

(2) Let $\mathfrak{a} \subseteq \pi_* A$ be a homogeneous ideal. We define the *$\mathfrak{a}$-depth of M* by
$$\mathrm{depth}_A(\mathfrak{a}, M) \coloneqq \mathrm{depth}_{\pi_* A}(\mathfrak{a}, \pi_* \mathrm{End}_{\mathcal{C}}(M)).$$
Thus $\mathrm{depth}_A(\mathfrak{a}, M)$ is the supremum of the lengths of $\pi_* \mathrm{End}_{\mathcal{C}}(M)$-regular sequences contained in $\mathfrak{a}$, provided $\mathfrak{a}\pi_* \mathrm{End}_{\mathcal{C}}(M) \neq \pi_* \mathrm{End}_{\mathcal{C}}(M)$, and it is ∞ otherwise.

Notice that $\mathrm{depth}_A(\mathfrak{a}, M)$ measures how many homogeneous scalars in $\mathfrak{a}$ can act successively as non-zero-divisors on the graded endomorphism module of M. The point of Letz's argument is that such a regular sequence produces a non-zero composition of ghost maps, and therefore a lower bound for generation time.

Definition 8.4.14. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathrm{Cat}^{\mathrm{perf}})$ be a A-linear category. We will say that $\mathcal{C}$ is *Ext-noetherian over A* if for every pair of objects $X, Y \in \mathcal{C}$ the graded $\pi_* A$-module $\pi_* \hom_{\mathcal{C}}(X, Y)$ is noetherian.

Lemma 8.4.15. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathrm{Cat}^{\mathrm{perf}})$ be a A-linear category. Fix $G \in \mathcal{C}$ and suppose that $\pi_* \hom_{\mathcal{C}}(G, M)$ is noetherian over $\pi_* A$. Then $\pi_* \hom_{\mathcal{C}}(G, N)$ is noetherian for every $N \in \mathrm{thick}(M)$. In particular, $\pi_* \hom_{\mathcal{C}}(G, M /\!\!/ \underline{x})$ is noetherian for every finite sequence $\underline{x}$ of homogeneous elements of $\pi_* A$.

Proof. Let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory of those $N \in \mathcal{C}$ such that $\pi_* \hom_{\mathcal{C}}(G, N)$ is noetherian over $\pi_* A$. Since noetherian graded modules are closed under shifts, finite direct sums, direct summands, and extensions, the subcategory $\mathcal{D}$ is thick. As $M \in \mathcal{D}$, it follows that $\mathrm{thick}(M) \subseteq \mathcal{D}$. ☐

Lemma 8.4.16. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathrm{Cat}^{\mathrm{perf}})$ be a A-linear category. Let $G, M \in \mathcal{C}$ and let $x \in \pi_d A$ be homogeneous. Assume that $\pi_* \hom_{\mathcal{C}}(G, M)$ is noetherian over $\pi_* A$. Then for $n \gg 0$ the composite
$$\Sigma^{-1}(M /\!\!/ x^{n+1}) \to M \xrightarrow{x_M^n} \Sigma^{nd} M$$
is G-ghost.

Proof. Since $\mathcal{C}$ is Ext-noetherian over A, the graded module $\pi_* \hom_{\mathcal{C}}(G, N)$ is noetherian over $\pi_* A$. In particular, its x-power torsion submodule $\Gamma_{(x)} \pi_* \hom_{\mathcal{C}}(G, N) \coloneqq \{\alpha \in \pi_* \hom_{\mathcal{C}}(G, N) \mid x^m \alpha = 0 \text{ for some } m \ge 0\}$ is finitely generated. Hence there exists $n \ge 0$ such that $x^n \Gamma_{(x)} \pi_* \hom_{\mathcal{C}}(G, M) = 0$. Consider now the exact sequence $M \to \Sigma^{(n+1)d} M \to M /\!\!/ x^{n+1}$ defining the Koszul object of $x^{n+1} \in \pi_{(n+1)d}$ on M. Rotating and applying $\pi_* \hom_{\mathcal{C}}(G, -)$ produces an exact sequence
$$\pi_* \hom_{\mathcal{C}}(G, \Sigma^{-1}(M /\!\!/ x^{n+1})) \to \pi_* \hom_{\mathcal{C}}(G, M) \xrightarrow{x^{n+1}} \pi_{*-(n+1)d} \hom_{\mathcal{C}}(G, M)$$

Thus the image of the first map is the kernel of multiplication by x^{n+1}, hence it is contained in $\Gamma_{(x)}\pi_* \operatorname{hom}_{\mathcal{C}}(G,M)$. Multiplication by x^n therefore kills this image, so the composite

$$\pi_* \operatorname{hom}_{\mathcal{C}}(G,\Sigma^{-1}(M/\!\!/x^{n+1})) \to \pi_* \operatorname{hom}_{\mathcal{C}}(G,M) \xrightarrow{x^n} \pi_{*-nd} \operatorname{hom}_{\mathcal{C}}(G,M)$$

is zero, and this shows the claim. □

Lemma 8.4.17. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}_A}(\mathrm{Cat}^{\mathrm{perf}})$ be a A-linear category. Let $M \in \mathcal{C}$ and let $x_1,\dots,x_t \in \pi_*A$ be an M-regular sequence. Then there is a natural isomorphism of graded π_*A-modules

$$\pi_* \operatorname{hom}_{\mathcal{C}}(\Sigma^{-t}M/\!\!/(x_1,\dots,x_t),M) \cong \pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1,\dots,x_t)$$

carrying the canonical morphism $\Sigma^{-t}M/\!\!/(x_1,\dots,x_t) \to M$ to the class of id_M.

Proof. The proof goes by induction on t. The case $t = 0$ is tautological. Assume therefore the claim true for $t-1$ and write $\underline{x}' = (x_1,\dots,x_{t-1})$. Consider the exact sequence $M/\!\!/\underline{x}' \to \Sigma^{|x_t|}M/\!\!/\underline{x}' \to M/\!\!/(x_1,\dots,x_t)$ defining the Koszul object of $(x_1,\dots,x_t)$ on M. Rotate it to get $\Sigma^{-1}M/\!\!/(x_1,\dots,x_t) \to M/\!\!/\underline{x}' \to \Sigma^{|x_t|}M/\!\!/\underline{x}'$ and apply $\Sigma^{-(t-1)}$ to get

$$\Sigma^{-t}M/\!\!/(x_1,\dots,x_t) \to \Sigma^{-(t-1)}M/\!\!/\underline{x}' \xrightarrow{x_t} \Sigma^{|x_t|-(t-1)}M/\!\!/\underline{x}'.$$

Since x_t is regular on $\pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1,\dots,x_{t-1})$, the relevant part of the associated long exact sequence shortens to a short exact sequence and yields an identification

$$\pi_* \operatorname{hom}_{\mathcal{C}}(\Sigma^{-t}M/\!\!/(x_1,\dots,x_t),M) \cong \frac{\pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1,\dots,x_{t-1})}{x_t} \cong \pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1,\dots,x_t).$$

Here the first step follows from induction, and the second one by definition of quotients. A diagram chase shows that the canonical map is sent to the class of id_M. □

Corollary 8.4.18. Let $M \in \mathcal{C}$ and let $x_1,\dots,x_t \in \pi_*A$ be an M-regular sequence. Then for every choice of integers $n_1,\dots,n_t \geq 0$, the composite

$$\Sigma^{-t}M/\!\!/(x_1^{n_1+1},\dots,x_t^{n_t+1}) \to M \xrightarrow{x_1^{n_1}\cdots x_t^{n_t}} \Sigma^{\sum_i n_i|x_i|}M$$

is non-zero in $h\mathcal{C}$.

Proof. By [?, Theorem 16.1], the sequence of powers $x_1^{n_1+1},\dots,x_t^{n_t+1}$ is again M-regular. Applying Lemma 8.4.17 to this sequence produces an isomorphism

$$\pi * \operatorname{hom}_{\mathcal{C}}(\Sigma^{-t}M/\!\!/(x_1^{n_1+1},\dots,x_t^{n_t+1}),M) \cong \pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1^{n_1+1},\dots,x_t^{n_t+1})$$

under which the canonical map to M corresponds to $[\mathrm{id}_M]$. Hence the displayed composite corresponds to the class $[x_1^{n_1}\cdots x_t^{n_t}] \in \pi_*\mathrm{End}_{\mathcal{C}}(M)/(x_1^{n_1+1},\dots,x_t^{n_t+1})$. Since $x_1,\dots,x_t$ is $\pi_*\mathrm{End}_{\mathcal{C}}(M)$-regular, this class is non-zero. □

We can finally start proving the lower bound.

