QUASI-COMPACT QUASI-SEPARATED SCHEMES ARE APPROXIMABLE

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Abstract. Let X be a quasi-compact quasi-separated scheme. We show that the derived stable ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ is approximable in the sense of Neeman. Furthermore, we show that if Z is a closed subscheme of X with quasi-compact open complement, then the stable ∞-category of quasi-coherent sheaves with support $\mathrm{QCoh}_Z(X)$ is approximable if and only if the underlying topological space of Z is open.

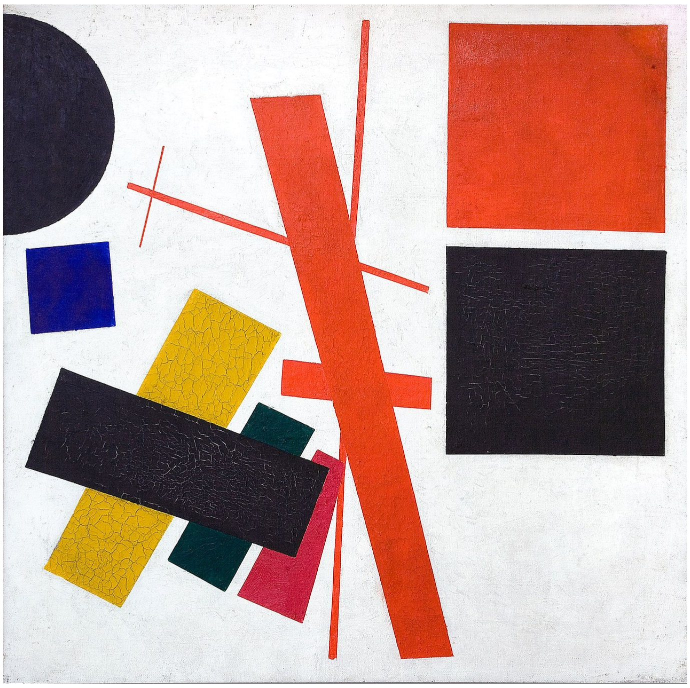

Figure 1. *Suprematism. Abstract Composition*, Kazimir Malevich, 1915.

1. Introduction

Compactly generated stable categories are ubiquitous in geometry and homotopy theory, and many finiteness questions about them can be understood as approximation problems. In such categories every object can be recovered from compact objects by filtered colimits. While this is a powerful finiteness statement, it does not say how efficient this recovery is. A t-structure makes this question sharper, since it allows one to measure the error of an approximation. In other words, compact objects provide the basic building blocks, while the t-structure controls the quality of the approximation. Neeman's notion of *weak approximability* [Nee25, Definition 0.25] makes this idea precise since it does not merely ask that a compact generator generate the category; it asks that connective objects admit approximations by arbitrary coproducts of bounded shifts of one compact generator.

For applications, weakly approximability is not enough; one needs the strength of *approximability*. The difference between the former and the latter is substantial, in that in the latter connective objects can be approximated by objects built from bounded shifts of a compact generator using arbitrary coproducts but finitely many extensions.

Regarding examples, Neeman showed in [Nee25, Example 4.6] that the (homotopy category of the) derived stable ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ is weakly approximable for every quasi-compact quasi-separated scheme X and approximable if X is furthermore separated. The first result of this note removes the separatedness hypothesis.

Theorem. Let X be a quasi-compact quasi-separated scheme. Then $\mathrm{QCoh}(X)$ is approximable.

The main idea is to show that approximability is Zariski local, so that the statement easily follows from the approximability of derived categories of modules. The main technical point is that, for a quasi-compact open immersion $j : U \hookrightarrow X$, the object $j_*(\mathcal{O}_U)$ is still uniformly controlled by perfect objects on X, or, equivalently, the open immersion is *extremely compactly bounded* (following the terminology of [DLR25, Definition 4.1]). In the affine case this follows from the telescope description of a principal localization and from a finite Čech complex. The general case is obtained by Zariski descent, using quasi-separatedness to keep all intersections quasi-compact. We point out that the methods used to prove the first theorem should be general enough to apply to more general categories than quasi-coherent sheaves on a scheme.

Another example of weakly approximabile category is given by categories of quasi-coherent sheaves with support $\mathrm{QCoh}_Z(X)$ for quasi-compact quasi-separated and $Z \subseteq X$ closed subscheme with quasi-compact open complement. Our second result shows that for these categories approximability is a very rigid condition: it occurs only in the evident split case.

Theorem. Let X be a quasi-compact quasi-separated scheme and let Z be a closed subscheme of X with quasi-compact open complement. Then $\mathrm{QCoh}_Z(X)$ is approximable if and only if the underlying topological space of Z is open. In this case, $\mathrm{QCoh}_Z(X)$ is equivalent to the derived category of quasi-coherent sheaves on the open subscheme of X with underlying topological space Z.

The proof uses the standard recollement $\mathrm{QCoh}_Z(X) \hookrightarrow \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$, where $U = X \smallsetminus Z$. If $|Z|$ is open, then X splits as the disjoint union of U and the corresponding open subscheme W, and the claim follows from the first theorem. Conversely, assume that $\mathrm{QCoh}_Z(X)$ is approximable. A compact generator of $\mathrm{QCoh}_Z(X)$ is perfect on X, hence killed by a fixed power of an ideal defining $|Z|$. Uniform approximability then forces the same power of the ideal to be idempotent, making $|Z|$ is open and closed. We would like to point out that this idea for the proof was partially contained in [Nee24, Remark 8.1].

Organization. The note is organized as follows. In section 2 we recall the definition of approximability and the basic examples. In section 3 we prove that quasi-compact open immersions of quasi-compact quasi-separated schemes are extremely compactly bounded. We apply this result in section 4 to prove approximability of derived ∞-categories of quasi-coherent sheaves on a scheme by Zariski descent. In section 5 we recall the recollement of derived categories associated to an open-closed decomposition of a scheme and use it to produce a characterization of those closed subschemes which are also open subschemes. Finally, in section 6 we prove the characterization of approximability for sheaves with support in a closed.

Convention. We use the following conventions.

Notation 1.1. We freely use the language of stable ∞-categories (also known as stable *categories*) and t-structures on them, which we grade homologically. If $\mathcal{C}$ is a stable ∞-category, we let $\hom_{\mathcal{C}}(-,-)$ denote the mapping spectrum. We denote by Sp the stable ∞-category of spectra and we equip it with the standard t-structure.

Notation 1.2. If A is a ring, we will denote by Mod_A the derived stable ∞-category of modules over A. Given a quasi-compact quasi-separated scheme X, we denote by $\mathrm{QCoh}(X)$ the derived stable ∞-category of quasi-coherent sheaves on X and by $\mathrm{Perf}(X)$ the corresponding category of compact objects. We equip $\mathrm{QCoh}(X)$ with the standard t-structure. We denote by $|X|$ the underlying topological space of X. We short the notation $\hom_{\mathrm{QCoh}(X)}(-,-)$ to $\hom_X(-,-)$.

Notation 1.3. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into a quasi-compact open U and closed Z.

1. Consider the adjoint functors $j^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(U) : j_*$. Then j^* is t-exact and j_* is fully-faithful and right t-exact up to a shift.
2. Consider the adjoint functors $i^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(Z) : i_*$. Then i_* is t-exact but not fully-faithful in general. Furthermore, the pullback functor i^* is not left t-exact, since the inclusion $i : Z \hookrightarrow X$ is not of finite tor-amplitude.

We let $\mathrm{QCoh}_Z(X)$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by those quasi-coherent sheaves $F \in \mathrm{QCoh}(X)$ such that $\pi_n(F) \in \mathrm{QCoh}(X)^\heartsuit$ is supported on Z (that is, their restriction to U vanishes).

2. The definition of approximability

The definition of approximability requires some more notation.

Notation 2.1. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a presentable stable category. Let $S \subseteq \mathcal{C}$ be a collection of objects. Given two (possibly infinite) integers $-\infty \le a \le b \le +\infty$, we let $S[a,b]$ denote the full subcategory of $\mathcal{C}$ spanned by the $\Sigma^{-i}s$ for $s \in S$ and $a \le i \le b$. Given $n \in \mathbb{N}$ we define inductively the full subcategories $\mathrm{Thick}_n(S)$ as follows. We let $\mathrm{Thick}_1(S)$ be the smallest subcategory containing arbitrary coproducts of S and closed under retracts and $\mathrm{Thick}_n(S)$ to be the smallest subcategory closed under retract on those $y \in \mathcal{C}$ such that there exists an exact sequence $x \to y \to z$ with $x \in \mathrm{Thick}_1(S)$ and $z \in \mathrm{Thick}_{n-1}(S)$. We let $\mathrm{Thick}(S)$ be the union of all the $\mathrm{Thick}_n(S)$. In the case where $S = \{G\}$ is a single object, we omit the parentheses. We also omit the brackets $[a,b]$ if a and b are infinite. We will refer to $\mathrm{Thick}_n(G[-n,n])$ as the *n-th proxy small closure of G*. We will also refer to objects therein as *proxy small.*

Neeman gave in [Nee25, Definition 0.25] the definition of a (weakly) approximable triangulated category. In ∞-categorical terms, it translates to the following.

Definition 2.2. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ be a compactly generated stable category. We will say that $\mathcal{C}$ is *weakly approximable* if there exist a compact generator $G \in \mathcal{C}^\omega$, a t-structure $(\mathcal{C}_{\ge 0}, \mathcal{C}_{\le 0})$ and an integer $A \in \mathbb{N}$ such that:

1. It is $\Sigma^A G \in \mathcal{C}_{\ge 0}$ and $\hom_{\mathcal{C}}(G, \mathcal{C}_{\ge A}) \in \mathrm{Sp}_{\ge 1}$.
2. For every connective object $x \in \mathcal{C}_{\ge 0}$ there exists an exact sequence $p \to x \to e$ in which $p \in \mathrm{Thick}(G[-A,A])$ and $e \in \mathcal{C}_{\ge 1}$.

We will furthermore say that $\mathcal{C}$ is *approximable* if condition (2) can be strengthened to:

(3) For every connective object $x \in \mathcal{C}_{\geq 0}$ there exists an exact sequence $p \to x \to e$ in which $p \in \mathrm{Thick}_A(G[-A, A])$ and $e \in \mathcal{C}_{\geq 1}$.

Remark 2.3. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ be a compactly generated stable category with a compact generator and a t-structure. Notice that if $A \in \mathbb{N}$ shows (weakly) approximability, then every index $A' \geq A$ will show (weakly) approximability.

Remark 2.4. The reason behind the nomenclature "approximability" is well explained in [Nee22, Remark 2.5 and Fact 2.6], in that condition (3) can be used to produce arbitrarily good approximations. To be precise, let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ be an approximable category and pick a compact generator G together with a t-structure and an integer $A \in \mathbb{N}$ satisfying the axioms. Let $x \in \mathcal{C}_{\geq 0}$ be connective. Then for every $n \geq 1$ there exists an exact sequence $p_n \to x \to e_n$ in which $p_n \in \mathrm{Thick}_{nA}(G[-A-n, A+n])$ and $e_n \in \mathcal{C}_{\geq n}$. This can be shown by induction. If $\mathcal{C}$ is assumed only to be weakly approximable, the same statement holds, but $p_n \in \mathrm{Thick}(G[-A-n, A+n])$ has no bound on the number of extensions.