Proposition 8.4.19. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}})$. Let $G,M \in \mathcal{C}$ and assume that $\mathrm{Ext}^*_{\mathcal{C}}(G,M)$ is noetherian over π_*A, and let $\mathfrak{a} \subseteq \pi_*A$ be a homogeneous ideal such that $\mathfrak{a}\,\pi_*\mathrm{End}_{\mathcal{C}}(M) \neq \pi_*\mathrm{End}_{\mathcal{C}}(M)$. Then

$$\mathrm{depth}_A(\mathfrak{a},M) + 1 \leq ⏱_{\mathcal{C}}(G).$$

Proof. Let $x_1,\dots,x_t \in \mathfrak{a}$ be an M-regular sequence. By Lemma 8.3.1 it suffices to construct a non-zero composite of t many G-ghosts. Set $M_1 := M$ and $M_s := \Sigma^{-s+1}M/\!\!/(x_1^{n_1+1},\dots,x_{s-1}^{n_{s-1}+1})$ for $2 \leq s \leq t$. Since each M_s lies in $\mathrm{thick}(M)$, Lemma 8.4.15 implies that $\pi^* \operatorname{hom}_{\mathcal{C}}(G,M_s)$ is noetherian for all s.

Choose now the integers $n_1,\dots,n_t$ inductively. Suppose that $n_1,\dots,n_{s-1}$ have already been chosen. Applying Lemma 8.4.16 to the object M_s and to the homogeneous element $x_s \in \pi_*A$ produces $n_s \geq 0$

such that the composite $\Sigma^{-1}(M_s /\!\!/ x_s^{n_s+1}) \to M_s \xrightarrow{x_s^{n_s}} \Sigma^{n_s|x_s|}M_s$ is G-ghost. After rewriting this in terms of iterated Koszul objects, it follows the existence of a G-ghost morphism

$$\Sigma^{-s}M /\!\!/ (x_1^{n_1+1}, \dots, x_s^{n_s+1}) \to \Sigma^{n_s|x_s|-s+1}M /\!\!/ (x_1^{n_1+1}, \dots, x_{s-1}^{n_{s-1}+1}).$$

Composing these t ghost maps for $s = t, t-1, \dots, 1$, and using the commutativity of the π_*A-action produces the composite

$$\Sigma^{-t}M /\!\!/ (x_1^{n_1+1}, \dots, x_t^{n_t+1}) \to M \xrightarrow{x_1^{n_1}\cdots x_t^{n_t}} \Sigma^{\sum_i n_i|x_i|}M.$$

By Corollary 8.4.18, this morphism is non-zero. This proves the claim since it forces $\text{⏱}_m athcalC(G) \geq t+1 = \operatorname{depth}_A(\mathfrak{a}, M) + 1$. □

Theorem 8.4.20. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}})$ be Ext-noetherian over A. Let $M \in \mathcal{C}$ and let $\mathfrak{a} \subseteq \pi_*A$ be a homogeneous ideal such that $\mathfrak{a}\,\pi_*\mathrm{End}_{\mathcal{C}}(M) \neq \pi_*\mathrm{End}_{\mathcal{C}}(M)$. Then

$$\operatorname{depth}_A(\mathfrak{a}, M) \leq \dim_R(\mathcal{C}).$$

Proof. If the right hand side is infinite, then there is nothing to prove. Assume therefore that it is finite and let $G \in \mathcal{C}$ be a strong generator. Since $\mathcal{C}$ is Ext-noetherian over A, the graded π_*A-module $\pi_* \hom_{\mathcal{C}}(G, M)$ is noetherian. Apply Proposition 8.4.19 to get $\operatorname{depth}_A(\mathfrak{a}, M) \leq \text{⏱}_{\mathcal{C}}(G) - 1$. Taking the infimum over all strong generators G gives $\operatorname{depth}_A(\mathfrak{a}, M) \leq \dim_R(\mathcal{C})$. □

We can finally compute the Rouquier dimension of a noetherian regular. Recall that a noetherian ring has finite global dimension n if and only if it is regular of Krull dimension n.

Theorem 8.4.21 ([Let25, Theorem 1])**.** Let R be a classical commutative noetherian regular ring. Then $\dim_{\mathrm{R}}(\mathrm{Perf}(R)) = \dim(R)$.

Proof. By confusing R with its Eilenberg-MacLane spectrum, it follows that $\mathrm{Perf}(R)$ is Ext-noetherian over R. Assume first that $(R, \mathfrak{m})$ is local so that $\operatorname{depth}_R(\mathfrak{m}, R) = \dim(R)$. Since $\pi_* \hom_{\mathrm{Perf}_R}(R, R) \simeq R$, an application of Theorem 8.4.20 with $M = R \in \mathrm{Perf}(R)$ shows that

$$\dim(R) = \operatorname{depth}_R(\mathfrak{m}, R) \leq \dim_{\mathrm{R}}(\mathrm{Perf}_R)$$

and since the opposite inequality holds from Corollary 8.4.6, the equality follows. For the general case, suppose first that $\dim(R) < \infty$. Let $G \in \mathrm{Perf}(R)$ be a strong generator and choose a maximal ideal $\mathfrak{m} \subseteq R$ such that $\dim(R) = \dim(R_{\mathfrak{m}})$. Localization induces an exact functor $\mathrm{Perf}(R) \to \mathrm{Perf}(R_{\mathfrak{m}})$ which does not increase generation time $\text{⏱}_{\mathrm{Perf}(R_{\mathfrak{m}})}(G_{\mathfrak{m}}) \leq \text{⏱}_{\mathrm{Perf}(R)}(G)$ by Lemma 8.2.2. Since $R_{\mathfrak{m}}$ is regular local, the local case gives

$$\dim(R) = \dim(R_{\mathfrak{m}}) \leq \dim_R(\mathrm{Perf}(R_{\mathfrak{m}})) \leq \text{⏱}_{\mathrm{Perf}(R_{\mathfrak{m}})}(G_{\mathfrak{m}}) - 1 \leq \text{⏱}_{\mathrm{Perf}(R)}(G) - 1.$$

Taking the infimum over all strong generators G yields $\dim(R) \leq \dim_R(\mathrm{Perf}(R))$, and so equality (by using Corollary 8.4.6). Finally, assume that $\dim(R) = \infty$. If G were a strong generator of $\mathrm{Perf}(R)$, say with $\text{⏱}_{\mathrm{Perf}(R)}(G) = n < \infty$, then for every maximal ideal $\mathfrak{m}$ one would have

$$\dim(R_{\mathfrak{m}}) \leq \text{⏱}_{\mathrm{Perf}(R_{\mathfrak{m}})}(G_{\mathfrak{m}}) - 1 \leq n - 1.$$

This is impossible because R has maximal ideals of arbitrarily large height. Hence $\mathrm{Perf}(R)$ admits no strong generator, and therefore $\dim_R(\mathrm{Perf}(R)) = \infty = \dim(R)$. □

8.5. **A representability result.**

9. The smooth (or diagonal) dimension

In this chapter we discuss the smooth, or diagonal, dimension of a stable category relative to a base, a notion of dimension relative to a base. We begin with the definition and with the first examples, emphasizing the relation with smoothness of the diagonal bimodule. We then study the formal properties of this invariant, such as its behaviour under tensor products and gluing constructions. Finally, we compare the diagonal dimension with the Rouquier dimension, obtaining inequalities in both directions (under suitable assumptions).

9.1. The definition.

Definition 9.1.1. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring and let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{A}}$ be an $\mathcal{A}$-module. The *diagonal dimension of* $\mathcal{C}$ *over* $\mathcal{A}$ is defined as

$$\dim_{\Delta}(\mathcal{C}/\mathcal{A}) = \inf\{n \in \mathbb{N} \text{ for which there are } F, G \in \mathcal{C} \text{ such that } \Delta_{\mathcal{C}} \in \mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G)\}.$$

Notice that $\dim_{\Delta}(\mathcal{C}/\mathcal{A}) \in \mathbb{N} \cup \{+\infty\}$.

Remark 9.1.2. It is clear that the diagonal dimension is invariant under equivalences of modules.

Lemma 9.1.3. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring and let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{A}}$ be an $\mathcal{A}$-module. Then the diagonal dimension of $\mathcal{C}$ is finite if and only if $\mathcal{C}$ is smooth.

Proof. Assume first that the diagonal dimension of $\mathcal{C}$ over $\mathcal{A}$ is finite. Then there are objects $F, G \in \mathcal{C}$ and a integer $n \in \mathbb{N}$ such that $\Delta \in \mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G) \subseteq \mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}$ and this implies that the diagonal bimodule is compact, thus showing smoothness. Conversely, assume that $\mathcal{C}$ is smooth as an $\mathcal{A}$-module. Then the diagonal bimodule $\Delta \in \mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C})$ is compact. By construction of the tensor product in $\mathrm{Cat}^{\mathrm{perf}}$, the category $\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}$ is the thick closure of the essential image of the exterior tensor product $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}$ given by $(x, y) \mapsto x \otimes_{\mathcal{A}} y$. Hence there exist finitely many objects $x_1, \dots, x_r, y_1, \dots, y_r \in \mathcal{C}$ and an integer $m \in \mathbb{N}$ such that $\Delta \in \mathrm{thick}_m(\{x_i \otimes_{\mathcal{A}} y_i\}_{i=1}^r)$. Set $F \coloneqq \bigoplus_{i=1}^r x_i$ and $G \coloneqq \bigoplus_{i=1}^r y_i$. Then $\mathrm{thick}_m(\{x_i \otimes_{\mathcal{A}} y_i\}_{i=1}^r) \subseteq \mathrm{thick}_m(F \otimes_{\mathcal{A}} G)$ since $F \otimes_{\mathcal{A}} G \simeq \bigoplus_{i,j} x_i \otimes_{\mathcal{A}} y_j$ and each $x_i \otimes_{\mathcal{A}} y_i$ is a retract of $F \otimes_{\mathcal{A}} G$. It follows that $\Delta \in \mathrm{thick}_m(F \otimes_{\mathcal{A}} G)$ and thus $\dim_{\Delta}(\mathcal{C}/\mathcal{A}) < \infty$. □

We now show that the diagonal dimension depends on the base.