The main examples of (weakly) approximable categories are given by derived categories of schemes.

Example 2.5. Let X be a quasi-compact quasi-separated scheme. Then [Nee25, Example 4.6.] shows that $\mathrm{QCoh}(X)$ is weakly approximable. More precisely, the compact generator is given by the construction of Bondal and Van den Bergh [BdB02, Theorem 3.1.1(ii)] and the t-structure may be chosen to be the standard one. The existence of the required index is shown in [Nee25, Lemma 4.5]. If X is moreover separated, then $\mathrm{QCoh}(X)$ is approximable.

Example 2.6. Let X be a quasi-compact quasi-separated scheme and let $Z \subseteq X$ be a closed subscheme with quasi-compact open complement. Then [Nee24, Theorem 3.2] states that $\mathrm{QCoh}_Z(X)$ is weakly approximable and the t-structure used to prove weakly approximability is equivalent to the standard t-structure.

3. Compactly bounded morphisms

Inspired by [DLR25, Definition 4.1], we give the following.

Definition 3.1. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ be the inclusion of a quasi-compact open. We will say that j is *extremely compactly bounded* if there exists an integer $n \in \mathbb{N}$ such that for every compact object $P \in \mathrm{Perf}(U)$ there exists a compact object $Q \in \mathrm{Perf}(X)$ such that $j_*(P) \in \mathrm{Thick}_n(Q[-n, n])$.

Example 3.2. Let X be a quasi-compact quasi-separated scheme and $j : U \hookrightarrow X$ be the inclusion of a quasi-compact open. To show that j is extremely compactly bounded, via the projection formula and Thomason-Trobaugh, it suffices to show that there exists an integer $n \in \mathbb{N}$ such that there exists a compact object $Q \in \mathrm{Perf}(X)$ for which $j_*(\mathcal{O}_U) \in \mathrm{Thick}_n(Q[-n, n])$. This was observed in [DLR25, Example 4.2]. In particular, if X is furthermore separated, then [Nee24, Theorem 6.2] shows that j is extremely compactly bounded.

Our first goal is to improve this result to all quasi-compact quasi-separated schemes. We begin with a technical observation.

Lemma 3.3. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ be the inclusion of a quasi-compact open. Then $\hom_X(j_*(\mathcal{O}_U), -) : \mathrm{QCoh}(X) \to \mathrm{Sp}$ is right t-exact up to a shift.

Proof. The claim is that there exists an integer $m \in \mathbb{N}$ such that $\hom_X(j_*(\mathcal{O}_U), -)$ sends $\mathrm{QCoh}(X)_{\geq 0}$ into $\mathrm{Sp}_{\geq -m}$. Assume first that X is affine, say $X = \mathrm{Spec}(R)$. Since U is a quasi-compact open, there are $f_1, \dots f_a \in R$ such that $U = D(f_1) \cup \dots \cup D(f_a)$. For every $1 \leq i \leq a$ let $j_i : D(f_i) \hookrightarrow X$ be the open inclusion. Then $(j_i)_*(\mathcal{O}_{D(f_i)}) \simeq R_{f_i}$ may be realized as a telescope, namely as the cofibre of a map $\bigoplus_{n\in\mathbb{N}} R \to \bigoplus_{n\in\mathbb{N}} R$. In particular, for every $M \in \mathrm{QCoh}(X)_{\geq 0} \simeq (\mathrm{Mod}_R)_{\geq 0}$ it follows that $\hom_X((j_i)_*(\mathcal{O}_{D(f_i)}), M) \simeq \mathrm{fib}(\prod_{n\in\mathbb{N}} M \to \prod_{n\in\mathbb{N}} M)$ is in $\mathrm{Sp}_{\geq -1}$. This follows by prestability plus the fact that the t-structure on Sp is compatible with arbitrary products. For every non-empty subset $I \subseteq \{1,\dots,a\}$, set $f_I := \prod_{i\in I} f_i$ and let $j_I : D(f_I) \hookrightarrow X$ be the corresponding principal open immersion. The Čech complex of the cover $U = \bigcup_i D(f_i)$ gives an equivalence

$$j_*(\mathcal{O}_U) \simeq \mathrm{Tot}\left[\prod_{|I|=1}(j_I)_*(\mathcal{O}_{D(f_I)}) \to \prod_{|I|=2}(j_I)_*(\mathcal{O}_{D(f_I)}) \to \cdots \to (j_{\{1,\dots,a\}})_*(\mathcal{O}_{D(f_1\cdots f_a)})\right].$$

Since all products are finite, it follows that $C^p := \bigoplus_{|I|=p+1}(j_I)_*(\mathcal{O}_{D(f_I)}) \simeq \bigoplus_{|I|=p+1} R_{f_I}$. Thus $j_*(\mathcal{O}_U)$ is obtained from the bounded Čech complex $C^0 \to C^1 \to \cdots \to C^{a-1}$. For every non-empty I, the object R_{f_I} is a principal localization of R. Hence, by the preceding paragraph, $\hom_X(R_{f_I}, -)$ is right t-exact up to a shift. Therefore $\hom_X(C^p, -) \simeq \prod_{|I|=p+1}\hom_X(R_{f_I}, -)$ is still right t-exact up to a shift. Since the Čech complex is bounded, $j_*(\mathcal{O}_U)$ admits a finite filtration whose graded pieces are shifts of the objects C^p. Applying $\hom_X(-, M)$, for $M \in \mathrm{QCoh}(X)_{\geq 0}$ connective, produces a finite tower whose successive fibres are shifts of the spectra $\hom_X(C^p, M)$. Thus the connectivity can decrease only by a bounded amount depending on a, showing that $\hom_X(j_*(\mathcal{O}_U), -)$ is right t-exact up to a shift in the affine case.

Assume now that X is quasi-compact separated. Let $X = V_1 \cup \cdots \cup V_a$ be an affine open cover and for every non-empty subset $I \subseteq \{1,\dots,a\}$ consider the quasi-compact open $V_I = \bigcap_{i\in I} V_i$. Since X is separated, the open $V_I \subseteq X$ is affine (being an intersection of affines). Now apply the preceding case to the quasi-compact open inclusion $i : U \cap V_I \hookrightarrow V_I$ to deduce that $\hom_{V_I}(i_*(\mathcal{O}_{U\cap V_I}), -)$ is right t-exact up to a shift. Notice that here the base change $(j_*(\mathcal{O}_U)|_{V_I} \simeq i_*(\mathcal{O}_U|_{U\cap V_I}) \simeq i_*(\mathcal{O}_{U\cap V_I})$ has been used. Since Zariski descent states that

$$\mathrm{QCoh}(X) \simeq \lim_{I\subseteq\{1,\dots a\}} \mathrm{QCoh}(V_I), \qquad \mathrm{QCoh}(X)_{\geq 0} \simeq \lim_{I\subseteq\{1,\dots a\}} \mathrm{QCoh}(V_I)_{\geq 0}$$

it follows that

$$\hom_X(j_*(\mathcal{O}_U), -) \simeq \lim_{I\subseteq\{1,\dots a\}} \hom_{V_I}(i_*(\mathcal{O}_{U\cap V_I}), (-)|_{V_I})$$

is still right t-exact up to a shift (since the limit is finite).

Finally, assume that X is quasi-compact quasi-separated. Let $X = W_1 \cup \cdots \cup W_b$ be an affine open cover and for every non-empty subset $I \subseteq \{1,\dots,b\}$ consider the quasi-compact open $W_I = \bigcap_{i\in I} W_i$. Then W_I is open in one of the affines W_i, making W_I separated. In particular, the previous paragraph applies to $i : U \cap W_I \hookrightarrow W_I$, making $\hom_{W_I}(i_*(\mathcal{O}_{U\cap W_I}), -)$ right t-exact up to a shift. Using again Zariski descent shows the claim. □

Then:

Proposition 3.4. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ be the inclusion of a quasi-compact open. Then j is extremely compactly bounded.

Proof. As already observed above, by using the projection formula it suffices to prove that there exists an integer $n \in \mathbb{N}$ such that there exists a compact object $Q \in \mathrm{Perf}(X)$ for which $j_*(\mathcal{O}_U) \in \mathrm{Thick}_n(Q[-n,n])$. Since j_* is right t-exact up to a shift, the pushforward of the unit $j_*(\mathcal{O}_U)$ is eventually connective, say $\Sigma^k j_*(\mathcal{O}_U) \in \mathrm{QCoh}(X)_{\geq 0}$ for $k \in \mathbb{N}$. Since $\mathrm{QCoh}(X)$ is weakly approximable, there exists a compact generator $G \in \mathrm{Perf}(X)$ and an integer $A \in \mathbb{N}$ satisfying the axioms with respect to the standard t-structure. Then for every $m \geq 1$ there exists an exact sequence $P \to \Sigma^k j_*(\mathcal{O}_U) \to E$ in which $P \in \mathrm{Thick}(G[-n,n])$ and $E \in \mathrm{QCoh}(X)_{\geq m}$ for some $n \in \mathbb{N}$. Notice now that it suffices to show that the map $\Sigma^k j_*(\mathcal{O}_U) \to E$ is null-homotopic, since it will make $j_*(\mathcal{O}_U)$ a retract of $\Sigma^{-k}P$, and hence an object of $\mathrm{Thick}(G[-n-k, n+k])$, which must therefore belong to a finite stage (and thus showing that j_* is extremely compactly bounded after taking the maximum between the stage and $n+k$). Now the existence of such m follows from Lemma 3.3. Indeed, since there exists an integer $w \in \mathbb{N}$ such that $\hom_X(j_*(\mathcal{O}_U), -)$ sends $\mathrm{QCoh}(X)_{\geq m-k}$ into $\mathrm{Sp}_{\geq m-k-w}$, it suffices to pick m such that $m - k - w \geq 1$. □

4. Approximability via Zariski descent

We now work towards the proof of the main result of these notes. We begin with the technical:

Lemma 4.1. Let X be a quasi-compact quasi-separated scheme. Let $N \in \mathbb{N}$ be an integer and let $G \in \mathrm{Perf}(X)$ be compact. Then there exists $w \in \mathbb{N}$ such that $\hom_X(P, \mathrm{QCoh}(X)_{\geq w}) \in \mathrm{Sp}_{\geq 1}$ for every $P \in \mathrm{Thick}(G[-N, N])$. In particular, w can be chosen ≥ 2.

Proof. Since X is quasi-compact and quasi-separated, the global sections functor $\Gamma(X, -) : \mathrm{QCoh}(X) \to \mathrm{Sp}$ is right t-exact up to a shift, in that there exists $d \in \mathbb{N}$ such that $\Gamma(X, \mathrm{QCoh}(X)_{\geq m}) \subseteq \mathrm{Sp}_{\geq m-d}$ for every $m \in \mathbb{Z}$. Since G is perfect, its dual $G^\vee$ is perfect and hence has finite tor-amplitude. In other terms, there exists $a \in \mathbb{N}$ such that $G^\vee \otimes \mathrm{QCoh}(X)_{\geq m} \subseteq \mathrm{QCoh}(X)_{\geq m-a}$ for every $m \in \mathbb{Z}$.