Example 9.1.4. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring. Since $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{A}$, and since the identity belongs to $\mathrm{thick}_1(\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}}) \simeq \mathrm{thick}_1(\mathrm{id}_{\mathcal{A}})$, it follows that $\dim_{\Delta}(\mathcal{A}/\mathcal{A}) = 0$.

Lemma 9.1.5. Let K/k be a finite field extension. Then

$$\dim_{\Delta}(\mathrm{Perf}_K/\mathrm{Perf}_k) = \begin{cases} 0 & \text{if } K/k \text{ is separable,} \\ +\infty & \text{if } K/k \text{ is inseparable.} \end{cases}$$

Proof. Let $\mu : K \otimes_k K \to K$ be the multiplication map. Under the equivalence $\mathrm{Perf}(K)^{\mathrm{op}} \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K) \simeq \mathrm{Perf}(K \otimes_k K)$ the diagonal bimodule of $\mathrm{Perf}(K)$ over $\mathrm{Perf}(k)$ corresponds to K, regarded as a $K \otimes_k K$-module via μ.

Assume first that K/k is separable. By [Aut18, Lemma 00U3], the map $k \to K$ is étale. Hence, after base change along $k \to K$, the map $K \to K \otimes_k K$ is again étale. Since R is a finite K-algebra, [Aut18, Proposition 03PC] implies that $K \otimes_k K \simeq \prod_{i=1}^n K_i$ for some finite separable field extensions K_i/K. Now the multiplication map $\mu : K \otimes_k K \to K$ is a K-algebra map. Therefore it factors through one of the factors $K_j \to K$. Since K_j is a field, this map is injective, hence an isomorphism. It follows that K is a direct summand of $K \otimes_k K$ as a $K \otimes_k K$-module. In particular, K is a finitely generated projective $K \otimes_k K$-module, hence $K \in \mathrm{Perf}(K \otimes_k K)$. Therefore the diagonal belongs to $\langle K \otimes_k K \rangle_1$, so that

$$\dim_{\Delta}(\mathrm{Perf}(K)/\mathrm{Perf}(k)) = 1.$$

Assume now that K/k is inseparable. Let $k_s \subseteq K$ be the maximal separable subextension of K/k. Then K/k_s is finite purely inseparable. Suppose for contradiction that $\dim_\Delta(\mathrm{Perf}(K)/\mathrm{Perf}(k)) < \infty$. By Lemma 9.1.3, this means that the diagonal is perfect, that is, that K is a perfect $K \otimes_k K$-module. Since k_s/k is finite separable, the K-algebra $k_s \otimes_k K$ is finite étale over K. Hence, by [Aut18, Proposition 03PC], it is a finite product of finite separable field extensions of K. The multiplication map $k_s \otimes_k K \to K$ is a K-algebra map, so one of these factors is isomorphic to K. Thus one may write $k_s \otimes_k K \simeq K \times S$ for some finite étale K-algebra S. Tensoring over k_s with K gives

$$K \otimes_k K \simeq K \otimes_{k_s} (k_s \otimes_k K) \simeq (K \otimes_{k_s} K) \times (K \otimes_{k_s} S).$$

Under this decomposition, the multiplication map $\mu : K \otimes_k K \to K$ factors through the first factor $K \otimes_{k_s} K \to K$, namely through the multiplication map for the extension K/k_s. Since perfect complexes over a product split componentwise, the assumption that K is perfect as an $K \otimes_k K$-module implies that K is perfect as a module over $K \otimes_{k_s} K$. But this is a contradiction[6] since K/k_s is a finite purely inseparable extension. Therefore K is not perfect as an $K \otimes_k K$-module and Lemma 9.1.3 implies that the diagonal dimension cannot be finite. Hence

$$\dim_\Delta(\mathrm{Perf}(K)/\mathrm{Perf}(k)) = \infty.$$

This proves the proposition. □

Example 9.1.6. Let $k = F_p(t)$ and $K = F_p(t^{1/p})$. Then K/k is an inseparable finite field extension, so that $\dim_\Delta(\mathrm{Perf}_{F_p(t^{1/p})}/\mathrm{Perf}_{F_p(t)}) = +\infty$ whereas $\dim_\Delta(\mathrm{Perf}_{F_p(t^{1/p})}/\mathrm{Perf}_{F_p(t^{1/p})}) = 0$ by Example 9.1.4.

9.2. **Formal properties of diagonal dimension.** We now discuss the structural properties of the diagonal dimension.

Lemma 9.2.1. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E} \in \mathrm{Cat}^{\mathrm{perf}}$ be small idempontent-complete stable ctageories and let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor exact in each variable separately. Then for every $n, m \in \mathbb{N}$ and every $x \in \mathcal{C}$ and $y \in \mathcal{D}$ the functor F restricts to $F : \mathrm{thick}_n(x) \times \mathrm{thick}_m(y) \to \mathrm{thick}_{n+m-1}(F(x,y))$.

Proof. Let $z \in \mathrm{thick}_n(x)$ and $w \in \mathrm{thick}_m(y)$. Then Lemma 4.2.9 implies the existence of a filtration $0 = z_0 \to \cdots \to z_n = z$ such that the cofibre $\mathrm{cofib}(z_{i-1} \to z_i)$ belongs to $\mathrm{thick}_1(x)$ for every $i = 1, \ldots, n$. Similarly, there exists a filtration $0 = w_0 \to \cdots \to w_m = w$ such that the cofibre $\mathrm{cofib}(w_{i-1} \to w_i)$ belongs to $\mathrm{thick}_1(y)$ for every $i = 1, \ldots, m$. By Lemma 4.2.9 it suffices to define a filtration on $F(z,w)$ of lenght $n + m - 1$ such that the successive cofibres lie in $\mathrm{thick}_1(F(x,y))$. To define such a filtration consider the poset on (i,j) with $0 \le i \le n$ and $0 \le j \le m$ ordered by $(i,j) \le (i',j')$ if and only if $i \le i'$ and $j \le j'$. For $k = 0, \ldots, n+m$, let $T_k = \{(i,j) \mid i + j \le k\}$ be the triangular part of the poset and define $v_n = \mathrm{colim}_{(i+j) \in T_k} F(z_i, w_j)$. Then the inclusions $T_k \hookrightarrow T_{k+1}$ produce a filtration

$$0 = v_0 \to v_1 \to \cdots \to v_{n+m} = F(z,w).$$

Notice that $v_1 = 0$ since the only points with $i + j \le 1$ have $i = 0$ or $j = 0$. Therefore the above is a filtration with lenght $n + m - 1$. It is not hard to see that

$$\mathrm{cofib}(v_{k-1} \to v_k) \simeq \bigoplus_{i+j=k} F(\mathrm{cofib}(z_{i-1} \to z_i), \mathrm{cofib}(w_{j-1} \to w_j))$$

for every $k = 0, \ldots n+m$. Now $F(\mathrm{thick}_1(x), \mathrm{thick}_1(y)) \subseteq \mathrm{thick}_1(F(x,y))$ proving the claim. □

The diagonal dimension plays well with tensor products.

Proposition 9.2.2. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring, and let $\mathcal{B}, \mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be two $\mathcal{A}$-modules. Then

$$\dim_\Delta(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}/\mathcal{A}) \le \dim_\Delta(\mathcal{B}/\mathcal{A}) + \dim_\Delta(\mathcal{C}/\mathcal{A}).$$

[6]Exercise!