Choose $w \geq d + a + N + 1$. Consider first the generators $\Sigma^{-n} G$, with $-N \leq n \leq N$. If $F \in \mathrm{QCoh}(X)_{\geq w}$, then

$$\hom_X(\Sigma^{-n} G, F) \simeq \hom_X(G, \Sigma^n F) \simeq \Gamma(X, G^\vee \otimes \Sigma^n F).$$

Since $\Sigma^n F \in \mathrm{QCoh}(X)_{\geq w+n}$ and $n \geq -N$, it follows that $\Sigma^n F \in \mathrm{QCoh}(X)_{\geq w-N}$. Therefore $G^\vee \otimes \Sigma^n F \in \mathrm{QCoh}(X)_{\geq w-N-a}$ since the t-structure is compatible with tensor products. Applying the cohomological amplitude bound for $\Gamma(X, -)$ gives $\Gamma(X, G^\vee \otimes \Sigma^n F) \in \mathrm{Sp}_{\geq w-N-a-d}$ and by the choice of w, this lies in $\mathrm{Sp}_{\geq 1}$.

Now let

$$\mathcal{S}_w = \{P \in \mathrm{QCoh}(X) \mid \hom_X(P, \mathrm{QCoh}(X)_{\geq w}) \subseteq \mathrm{Sp}_{\geq 1}\}.$$

The previous paragraph shows that $\Sigma^{-n} G \in \mathcal{S}_w$ for every $-N \leq n \leq N$. Notice now that the class $\mathcal{S}_w$ is closed under arbitrary coproducts (which follows since the connective half of the t-structure on spectra is closed under arbitrary products), that it is closed under retracts (which follows since every connective half of a t-structure is closed under retracts) and that is closed under extensions (which follows since every connective half of a t-structure has this property). It follows that $\mathcal{S}_w$ contains the closure of $\{\Sigma^{-n} G \mid -N \leq n \leq N\}$ under arbitrary coproducts, retracts and extensions, thus proving the claim. If necessary, replacing w by $\max\{w, 2\}$ gives $w \geq 2$. □

We have then the following descent result.

Lemma 4.2. Let X be a quasi-compact quasi-separated scheme. Assume that $X = U \cup V$ is a union of quasi-compact open subschemes with intersection $W = U \cap V$ and denote by

$$\begin{array}{ccc} W & \longrightarrow & U \\ \big\downarrow & \searrow^{k} & \big\downarrow{\scriptstyle i} \\ V & \xrightarrow[j]{} & X \end{array}$$

the various inclusions. If $\mathrm{QCoh}(U)$, $\mathrm{QCoh}(V)$ and $\mathrm{QCoh}(W)$ are approximable, then $\mathrm{QCoh}(X)$ is approximable.

Proof. As observed in Example 2.5, the category $\mathrm{QCoh}(X)$ is weakly approximable. By [Nee25, Fact 0.26] for every compact generator $G' \in \mathrm{Perf}(X)$ there exists an integer $A' \in \mathbb{N}$ such that $\Sigma^{A'} G' \in \mathrm{QCoh}(X)_{\geq 0}$ and $\hom_X(G', \mathrm{QCoh}(X)_{\geq A'}) \in \mathrm{Sp}_{\geq 1}$. The claim is then that, after changing the compact generator to G and A if necessary, every connective quasi-coherent sheaf $F \in \mathrm{QCoh}(X)_{\geq 0}$ fits inside an exact sequence $P \to F \to E$ with $P \in \mathrm{Thick}_A(G[-A, A])$ and $E \in \mathrm{QCoh}(X)_{\geq 1}$.

Consider now the compact object $G_U = i^*(G') \in \mathrm{QCoh}(U)$. Since G_U is a compact generator (by adjunction and fully-faithfulness of i_*) and since $\mathrm{QCoh}(U)$ is assumed approximable, [Nee25, Fact

0.26] implies the existence of an integer $A_U \in \mathbb{N}$ for which the definition of approximability holds with the integer A_U. The same observation applied to $G_V = j^*(G')$ and $G_W = k^*(G')$ produces integers $A_V \in \mathbb{N}$ and $A_W \in \mathbb{N}$ for which the same conclusion holds. Let B be the maximum of $\{A_U, A_V, A_W\}$ so that the definition of approximability holds for $\mathrm{QCoh}(U), \mathrm{QCoh}(V)$ and $\mathrm{QCoh}(W)$ with the integer $B \in \mathbb{N}$, as observed in Remark 2.3.

By Proposition 3.4, the pushforwards i_*, j_* and k_* are extremely compactly bounded. In particular, there exist integers $n_U, n_V, n_W \in \mathbb{N}$ such that

$$i_*(G_U) \in \mathrm{Thick}_{n_U}(H_U[-n_U, n_U]),$$

$$j_*(G_V) \in \mathrm{Thick}_{n_V}(H_V[-n_V, n_V]),$$

$$k_*(G_W) \in \mathrm{Thick}_{n_W}(H_W[-n_W, n_W]),$$

for some compact objects $H_U, H_V, H_W \in \mathrm{Perf}(X)$. Set now $G := G' \oplus H_U \oplus H_V \oplus H_W \in \mathrm{Perf}(X)$ and notice that it is still a compact generator for $\mathrm{QCoh}(X)$ since it contains a compact generator as a direct summand. If N is the maximum of $\{n_U, n_V, n_W\}$, then the above can be rephrased as

$$i_*(G_U) \in \mathrm{Thick}_N(G[-N, N]),$$

$$j_*(G_V) \in \mathrm{Thick}_N(G[-N, N]),$$

$$k_*(G_W) \in \mathrm{Thick}_N(G[-N, N]),$$

as follows from the definition. Finally, let r_i, r_j and $r_k \in \mathbb{N}$ be the errors of right t-exactness of i_*, j_* and k_*. Set r to be the maximum of the three, so that i_*, j_* and k_* will send connective objects to $(-r)$-connective objects.