Proof. If one of the two dimensions on the right-hand side is infinite, there is nothing to prove. Thus assume that $n \coloneqq \dim_{\Delta}(\mathcal{B}/\mathcal{A})$ and $m \coloneqq \dim_{\Delta}(\mathcal{C}/\mathcal{A})$ are finite. By definition, there exist objects $F_{\mathcal{B}}, G_{\mathcal{B}} \in \mathcal{B}$ and $F_{\mathcal{C}}, G_{\mathcal{C}} \in \mathcal{C}$ such that $\Delta_{\mathcal{B}} \in \mathrm{thick}_{n+1}(F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{B}})$ and $\Delta_{\mathcal{C}} \in \mathrm{thick}_{m+1}(F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}})$. Consider the exact bifunctor

$$- \otimes_{\mathcal{A}} - : (\mathcal{B}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{B}) \times (\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}) \to (\mathcal{B}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{B}) \otimes_{\mathcal{A}} (\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}) \simeq (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C})^{\mathrm{op}} \otimes_{\mathcal{A}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C})$$

The diagonal bimodule of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$ is $\Delta_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}} \simeq \Delta_{\mathcal{B}} \otimes_{\mathcal{A}} \Delta_{\mathcal{C}}$. Applying Lemma 9.2.1 to the bifunctor $- \otimes_{\mathcal{A}} -$, shows that

$$\Delta_{\mathcal{B}} \otimes_{\mathcal{A}} \Delta_{\mathcal{C}} \in \mathrm{thick}_{n+m+1}((F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{B}}) \otimes_{\mathcal{A}} (F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}})).$$

Using again the symmetry constraints, the generator on the right identifies with $(F_{\mathcal{B}} \otimes_{\mathcal{A}} F_{\mathcal{C}}) \otimes_{\mathcal{A}} (G_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{C}}) \in (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C})^{\mathrm{op}} \otimes_{\mathcal{A}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C})$. Therefore

$$\Delta_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}} \in \mathrm{thick}_{n+m+1}((F_{\mathcal{B}} \otimes_{\mathcal{A}} F_{\mathcal{C}}) \otimes_{\mathcal{A}} (G_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{C}}))$$

and hence $\dim_{\Delta}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}/\mathcal{A}) \leq n + m$. □

Lemma 9.2.3. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring, and let $\mathcal{B}, \mathcal{C} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be two $\mathcal{A}$-modules. Then

$$\dim_{\Delta}(\mathcal{B} \oplus \mathcal{C}/\mathcal{A}) = \max\{\dim_{\Delta}(\mathcal{B}/\mathcal{A}), \dim_{\Delta}(\mathcal{C}/\mathcal{A})\}.$$

Proof. Set $n \coloneqq \dim_{\Delta}(\mathcal{B}/\mathcal{A})$ and $m \coloneqq \dim_{\Delta}(\mathcal{C}/\mathcal{A})$. Consider first the upper bound. If one of n, m is infinite there is nothing to prove, so assume they are finite and, without loss of generality, $n \geq m$. Choose objects $F_{\mathcal{B}}, G_{\mathcal{B}} \in \mathcal{B}$ and $F_{\mathcal{C}}, G_{\mathcal{C}} \in \mathcal{C}$ such that $\Delta_{\mathcal{B}} \in \mathrm{thick}_{n+1}(F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{B}})$ and $\Delta_{\mathcal{C}} \in \mathrm{thick}_{m+1}(F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}})$. Set $F \coloneqq (F_{\mathcal{B}}, F_{\mathcal{C}}) \in \mathcal{B} \oplus \mathcal{C}$ and $G \coloneqq (G_{\mathcal{B}}, G_{\mathcal{C}}) \in \mathcal{B} \oplus \mathcal{C}$. Then

$$F \otimes_{\mathcal{A}} G \simeq (F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{B}}) \oplus (F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{C}}) \oplus (F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{B}}) \oplus (F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}}),$$

so both $F_{\mathcal{B}} \otimes_{\mathcal{A}} G_{\mathcal{B}}$ and $F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}}$ are retracts of $F \otimes_{\mathcal{A}} G$. Hence $\Delta_{\mathcal{B}} \in \mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G)$ and $\Delta_{\mathcal{C}} \in \mathrm{thick}_{m+1}(F \otimes_{\mathcal{A}} G) \subseteq \mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G)$. Under the canonical decomposition

$$(\mathcal{B} \oplus \mathcal{C})^{\mathrm{op}} \otimes_{\mathcal{A}} (\mathcal{B} \oplus \mathcal{C}) \simeq (\mathcal{B}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{B}) \oplus (\mathcal{B}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}) \oplus (\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{B}) \oplus (\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}} \mathcal{C}),$$

the diagonal bimodule identifies with $\Delta_{\mathcal{B} \oplus \mathcal{C}} \simeq \Delta_{\mathcal{B}} \oplus \Delta_{\mathcal{C}}$. Since $\mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G)$ is a thick subcategory, it is closed under finite direct sums. Therefore $\Delta_{\mathcal{B} \oplus \mathcal{C}} \in \mathrm{thick}_{n+1}(F \otimes_{\mathcal{A}} G)$ and hence

$$\dim_{\Delta}(\mathcal{B} \oplus \mathcal{C}/\mathcal{A}) \leq n = \max\{n, m\}.$$

To prove the lower bound, let $\pi_{\mathcal{B}} : \mathcal{B} \oplus \mathcal{C} \to \mathcal{B}$ and $\pi_{\mathcal{C}} : \mathcal{B} \oplus \mathcal{C} \to \mathcal{C}$ be the projections. If $\Delta_{\mathcal{B} \oplus \mathcal{C}} \in \mathrm{thick}_r(X \otimes_{\mathcal{A}} Y)$ for some $X, Y \in \mathcal{B} \oplus \mathcal{C}$, then applying the exact functors $\pi_{\mathcal{B}}^{\mathrm{op}} \otimes_{\mathcal{A}} \pi_{\mathcal{B}}$ and $\pi_{\mathcal{C}}^{\mathrm{op}} \otimes_{\mathcal{A}} \pi_{\mathcal{C}}$ gives $\Delta_{\mathcal{B}} \in \mathrm{thick}_r(\pi_{\mathcal{B}}(X) \otimes_{\mathcal{A}} \pi_{\mathcal{B}}(Y))$ and $\Delta_{\mathcal{C}} \in \mathrm{thick}_r(\pi_{\mathcal{C}}(X) \otimes_{\mathcal{A}} \pi_{\mathcal{C}}(Y))$. Thus $n \leq r$ and $m \leq r$, so $\max\{n, m\} \leq r$ and taking the infimum over all such r yields

$$\max\{\dim_{\Delta}(\mathcal{B}/\mathcal{A}), \dim_{\Delta}(\mathcal{C}/\mathcal{A})\} \leq \dim_{\Delta}(\mathcal{B} \oplus \mathcal{C}/\mathcal{A}).$$

□

We have a similar statement for split Verdier sequences.

Lemma 9.2.4. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring, and let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a right split Verdier sequence in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$. Then

$$\dim_{\Delta}(\mathcal{D}/\mathcal{A}) \leq \dim_{\Delta}(\mathcal{C}/\mathcal{A}) + \dim_{\Delta}(\mathcal{E}/\mathcal{A}) + 1.$$

Proof. If one of the two dimensions on the right-hand side is infinite, there is nothing to prove. Thus assume that $n \coloneqq \dim_{\Delta}(\mathcal{C}/\mathcal{A})$ and $m \coloneqq \dim_{\Delta}(\mathcal{E}/\mathcal{A})$ are finite. By definition, there exist objects $F_{\mathcal{C}}, G_{\mathcal{C}} \in \mathcal{C}$ and $F_{\mathcal{E}}, G_{\mathcal{E}} \in \mathcal{E}$ such that $\Delta_{\mathcal{C}} \in \mathrm{thick}_{n+1}(F_{\mathcal{C}} \otimes_{\mathcal{A}} G_{\mathcal{C}})$ and $\Delta_{\mathcal{E}} \in \mathrm{thick}_{m+1}(F_{\mathcal{E}} \otimes_{\mathcal{A}} G_{\mathcal{E}})$. Let $q : \mathcal{E} \to \mathcal{D}$ be the fully faithful right adjoint of p. Since right split Verdier sequences are equivalent to semiorthogonal decompositions by Lemma 3.3.7 the essential image of i is a right admissible subcategory of $\mathcal{D}$. Denote by $i^! : \mathcal{D} \to \mathcal{C}$ the right adjoint of i. Then there is an exact sequence of exact $\mathcal{A}$-linear endofunctors $ii^! \to \mathrm{id}_\mathcal{D} \to qp$. Passing to the corresponding kernels in $\mathrm{Ind}(\mathcal{D}^{\mathrm{op}} \otimes_\mathcal{A} \mathcal{D})$, one gets an exact triangle

$$K_\mathcal{C} \to \Delta_\mathcal{D} \to K_\mathcal{E},$$

where $K_\mathcal{C} \simeq (i^{\mathrm{op}} \otimes_\mathcal{A} i)(\Delta_\mathcal{C})$ and $K_\mathcal{E} \simeq (q^{\mathrm{op}} \otimes_\mathcal{A} q)(\Delta_\mathcal{E})$. Since $i^{\mathrm{op}} \otimes_\mathcal{A} i$ and $q^{\mathrm{op}} \otimes_\mathcal{A} q$ are exact, they preserve thickenings. Therefore $K_\mathcal{C} \in \mathrm{thick}_{n+1}(i(F_\mathcal{C}) \otimes_\mathcal{A} i(G_\mathcal{C}))$ and $K_\mathcal{E} \in \mathrm{thick}_{m+1}(q(F_\mathcal{E}) \otimes_\mathcal{A} q(G_\mathcal{E}))$. Set $F := i(F_\mathcal{C}) \oplus q(F_\mathcal{E})$ and $G := i(G_\mathcal{C}) \oplus q(G_\mathcal{E})$. Then

$$F \otimes_\mathcal{A} G \simeq (i(F_\mathcal{C}) \otimes_\mathcal{A} i(G_\mathcal{C})) \oplus (i(F_\mathcal{C}) \otimes_\mathcal{A} q(G_\mathcal{E})) \oplus (q(F_\mathcal{E}) \otimes_\mathcal{A} i(G_\mathcal{C})) \oplus (q(F_\mathcal{E}) \otimes_\mathcal{A} q(G_\mathcal{E})),$$

so both $i(F_\mathcal{C}) \otimes_\mathcal{A} i(G_\mathcal{C})$ and $q(F_\mathcal{E}) \otimes_\mathcal{A} q(G_\mathcal{E})$ are retracts of $F \otimes_\mathcal{A} G$. Hence $K_\mathcal{C} \in \mathrm{thick}_{n+1}(F \otimes_\mathcal{A} G)$ and $K_\mathcal{E} \in \mathrm{thick}_{m+1}(F \otimes_\mathcal{A} G)$. Since $\Delta_\mathcal{D}$ sits in an exact triangle with left term in $\mathrm{thick}_{n+1}(F \otimes_\mathcal{A} G)$ and right term in $\mathrm{thick}_{m+1}(F \otimes_\mathcal{A} G)$, it follows from the additivity of the thickening filtration that

$$\Delta_\mathcal{D} \in \mathrm{thick}_{n+1}(F \otimes_\mathcal{A} G) * \mathrm{thick}_{m+1}(F \otimes_\mathcal{A} G) \subseteq \mathrm{thick}_{n+m+2}(F \otimes_\mathcal{A} G).$$