Enough with notation, now it is time for the actual proof. Pick a connective object $F \in \mathrm{QCoh}(X)_{\geq 0}$. Since the categories $\mathrm{QCoh}(U)$, $\mathrm{QCoh}(V)$ are approximable by assumption, Remark 2.4 provides exact sequences

$$P_U \to i^*(F) \to E_U, \qquad P_V \to j^*(F) \to E_V$$

in which

$$P_U \in \mathrm{Thick}_{(r+1)B}(G_U[-B-r-1, B+r+1])$$

$$P_V \in \mathrm{Thick}_{(r+1)B}(G_V[-B-r-1, B+r+1])$$

where the cofibres $E_U \in \mathrm{QCoh}(U)_{\geq r+1}$ and $E_V \in \mathrm{QCoh}(V)_{\geq r+1}$ are $(r+1)$-connective. Pushing to X and summing the two exact sequences produces an exact sequence

$$i_*(P_U) \oplus j_*(P_V) \to i_*i^*(F) \oplus j_*j^*(F) \to i_*(E_U) \oplus j_*(E_V)$$

in which the first term is a sum of objects in

$$i_*(P_U) \in \mathrm{Thick}_{(r+1)B}(i_*(G_U)[-B-r-1, B+r+1]) \subseteq \mathrm{Thick}_{(r+1)BN}(G[-B-r-1-N, B+r+1+N])$$

$$j_*(P_V) \in \mathrm{Thick}_{(r+1)B}(j_*(G_V)[-B-r-1, B+r+1]) \subseteq \mathrm{Thick}_{(r+1)BN}(G[-B-r-1-N, B+r+1+N])$$

and hence it lives in $\mathrm{Thick}_{(r+1)BN}(G[-B-r-1-N, B+r+1+N])$. On the other side, the cofibre $i_*(E_U) \oplus j_*(E_V)$ lives in $\mathrm{QCoh}(X)_{\geq 1}$ by definition of the error r of right t-exactness. Since the sum $i_*(P_U) \oplus j_*(P_V)$ lives in the above thickening, Lemma 4.1 produces an integer $w \in \mathbb{N}$, assumed $w \geq 2$, such that

$$\hom_X(i_*(P_U) \oplus j_*(P_V), \mathrm{QCoh}(X)_{\geq w}) \in \mathrm{Sp}_{\geq 1}.$$

This $w \in \mathbb{N}$ can now be fed into the definition of approximability of $\mathrm{QCoh}(W)$ to produce an exact sequence

$$P_W \to k^*(F) \to E_W$$

in which $P_W \in \mathrm{Thick}_{(w+r)B}(G_W[-B-w-r, B+w+r])$ and, more importantly, for which the cofibre $E_W \in \mathrm{QCoh}(W)_{\geq w+r}$ is $(w+r)$-connective. Consider now the diagram

$$\begin{array}{ccccc} & & i_*(P_U) \oplus j_*(P_V) & \dashrightarrow & k_*(P_W) \\ & & \downarrow & & \downarrow \\ F & \longrightarrow & i_*i^*(F) \oplus j_*j^*(F) & \longrightarrow & k_*k^*(F) \end{array}$$

where the bottom row is the standard Mayer-Vietoris exact sequence. The goal is to produce the dotted map making the diagram commute. This follows since the obstruction to such a lift lives in $\pi_0 \hom_X(i_*(P_U) \oplus j_*(P_V), k_*(E_W))$ which is zero since $k_*(E_W) \in \mathrm{QCoh}(X)_{\geq w}$ by the definition of E_W and the error r of right t-exactness. Thus the above diagram can be completed to a morphism of exact sequences

$$\begin{array}{ccccc} P & \longrightarrow & i_*(P_U) \oplus j_*(P_V) & \longrightarrow & k_*(P_W) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & i_*i^*(F) \oplus j_*j^*(F) & \longrightarrow & k_*k^*(F) \end{array}$$

Let E be the cofibre of $P \to F$ and notice that the 3×3-lemma provides the exact sequence $E \to i_*(E_U) \oplus j_*(E_V) \to k_*(E_W)$. Since $w \geq 2$, it is the suspension of a 1-connective object, so that $E \in \mathrm{QCoh}(X)_{\geq 1}$ is 1-connective by prestability. It remains to show that P is proxy small. However, rotating the exact sequence defining P shows that it is an extension

$$\Sigma^{-1} k_*(P_W) \to P \to i_*(P_U) \oplus j_*(P_V)$$

where the first object lives in

$$\begin{aligned} \Sigma^{-1} k_*(P_W) &\in \mathrm{Thick}_{(w+r)B}(k_*(G_W)[-B-w-r-1, B+w+r+1]) \\ &\subseteq \mathrm{Thick}_{(w+r)BN}(G[-B-w-r-1-N, B+w+r+N+1]) \end{aligned}$$

and the second one in $\mathrm{Thick}_{(r+1)BN}(G[-B-r-1-N, B+r+1+N])$. Let A_0 be the maximum of $\{(w+r)BN, B+w+r+1+N, (r+1)BN, B+r+1+N\}$ so that both terms live in $\mathrm{Thick}_{A_0}(G[-A_0, A_0])$, thus proving that $P \in \mathrm{Thick}_{2A_0}(G[-A_0, A_0])$. To sum up, the construction has produced for every $F \in \mathrm{QCoh}(X)_{\geq 0}$ an exact sequence $P \to F \to E$ in which $P \in \mathrm{Thick}_{2A_0}(G[-2A_0, 2A_0])$ and $E \in \mathrm{QCoh}(X)_{\geq 1}$. Finally, if A is the maximum of A_0 and A', then A will show both conditions in the definition of approximability, thus proving that $\mathrm{QCoh}(X)$ is approximable. □

We deduce the main result of these notes.