Therefore

$$\dim_\Delta(\mathcal{D}/\mathcal{A}) \le n + m + 1 = \dim_\Delta(\mathcal{C}/\mathcal{A}) + \dim_\Delta(\mathcal{E}/\mathcal{A}) + 1,$$

as claimed. □

We are left to study the behaviour of the diagonal dimension under essentially surjective functors.

Remark 9.2.5. Mere essential surjectivity up to direct summands does not control the diagonal dimension.

(1) There exists an exact essentially surjective functor $\mathrm{Perf}(k[x]) \to \mathrm{Perf}(k[x]/(x^2))$ such that $\dim_\Delta(\mathrm{Perf}(k[x])/\mathrm{Perf}(k)) < \infty$ and $\dim_\Delta(\mathrm{Perf}(k[x]/(x^2))/\mathrm{Perf}(k)) = \infty$. Indeed, the extension of scalars $- \otimes_{k[x]} k[x]/(x^2) : \mathrm{Perf}(k[x]) \to \mathrm{Perf}(k[x]/(x^2))$ is essentially surjective, since its image contains the unit $k[x]/(x^2)$, which generates $\mathrm{Perf}(k[x]/(x^2))$ as a thick subcategory. On the other hand, $k[x]$ is smooth over k, whereas $k[x]/(x^2)$ is not smooth over k. The claim then follows from Lemma 9.1.3.

(2) There exists an exact essentially surjective functor $\mathrm{Perf}(k[\varepsilon]/(\varepsilon^2)) \to \mathrm{Perf}(k)$ such that $\dim_\Delta(\mathrm{Perf}(k[\varepsilon]/(\varepsilon^2))/\mathrm{Perf}(k)) = \infty$ and $\dim_\Delta(\mathrm{Perf}(k)/\mathrm{Perf}(k)) = 0$. Indeed, the extension of scalars $- \otimes_{k[\varepsilon]/(\varepsilon^2)} k : \mathrm{Perf}(k[\varepsilon]/(\varepsilon^2)) \to \mathrm{Perf}(k)$ is essentially surjective, since its image contains the unit k. Moreover, $k[\varepsilon]/(\varepsilon^2)$ is not smooth over k, while $\mathrm{Perf}(k)$ has diagonal dimension 0.

9.3. **The comparison with the Rouquier dimension.** We now explore the relations between the Rouquier and diagonal dimension.

Example 9.3.1. One may expect that the diagonal dimension bounds for above the Rouquier one, in that $\dim_\mathrm{R}(\mathcal{C}) \le \dim_\Delta(\mathcal{C}/\mathcal{A})$ for $\mathcal{C} \in \mathrm{Mod}_\mathcal{A}(\mathrm{Cat}^{\mathrm{perf}})$ and $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ a small rigid 2-ring. This is in general false. Indeed, let k be a field and let $R = k[x,y]$ and set $\mathcal{A} = \mathcal{C} = \mathrm{Perf}(R)$. Thne Example 9.1.4 implies that $\dim_\Delta(\mathcal{A}/\mathcal{A}) = 0$. On the other hand, R is a classical commutative noetherian regular ring of Krull dimension 2, so Theorem 8.4.21 gives $\dim_\mathrm{R}(\mathrm{Perf}(R)) = \dim(R) = 2$ so that $\dim_\mathrm{R}(\mathcal{C}) = 2 > 0 = \dim_\Delta(\mathcal{C}/\mathcal{A})$.

Proposition 9.3.2. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring, and let $\mathcal{C} \in \mathrm{Mod}_\mathcal{A}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Assume that $\mathcal{C}$ has finite diagonal dimension over $\mathcal{A}$ and that $\mathcal{A}$ has finite Rouquier dimension. Then

$$\dim_\mathrm{R}(\mathcal{C}) \le (\dim_\Delta(\mathcal{C}/\mathcal{A}) + 1)(\dim_\mathrm{R}(\mathcal{A}) + 1) - 1.$$

Proof. Let $m = \dim_\mathrm{R}(\mathcal{A})$ be the Rouquier dimension of $\mathcal{A}$. Then there exists $a \in \mathcal{A}$ such that $\mathcal{A} = \mathrm{thick}_{m+1}(a)$. Similarly, let $n = \dim_\Delta(\mathcal{C}/\mathcal{A})$ be the diagonal dimension of $\mathcal{C}$ over $\mathcal{A}$ and pick objects $F, G \in \mathcal{C}$ such that $\Delta_\mathcal{C} \in \mathrm{thick}_{n+1}(F \otimes_\mathcal{A} G)$ inside $\mathcal{C}^{\mathrm{op}} \otimes_\mathcal{A} \mathcal{C}$. Under the identification $\mathcal{C}^{\mathrm{op}} \otimes_\mathcal{A} \mathcal{C} \simeq$

$\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{C},\mathcal{C})$ of Proposition 2.3.15 the diagonal bimodule $\Delta_{\mathcal{C}}$ corresponds to the identity functor, while $F\otimes_{\mathcal{A}} G$ corresponds to the functor $x\mapsto \hom_{\mathcal{C}}(F,x)\otimes_{\mathcal{A}} G$. Since exact functors preserve the thickening filtration, it follows that for every $x\in\mathcal{C}$ one has

$$x\in \mathrm{thick}_{n+1}(\hom_{\mathcal{C}}(F,x)\otimes_{\mathcal{A}} G).$$

Now $\hom_{\mathcal{C}}(F,x)$ is an object of $\mathcal{A}$, hence $\hom_{\mathcal{C}}(F,x)\in\mathrm{thick}_{m+1}(a)$. Applying the exact functor $(-)\otimes_{\mathcal{A}} G$ gives $\hom_{\mathcal{C}}(F,x)\otimes_{\mathcal{A}} G\in\mathrm{thick}_{m+1}(a\otimes_{\mathcal{A}} G)$. Therefore the multiplicative behaviour of thickenings (proved in Lemma 4.2.9) yields $x\in\mathrm{thick}_{(n+1)(m+1)}(a\otimes_{\mathcal{A}} G)$. In particular, it follows that $\mathcal{C}=\mathrm{thick}_{(n+1)(m+1)}(a\otimes_{\mathcal{A}} G)$ and thefore that $\dim_{\mathrm{R}}(\mathcal{C})\leq (n+1)(m+1)-1$. □

We deduce the following bound.

Corollary 9.3.3. Let $\mathcal{A}\in\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring such that $\dim_{\mathrm{R}}(\mathcal{A})=0$, and let $\mathcal{C}\in\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Then

$$\dim_{\mathrm{R}}(\mathcal{C})\leq \dim_{\Delta}(\mathcal{C}/\mathcal{A}).$$

Proof. This is the special case $m=0$ of Proposition 9.3.2. □

We now prove a mild converse.

Remark 9.3.4. Let $\mathcal{C}\in\mathrm{Cat}^{\mathrm{st}}$ be a stable category and let $G\in\mathcal{C}$ be a strong generator with generation time n, so that $\mathcal{C}=\mathrm{thick}_n(G)$. Since the construction of shifts, finite coproducts, retracts and extensions is stale under taking duals, it follows that $\mathcal{C}^{\mathrm{op}}=\mathrm{thick}_n(G^{\mathrm{op}})$.