Theorem 4.3. Let X be a quasi-compact quasi-separated scheme. Then $\mathrm{QCoh}(X)$ is approximable.

Proof. Since approximability is known for affine schemes, the result follows from Lemma 4.2. □

Remark 4.4. Note that the proof of Lemma 4.1 has nothing to do with schemes. Indeed, the proof can be easily adapted to show that a weakly approximable category $\mathcal{P}$, which is the pullback in $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ of a span $\mathcal{C} \to \mathcal{E} \leftarrow \mathcal{D}$ of approximable categories in which the right adjoints are colimit preserving and extremely compactly bounded (and right t-exact up to a shift), is approximable. To be more precise, it is sufficient to assume that $\mathcal{P}$ has a compact generator satisfyin assumption (1) of the definition of approximability for a given t-structure and integer.

5. The standard recollement

We start by recalling some terminology.

Remark 5.1. Recall from [CDH+25, Appendix A] that a sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with vanishing composite is called a *Verdier sequence* if $\mathcal{D} \to \mathcal{E}$ is a localization with kernel given by $\mathcal{C} \to \mathcal{D}$. In the presentable case, every Verdier sequence is right split: by definition this means that $\mathcal{D} \to \mathcal{E}$ admits a right adjoint (which can be shown to be fully-faithful).

The following result is well-known; we include a proof for completeness.

Lemma 5.2. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into a quasi-compact open U and closed Z. Then there exists a right split Verdier sequence

$$\mathrm{QCoh}_Z(X) \to \mathrm{QCoh}(X) \xrightarrow{j^*} \mathrm{QCoh}(U)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Furthermore, $\mathrm{QCoh}_Z(X) \in \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ is compactly generated and the inclusion into $\mathrm{QCoh}(X)$ preserves compact objects.

Proof. Since the right adjoint j_* is fully-faithful, the restriction j^* is a localization. An application of [CDH+25, Corollary A.1.10] shows the existence of a Verdier sequence $\ker(j^*) \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ which is automatically right split. Thus the claim will be proved if $\mathrm{QCoh}_Z(X) \simeq \ker(j^*)$. Notice, first of all, that $\pi_n(j^*(F)) \simeq j^*\pi_n(F)$ for every $F \in \mathrm{QCoh}(X)$. This follows since j^* is t-exact. For $(\subseteq)$, let $F \in \mathrm{QCoh}_Z(X)$. Then $\pi_n(F)$ is supported on Z, so that $j^*\pi_n(F) \simeq 0$ in $\mathrm{QCoh}(U)^{\heartsuit}$ for every $n \in \mathbb{Z}$. But this implies that $\pi_n(j^*(F)) \simeq j^*\pi_n(F) \simeq 0$, and since the t-structure on $\mathrm{QCoh}(U)$ is non-degenerate (see for example [BNP23, Lemma 3.6]) it follows that $j^*(F) \simeq 0$, and hence $F \in \ker(j^*)$. Conversely $(\supseteq)$, if $F \in \ker(j^*)$ then $j^*F \simeq 0$, so that $\pi_n(j^*F) \simeq j^*\pi_n(F) \simeq 0$, implying that $F \in \mathrm{QCoh}_Z(X)$. The last claim follows since $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ is closed under all limits. □

We deduce the following:

Corollary 5.3. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then there exists a recollement

$$\mathrm{QCoh}_Z(X) \underset{\Gamma_Z}{\overset{\iota^*}{\leftleftarrows}} \!\!\xrightarrow{\ \iota_*\ }\! \mathrm{QCoh}(X) \underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\!\xrightarrow{\ j^*\ }\! \mathrm{QCoh}(U)$$

Proof. It follows since every right split Verdier sequence extends to a recollement by [CDH+25, Proposition A.2.19]. □

We now use the above recollement to deduce topological information on the closed complement.

Proposition 5.4. Let X be a quasi-compact quasi-separated scheme and let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be a decomposition of X into an open U and closed Z. Then Z is an open subscheme if and only if the pullback i^* induces an equivalence $\mathrm{QCoh}_Z(X) \simeq \mathrm{QCoh}(Z)$.

Proof. Assume first that Z is an open subscheme. Then $X \simeq U \sqcup Z$ as schemes and Zariski descent implies that $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(U) \oplus \mathrm{QCoh}(Z)$. Under this identification, the kernel of $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$ identifies with $\mathrm{QCoh}(Z)$, so that $\mathrm{QCoh}_Z(X) = \ker(j^*) \simeq \mathrm{QCoh}(Z)$. Moreover, the restriction of i^* to this kernel is exactly the projection onto the second factor, hence inducing the required equivalence.

Conversely, let $\mathcal{K} := \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ be the quasi-coherent ideal defining the closed immersion $i : Z \hookrightarrow X$. Since U is quasi-compact, we may choose a finitely generated quasi-coherent ideal $\mathcal{J} \subseteq \mathcal{O}_X$ such that $V(\mathcal{J}) = |Z| = V(\mathcal{K})$. This is possible by [Aut18, Lemma 01PH]. Since $\mathcal{J}$ is finitely generated and $\mathcal{J} \subseteq \sqrt{\mathcal{K}}$, there exists $n \geq 1$ such that $\mathcal{J}^n \subseteq \mathcal{K}$. Assume now that i^* induces an equivalence $i^* : \mathrm{QCoh}_Z(X) \to \mathrm{QCoh}(Z)$. Since the inverse must be given by i_*, it follows that every object of $\mathrm{QCoh}_Z(X)$ lies in the essential image of i_*.