Proposition 9.3.5. Let $\mathcal{A}\in\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring, and let $\mathcal{C}\in\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ be an $\mathcal{A}$-module. Assume that $\mathcal{C}$ is smooth over $\mathcal{A}$ and that $\dim_{\mathrm{R}}(\mathcal{C})<\infty$ is finite. Then

$$\dim_{\Delta}(\mathcal{C}/\mathcal{A})\leq 2\dim_{\mathrm{R}}(\mathcal{C}).$$

Proof. Let $n=\dim_{\mathrm{R}}(\mathcal{C})$ be the Rouquier dimension of $\mathcal{C}$ and let $G\in\mathcal{C}$ be such that $\mathcal{C}=\mathrm{thick}_{n+1}(G)$. Then $\mathcal{C}^{\mathrm{op}}=\mathrm{thick}_{n+1}(G^{\mathrm{op}})$ by Remark 9.3.4. Apply now Lemma 9.2.1 (to the functor $-\otimes_{\mathcal{A}}-$) to get $\mathcal{C}^{\mathrm{op}}\otimes_{\mathcal{A}}\mathcal{C}=\mathrm{thick}_{2n+1}(G^{\mathrm{op}}\otimes G)$. Since $\mathcal{C}$ is smooth over $\mathcal{A}$, the diagonal bimodule is compact, and hence it belongs to $\Delta\in\mathcal{C}^{\mathrm{op}}\otimes_{\mathcal{A}}\mathcal{C}=\mathrm{thick}_{2n+1}(G^{\mathrm{op}}\otimes G)$, so that $\dim_{\Delta}(\mathcal{C}/\mathcal{A})\leq 2n$, which is the claim. □

The next goal is to understand how to force a module to be smooth, by using Hoschild (co)homology.

9.4. Examples.

10. Orlov's conjecture

In this chapter we turn to Orlov's conjecture. We begin with an approximation result by compact objects, following Lank and Olander, which provides an important tool for computing the Rouquier dimension of categories with a bounded t-structure. We then combine the results of the previous chapters to obtain the universal upper and lower bounds predicted by Rouquier's theory. We conclude with a proof of Orlov's conjecture in the quasi-affine regular case, following Olander's argument.

10.1. **Approximation by compact objects.** We now discuss an important tool to compute Rouquier dimension due to Lank and Olander [LO25, Proposition 3.5].

Lemma 10.1.1. Let $\mathcal{C}\in\mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure. Let $x\to y$ be a map in $\mathcal{C}$ whose cofibre lies in $\mathcal{C}_{\geq a}$. Then the induced map $\hom_{\mathcal{C}}(y,c)\to\hom_{\mathcal{C}}(x,c)$ is an equivalence for all $c\in\mathcal{C}_{\leq a-1}$.

Proof. Consider the exact sequence $x\to y\to z$ in which the cofibre lies in $\mathcal{C}_{\geq a}$. Apply $\hom_{\mathcal{C}}(-,c)$ to get the exact sequence $\hom_{\mathcal{C}}(z,c)\to\hom_{\mathcal{C}}(y,c)\to\hom_{\mathcal{C}}(x,c)$. Use that $\hom_{\mathcal{C}}(z,c)\simeq 0$ since $c\in\mathcal{C}_{\leq a-1}$. □

Lemma 10.1.2. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ and let $x \to x'$ and $y \to y'$ be two maps. Assume:

(1) The cofibre of $x \to x'$ lies in $\mathcal{C}_{\geq a+N}$ and $x' \in \mathcal{C}_{\leq a-1}$.
(2) The cofibre of $y \to y'$ lies in $\mathcal{C}_{\geq a}$ and $y' \in \mathcal{C}_{\leq a+N-1}$.

If x is a retract of y, then x' is a retract of y'.

Proof. By assumption there exists a retract $x \xrightarrow{i} y \xrightarrow{r} x$. It suffices to complete the dotted diagram

$$\begin{array}{ccccc} x & \xrightarrow{i} & y & \xrightarrow{r} & x \\ \downarrow & & \downarrow & & \downarrow \\ x' & \dashrightarrow & y' & \dashrightarrow & x' \end{array}$$

to a solid one, ensuring that the composite is the identity. The goal is to apply Lemma 10.1.1 to construct the dotted maps.

(1) To construct $x' \to y'$, consider the map $x \to x'$. By assumption its cofibre lies in $\mathcal{C}_{\geq a+N}$ and since $y' \in \mathcal{C}_{\leq a+N-1}$, Lemma 10.1.1 allows to lift the map $x \xrightarrow{i} y \to y'$ to a map $x' \to y'$ making the diagram commute.
(2) Similary, to construct $y' \to x'$, consider the map $y \to y'$. Its cofibre lies in $\mathcal{C}_{\geq a}$ and since $x' \in \mathcal{C}_{\leq a-1}$ Lemma 10.1.1 allows to lift the map $y \xrightarrow{r} x \to x'$ to a map $y' \to x'$ making the diagram commute.

The same can be done to ensure that $x' \to y' \to x'$ is the identity since $N \in \mathbb{N}$. □

Lemma 10.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure. Let $G \in \mathcal{C}$ be a bounded object and $d \in \mathbb{N}$. There exists an $N \in \mathbb{N}$ such that, for every $a \in \mathbb{Z}$ and every $x \in \mathrm{coprod}_d(G)$, then there exist an object $x' \in \mathrm{coprod}_d(G) \cap \mathcal{C}_{\leq a+N-1}$ and a map $x \to x'$ whose cofibre lies in $\mathcal{C}_{\geq a}$.

Proof. Let $\mathrm{amp}(G) \in \mathbb{N}$ the amplitude of G, introduced in Lemma 3.5.5. The proof goes by induction on d. For $d = 1$ the integer $\mathrm{amp}(G)$ works. Indeed, if $x \in \mathrm{coprod}_1(G)$ then there exists an equivalence $x \simeq \bigoplus_i \Sigma^{n_i} G$ where the sum is finite. Notice that every shift $\Sigma^{n_i} G$ has amplitude $\mathrm{amp}(G)$. Given $a \in \mathbb{Z}$, define.

(1) The obejct x' to be the sum over those indexes i such that $\Sigma^{n_i} G \notin \mathcal{C}_{\geq a}$. In this case, $x' \in \mathcal{C}_{\leq N-a}$.
(2) The object x'' to be the sum over those indexes i such that $\Sigma^{n_i} G \in \mathcal{C}_{\geq a}$.

Thus $x \simeq x' \oplus x''$ and the projection map $x \to x''$ does the job. Assume thefore the claim for $d-1$. To prove the claim for d, set $N = N_{d-1} + \mathrm{amp}(G) + 1$. Let $x \in \mathrm{coprod}_d(G)$ and $a \in \mathbb{Z}$. Then x fits into an exact sequence $y \to x \to z$ in which $z \in \mathrm{coprod}_1(G)$ and $y \in \mathrm{coprod}_{d-1}(G)$. Rotating produces an exact sequnce $\Sigma^{-1} z \to y \to x$ in which $\Sigma^{-1} z \in \mathrm{coprod}_1(G)$. By the inductive hypothesis, there exists a map $y \to y'$ with $y' \in \mathrm{coprod}_{d-1}(G) \cap \mathcal{C}_{\leq N_{d-1}-a}$ whose cofibre lies in $\mathcal{C}_{\geq a}$. Split $w = \Sigma^{-1} z$ as in the base case in $w' \in \mathrm{coprod}_1(G) \cap \mathcal{C}_{\leq N_{d-1}-a}$ and $w'' \in \mathrm{coprod}_1(G) \cap \mathcal{C}_{\geq N_{d-1}-a+1}$. Consider then the solid diagram

$$\begin{array}{ccccc} w & \longrightarrow & y & \longrightarrow & x \\ \downarrow & & \downarrow & & \vdots \\ w' & \dashrightarrow & y' & \dashrightarrow & x' \end{array}$$

First, there is a map $w' \to y'$. This can be proved via Lemma 10.1.1: the cofibre of $w \to w'$ is $w'' \in \mathcal{C}_{\geq N_{d-1}-a+1}$ and y' lies in $\mathcal{C}_{\leq N_{d-1}-a}$. Complete now the diagram and conclude. □

Lemma 10.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure. Let $G \in \mathcal{C}$ be a bounded object and let $d \in \mathbb{N}$. Let $x \in \mathcal{C}$ be an object. There exists an $N \in \mathbb{N}$ such that the followings:

(1) The object $x \in \mathcal{C}_{\leq a-1}$ for some $a \in \mathbb{Z}$.
(2) There is a map $y \to x$ whose cofibre lies in $\mathcal{C}_{\geq a+N}$.

(3) The object y belongs to $\mathrm{thick}_d(G)$.

imply that $x \in \mathrm{thick}_d(G)$.

Proof. Suppose (1)-(2) and (3) are true for the integer $N \in \mathbb{N}$ found in Lemma 10.1.3. Since $y \in \mathrm{thick}_d(G)$, the object y is a retract of an object y' in $\mathrm{coprod}_d(G)$ by Lemma 4.2.11. Hence Lemma 10.1.3 provides a map $y' \to y''$ with $y'' \in \mathrm{coprod}_d(G) \cap \mathcal{C}_{\leq a+N-1}$ and cofibre in $\mathcal{C}_{\geq a}$. Apply Lemma 10.1.2 to get that x is a retract of y''. □

Definition 10.1.5. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure and let $\mathcal{P} \subseteq \mathcal{C}$ be a collection of objects. We will say that $\mathcal{P}$ *approximates* $\mathcal{C}$ if for every object $x \in \mathcal{C}$ and integer $a \in \mathbb{N}$, there is a map $p \to x$ with $p \in \mathcal{P}$ whose cofibre lies in $\mathcal{C}_{\geq a}$.

Example 10.1.6. Let X be a quasi-compact quasi-separated scheme. One of the main results of [LN07] is that $F \in \mathrm{PCoh}(X)$ and integers $a \in \mathbb{N}$, there exists a map $P \to F$ in $\mathrm{QCoh}(X)$ with $P \in \mathrm{Perf}(X)$ perfect whose cofibre lies in $\mathrm{QCoh}(X)_{\geq a}$. In particular, $\mathrm{Perf}(X)$ apporximates $\mathrm{PCoh}(X)$.

Here it is the result of Lank and Olander.

Proposition 10.1.7. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a bounded t-structure. Let $G \in \mathcal{C}$ be an object and $n \in \mathbb{N}$ an integer. Let $\mathcal{P} \subseteq \mathcal{C}$ be a collection of objects that approximates $\mathcal{C}$. Then the following are equivalent:

(1) $\mathcal{C} = \mathrm{thick}_n(G)$.
(2) $\mathcal{P} \subseteq \mathrm{thick}_n(G)$.

Proof. The implication (1) ⇒ (2) is obvious. For (2) ⇒ (1), let $x \in \mathcal{C}$. Since the t-structure is bounded, there exists $a \in \mathbb{Z}$ such that $x \in \mathcal{C}_{\leq a-1}$. Take N as in Lemma 10.1.3. Since $\mathcal{P} \subseteq \mathcal{C}$ approximates, there exists a map $y \to x$ whose cofibre lies in $\mathcal{C}_{\geq a+N}$ and since $y \in \mathcal{P} \subseteq \mathrm{thick}_n(G)$ it follows that $x \in \mathrm{thick}_n(G)$ by Lemma 10.1.3. □

We deduce the following.

Theorem 10.1.8 ([LO25, Theorem 1.1])**.** Let X be a noetherian scheme. Let $G \in \mathrm{Coh}(X)$ and $d \in \mathbb{N}$. The following are equivalent:

(1) $\mathrm{Coh}(X) = \mathrm{thick}_d(G)$.
(2) $\mathrm{Perf}(X) \subseteq \mathrm{thick}_d(G)$.

Proof. Apply the previous proposition together with Example 10.1.6. □

10.2. Towards the conjecture.

Theorem 10.2.1 (Rouquier's universal upper bound)**.** Let X be a quasi-compact smooth separated scheme over a field k. Then $\dim_{\mathrm{R}}(\mathrm{Coh}(X)) \leq 2\dim(X)$.

Proof. Let d be the dimension of X. By Lemma 5.4.2 it is $\dim_{\Delta}(\mathrm{Coh}(X)/\mathrm{Perf}_k) \leq 2d$. Since the Rouquier dimension of Perf_k is zero, Corollary 9.3.3 applies and shows the required bound. □

we may want to know who is the strong generator: see Example 2.6. approx by perfect complexes, use Proposition 5.4.4

Theorem 10.2.2 (Rouquier's universal lower bound)**.** Let X be a reduced separated scheme of finite type over a field k. Then $\dim(X) \leq \dim_R(\mathrm{Coh}(X))$.

Proof. If $\dim_R(\mathrm{Coh}(X)) = \infty$ there is nothing to prove. Set $n := \dim_R(\mathrm{Coh}(X))$, and choose $G \in \mathrm{Coh}(X)$ such that $\mathrm{Coh}(X) = \mathrm{thick}_{n+1}(G)$.

Let $Y \subseteq X$ be an irreducible component of dimension $\dim(X)$. Since X is reduced, the generic point of Y is regular (see [Aut18, 00ER, 02LV]). Moreover, because X is of finite type over a field, the regular locus $\mathrm{Reg}(X)$ is open [Aut18, 07R5]. Hence $\mathrm{Reg}(X) \cap Y$ is a non-empty open subset of Y. Choose a non-empty affine open $U \subseteq \mathrm{Reg}(X) \cap Y$, and write $U = \mathrm{Spec}(R)$. Then R is a classical commutative noetherian regular ring, and $\dim(R) = \dim(U) = \dim(Y) = \dim(X)$. By Theorem 10.1.8 one has $\mathrm{Perf}(X) \subseteq \mathrm{thick}_{n+1}(G)$. Let $j : U \hookrightarrow X$ be the open immersion. Then

$$\mathrm{Perf}(U) \subseteq \mathrm{thick}_{n+1}(j^*G).$$

Indeed, let $P \in \mathrm{Perf}(U)$. By Corollary 5.3.2, P is a direct summand of the restriction j^*Q of some $Q \in \mathrm{Perf}(X)$. Since $Q \in \mathrm{Perf}(X) \subseteq \mathrm{thick}_{n+1}(G)$ and j^* is exact, it follows that $j^*Q \in \mathrm{thick}_{n+1}(j^*G)$. As $\mathrm{thick}_{n+1}(j^*G)$ is thick, it is closed under direct summands, hence $P \in \mathrm{thick}_{n+1}(j^*G)$. Applying Theorem 10.1.8 to the noetherian scheme U shows that $\mathrm{Coh}(U) = \mathrm{thick}_{n+1}(j^*G)$. Since U is regular noetherian, one has $\mathrm{Coh}(U) = \mathrm{Perf}(U)$ and therefore $\dim_R(\mathrm{Perf}(U)) = \dim_R(\mathrm{Coh}(U)) \leq n$. Apply now Letz's theorem Theorem 8.4.21 to the commutative noetherian regular ring R to get

$$\dim(X) = \dim(R) = \dim_R(\mathrm{Perf}(U)) = \dim_R(\mathrm{Coh}(U)) \leq n$$

so that $\dim(X) \leq \dim_{\mathrm{R}}(\mathrm{Coh}(X))$. □

Putting these two results together proves the following.

Corollary 10.2.3. Let X be a quasi-compact smooth separated scheme of finite type over a field k. Then $\dim(X) \leq \dim_{\mathrm{R}}(\mathrm{Coh}(X)) \leq 2\dim(X)$.

Orlov's conjecture [Orl08, Conjecture 10] asks for equality.

Conjecture 10.2.4. Let X be a quasi-projective projective over a field k. Then $\dim_{\mathrm{R}}(\mathrm{Coh}(X)) = \dim(X)$.

Example 10.2.5. Notice that Orlov's conjecture is not true for non-regular schemes. Indeed, let k be a field and consider the zero dimensional scheme $X = \mathrm{Spec}(k[x]/(x^2))$. Then $\dim_{\mathrm{R}}(\mathrm{Coh}(X)) = 1$.

Remark 10.2.6. As already observed, there are schemes that are not quasi-projective projective over a field for which the statement of Orlov's conjecture hold. For example:

(1) Letz's result [Let25, Theorem A] (proved in these notes in Theorem 8.4.21) for classical commutative noetherian regular rings.
(2) Olander's result [Ola21, Theorem 2] (proved in these notes in Corollary 10.3.2) for quasi-affine noetherian regular schemes.

10.3. **Olander's result.** We now prove Orlov's conjeture in the case of quasi-affine noetherian regular schemes, a result due to Olander [Ola21].

Theorem 10.3.1. Let X be a noetherian regular scheme of dimension $d < \infty$. Assume X has an ample invertible sheaf L. Then $\mathrm{Coh}(X) = \mathrm{thick}_{d+1}(\{L^{\otimes -n}\}_{n\geq 0})$.

Corollary 10.3.2 ([Ola21, Theorem 2])**.** Let X be a quasi-affine noetherian regular scheme of dimension $d \leq \infty$. Then $\mathrm{Coh}(X) = \mathrm{thick}_{d+1}(\mathcal{O}_X)$.

Excercises

We list some useful exercises.

E.1. **Stable categories.**

Exercise E.1.1. Recall the definition of triangulated category. Prove that the homotopy category of a stable category is triangulated. Can you find an example of a triangulated category which is not the homotopy category of a stable category?

Exercise E.1.2. Construct the spectrum object functor $\mathrm{Sp} : \mathrm{Cat}^{\mathrm{lex,pt}} \to \mathrm{Cat}^{\mathrm{st}}$ as a right adjoint to the inclusion of stable categories into $\mathrm{Cat}^{\mathrm{lex,pt}}$.

Exercise E.1.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category. Show that $\mathcal{C}$ is enriched in Sp in the following sense: there exists a functor $\hom_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ such that $\Omega^\infty \hom_{\mathcal{C}} \simeq \mathrm{Hom}_{\mathcal{C}}$. Is every Sp-enriched category stable?

Exercise E.1.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Show that every exact sequence $x \to y \to z$ in $\mathcal{C}$ induces a long exact sequence

$$\dots \to \pi_{n+1} z \to \pi_n x \to \pi_n y \to \pi_n z \to \pi_{n-1} x \to \dots$$

in $\mathcal{C}^{\heartsuit}$.

E.2. **Idempotent-completion.**

Exercise E.2.1. Recall the construction of the Grothendieck group K_0 of a small stable category.

Exercise E.2.2. Let $\mathcal{A} \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{perf}})$ be a small rigid 2-ring. Show that an $\mathcal{A}$-module $\mathcal{C}$ is dualizable in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{perf}})$ if and only if it is smooth and proper.

Exercise E.2.3. Recall the construction of the Grothendieck construction and show explicitly that the category of sections of the Grothendieck construction of a diagram $I \to \mathbf{Pr}^{\mathrm{L}}$ exhibits the lax limit.

E.3. **Verdier sequences and recollements.**

Exercise E.3.1. Is the notion of stable recollement symmetric in its inputs?

Exercise E.3.2. Let $i : \mathcal{C} \to \mathcal{D}$ be a fully-faithful functor in $\mathrm{Cat}^{\mathrm{st}}$. Assume that both $\mathcal{C}$ and $\mathcal{D}$ come equipped with bounded t-structures and that i is t-exact. Show that the quotient functor $q : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is the initial t-exact functor to a stable category with a bounded t-structure.

Exercise E.3.3. Let X be a noetherian scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then j^* induces an equivalence

$$\mathrm{Coh}^{\heartsuit}(X)/\mathrm{Coh}^{\heartsuit}_Z(X) \to \mathrm{Coh}^{\heartsuit}(U)$$

of abelian categories.

Exercise E.3.4 (The wrong way recollement)**.** Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ be a compactly generated stable category with a set K which consists of compact objects and is closed under shift. Then there is a recollement

$$K^{\perp} \xrightarrow{\;i_*\;} \mathcal{C} \xrightarrow{\;j^*\;} \langle K \rangle \qquad i^* , i^! : \mathcal{C} \to K^{\perp}, \quad j_!, j_* : \langle K \rangle \to \mathcal{C}$$

where i_* and $j_!$ are inclusions of full subcategories. Here

$$K^{\perp} = \{x \in \mathcal{C} \,|\, \pi_0 \hom_{\mathcal{C}}(K, x) = 0 \text{ for each } k \in K\}$$

and $\langle K \rangle$ is the smallest stable subcategory of $\mathcal{C}$ which contains K and is closed under coproducts. Deduce Proposition 3.5.8.

E.4. **Calculus of subcategories.**

Exercise E.4.1 ([Nee21, Proposition 1.9])**.** Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $B \subseteq \mathcal{C}^{\omega}$ be a full subcategory. Show that:

(1) Every compact object in $\mathrm{Coprod}_n(B)$ belongs to $\mathrm{smd}(\mathrm{coprod}_n(B))$.
(2) Every compact object in $\mathrm{Coprod}(B)$ belongs to $\mathrm{smd}(\mathrm{coprod}(B))$.

Exercise E.4.2 ([Nee21, Lemma 2.6])**.** Let X be a noetherian separated scheme and let $F \in \mathrm{Coh}(X)$. Prove that for every $n \in \mathbb{N}$ one has

$$\mathrm{Coh}(X) \cap \mathrm{Thick}_n^{\mathrm{QCoh}(X)}(F) \subseteq \mathrm{thick}_{2n}^{\mathrm{Coh}(X)}(F).$$

E.5. **Generators.**

Exercise E.5.1. Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{\heartsuit}$ be a classical ring.

Exercise E.5.2. Let X, Y be smooth projective

Exercise E.5.3 (Splitting lemma)**.** Let X be a regular noetherian scheme of dimension d, and let $K \in \mathrm{Coh}(X)$. Fix an integer $b \in \mathbb{Z}$, and assume that $\pi_i(K) = 0$ for every i with $b - d < i < b$. Show that the exact sequence

$$\tau_{\le b-d}K \to K \to \tau_{\ge b}K$$

splits. Equivalently, prove that $K \simeq \tau_{\le b-d}K \oplus \tau_{\ge b}K$.

E.6. **Projective classes.**

Exercise E.6.1. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction in $\mathrm{Cat}^{\mathrm{st}}$. Let $\mathcal{P} = \mathrm{thick}_1(\mathrm{im}(F))$ and $\mathcal{I} = \{f \in \mathcal{D}^{\Delta^1} \mid R(f) \simeq 0\}$. Show that $(\mathcal{P}, \mathcal{I})$ defines a projective class on $\mathcal{D}$.

Exercise E.6.2.

Exercise E.6.3. Exact sequence and surjectivity

Exercise E.6.4 ([Mod10, Lemma 3.4])**.** Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category and let $(\mathcal{P}, \mathcal{I})$ generated by a small category. Let i be an ordinal and let $F \in \mathrm{Ab}^{\oplus}[\mathcal{C}]$ be a cohomological functor which sends coproducts into products. Show that the category $\mathrm{El}(F, \mathcal{P}^{*i})$ has a weak terminal object.

Hint: use transfine induction.

E.7. **Strong generation and descent.**

Exercise E.7.1. Recall that a map $A \to B$ in $\mathrm{CAlg}(\mathrm{Sp})$ is *faithfully flat* if:

(1) The map $\pi_0(A) \to \pi_0(B)$ is faithfully flat.
(2) The natural map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism.

Show that for every A-module N, the natural map $\pi_*(N) \to \pi_*(N \otimes_A B)$ identifies with the base-change map $\pi_*(N) \to \pi_*(N) \otimes_{\pi_0(A)} \pi_0(B)$ and in particular is injective. Equivalently, prove that there is a natural isomorphism $\pi_*(N \otimes_A B) \simeq \pi_*(N) \otimes_{\pi_0(A)} \pi_0(B)$. Deduce that if $f : P \to M$ is a map of A-modules with P perfect, and if $f \otimes_A B : P \otimes_A B \to M \otimes_A B$ is nullhomotopic, then f is already nullhomotopic.

Hint: use the Künneth spectral sequence $E^2_{p,q} = \mathrm{Tor}^{\pi_*(A)}_{p,q}(\pi_*(N), \pi_*(B)) \Rightarrow \pi_{p+q}(N \otimes_A B)$, together with the identification $\pi_*(B) \simeq \pi_*(A) \otimes_{\pi_0(A)} \pi_0(B)$ and the flatness of $\pi_0(B)$ over $\pi_0(A)$. Then use the duality.

E.8. **The Rouquier dimension.**

Exercise E.8.1. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Mod}_A}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$. Show that for every pair of objects $x, y \in \mathcal{C}$ there exists an object $\underline{\mathrm{hom}}_{\mathcal{C}}(x, y) \in \mathrm{Mod}_A$ such that, functorially in $M \in \mathrm{Mod}_A$, there is a natural equivalence

$$\mathrm{hom}_{\mathrm{Mod}_A}(M, \underline{\mathrm{hom}}_{\mathcal{C}}(x, y)) \simeq \mathrm{hom}_{\mathcal{C}}(M \otimes x, y).$$

Deduce that the mapping space $\mathrm{hom}_{\mathcal{C}}(x, y)$ refines functorially to an A-module spectrum.

Exercise E.8.2 (The graded center of a triangulated category)**.** Let $\mathcal{T}$ be a triangulated category with suspension functor Σ.

(1) For $d \in \mathbb{Z}$, define $Z^d(\mathcal{T})$ to be the set of natural transformations $\eta : \mathrm{id}_{\mathcal{T}} \Rightarrow \Sigma^d$ such that $\eta_{\Sigma x} = (-1)^d \Sigma(\eta_x)$ for every $x \in \mathcal{T}$. Show that $Z^*(\mathcal{T}) = \bigoplus_{d \in \mathbb{Z}} Z^d(\mathcal{T})$ carries a natural structure of graded ring, where for $\eta \in Z^d(\mathcal{T})$ and $\theta \in Z^e(\mathcal{T})$ the product is defined by $(\eta\theta)_x := \Sigma^e(\eta_x) \circ \theta_x : x \to \Sigma^{d+e} x$.

(2) Let $x, y \in \mathcal{T}$, let $n \in \mathbb{Z}$, and let $f : x \to \Sigma^n y$. Show that if $\eta \in Z^d(\mathcal{T})$, then $\Sigma^d(f) \circ \eta_x = \eta_{\Sigma^n y} \circ f = (-1)^{dn} \Sigma^n(\eta_y) \circ f$.
(3) Deduce that every $\eta \in Z^d(\mathcal{T})$ defines a degree-d endomorphism of $\mathrm{Ext}^*_{\mathcal{T}}(x, y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(x, \Sigma^n y)$, and therefore $\mathrm{Ext}^*_{\mathcal{T}}(x, y)$ is naturally a graded $Z^*(\mathcal{T})$-module.
(4) Deduce that if R^* is a graded ring and there is a morphism of graded rings $R^* \to Z^*(\mathcal{T})$, then every $\mathrm{Ext}^*_{\mathcal{T}}(x, y)$ is naturally a graded R^*-module.

E.9. **The smooth (or diagonal) dimension.**

E.10. **Orlov's conjecture.**

Exercise E.10.1 (The countable dimension)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ be small idempotent-complete stable category. We define the *countable dimension of* $\mathcal{C}$ to be the smallest integer $d \in \mathbb{N}$ such that there exists a countable subcategory $\mathcal{G} \subseteq \mathcal{C}$ such that $\mathcal{C} = \mathrm{thick}_{d+1}(\mathcal{G})$.

(1) Which properties of **??** do still hold for the countable dimension?
(2) Can you formulate a ghost lemma for the countable dimension?
(3) Show that the countable dimension of Sp^{ω} is zero.

Notice that by **??** a smooth projective variety over a field

Exercise E.10.2 (Very difficult!)**.** Prove Orlov's conjecture.

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