For every $m \geq 1$, the discrete quasi-coherent sheaf $\mathcal{O}_X/\mathcal{J}^m$ belongs to $\mathrm{QCoh}_Z(X)$, since its restriction to U vanishes, and hence it lies in the essential image of i_*. Since i_* is t-exact, it follows that $\mathcal{O}_X/\mathcal{J}^m$ is the pushforward of a discrete quasi-coherent sheaf on Z. Thus the ideal $\mathcal{K}$ acts trivially on $\mathcal{O}_X/\mathcal{J}^m$, and hence $\mathcal{K} \subseteq \mathcal{J}^m$. Applying this with $m = n$ gives $\mathcal{K} \subseteq \mathcal{J}^n \subseteq \mathcal{K}$ and hence $\mathcal{K} = \mathcal{J}^n$. Applying the same argument with $m = n + 1$ gives $\mathcal{K} \subseteq \mathcal{J}^{n+1} \subseteq \mathcal{J}^n = \mathcal{K}$ and hence $\mathcal{J}^n = \mathcal{J}^{n+1}$. Therefore $\mathcal{K}^2 = \mathcal{J}^{2n} = \mathcal{J}^n = \mathcal{K}$. Thus $\mathcal{K}$ is a finitely generated idempotent quasi-coherent ideal. By [Aut18, Lemma 00EH], $\mathcal{K}$ is locally generated by an idempotent, and therefore the closed subscheme $Z = V(\mathcal{K})$ is open and closed in X. Hence Z is an open subscheme. □

Remark 5.5. Notice that the above claim is false if Z is assumed to be only topologically open. Indeed, let k be a field and let $X = \mathrm{Spec}(k[x]/(x^2))$ and $Z = \mathrm{Spec}(k)$. Then Z is a closed subscheme which is topologically open (since the underlying spaces of X and Z agree), but $\mathrm{QCoh}(Z) \simeq \mathrm{Mod}_k$ is not equivalent to $\mathrm{QCoh}_Z(X) \simeq \mathrm{Mod}_{k[x]/(x^2)}$.

6. Approximability for quasi-coherent sheaves with support

We can finally prove the main result of this note.

Theorem 6.1. Let X be a quasi-compact quasi-separated scheme and let Z be a closed subscheme of X with quasi-compact open complement. Then $\mathrm{QCoh}_Z(X)$ is approximable if and only if the underlying topological space of Z is open. In this case, $\mathrm{QCoh}_Z(X)$ is equivalent to the derived category of quasi-coherent sheaves on the open subscheme of X with underlying topological space Z.

Proof. Assume first that $|Z|$ is open. Let $W \subseteq X$ be the open subscheme with underlying space $|Z|$. Since $|Z|$ is also closed, there exists a decomposition $X \simeq U \sqcup W$. Moreover $\mathrm{QCoh}_Z(X) = \mathrm{QCoh}_W(X)$, since both subcategories are defined by the same support condition. Since $\mathrm{QCoh}(W) \simeq \mathrm{QCoh}_W(X)$ by Proposition 5.4, it follows that $\mathrm{QCoh}_Z(X) \simeq \mathrm{QCoh}(W)$. Since $\mathrm{QCoh}(W)$ is approximable by Theorem 4.3, it follows that $\mathrm{QCoh}_Z(X)$ is approximable.

Conversely, assume that $\mathrm{QCoh}_Z(X)$ is approximable. By using again [Aut18, Lemma 01PH], choose a finitely generated quasi-coherent ideal $\mathcal{J} \subseteq \mathcal{O}_X$ such that $V(\mathcal{J}) = |Z|$. Choose a compact generator $G \in \mathrm{QCoh}_Z(X)^\omega$. Since $\mathrm{QCoh}_Z(X)$ is approximable and, by Example 2.6, the relevant t-structure is equivalent to the standard one, [Nee25, Fact 0.26] implies that there exists an integer $A \geq 1$ such that $\mathrm{QCoh}_Z(X)$ satisfies conditions (1) and (3) of Definition 2.2 with respect to the standard t-structure.

Since G is compact, it is perfect as an object of $\mathrm{QCoh}(X)$ by Lemma 5.2. In particular, it has only finitely many nonzero homotopy sheaves $\pi_i(G)$, and these are of finite type. Each $\pi_i(G)$ is supported on $V(\mathcal{J})$, so, since $\mathcal{J}$ is finitely generated, there exists $s \geq 1$ such that $\mathcal{J}^s\pi_i(G) = 0$ for every $i \in \mathbb{Z}$. This implies that every object $E \in \mathrm{Thick}_A(G)$ is such that $\mathcal{J}^{sA}\pi_0(E) = 0$. Indeed, the arbitrary shifts of G clearly satisfy the required vanishing, and arbitrary coproducts of objects with that vanishing again satisfy it (since the t-structure on $\mathrm{QCoh}_Z(X)$ is compatible with coproducts). Furthermore if an object D is an extension $E \to D \to F$ of two objects with the vanishing property, then the long exact sequence in homotopy implies the claim after summing the indices. Finally, it clearly extends to retracts.

Consider now $F := \mathcal{O}_X/\mathcal{J}^{sA+1}$. Since $j^*\mathcal{J} \simeq \mathcal{O}_U$, it follows that the restriction $j^*F \simeq 0$ vanishes, so that $F \in \mathrm{QCoh}_Z(X)_{\geq 0}$ (the connectivity is by construction). By approximability, there exists an exact sequence $E \to F \to D$ with $E \in \mathrm{Thick}_A(G[-A, A])$ and $D \in \mathrm{QCoh}_Z(X)_{\geq 1}$. The long exact sequence then shows that $\pi_0(E) \twoheadrightarrow \pi_0(F) \simeq \mathcal{O}_X/\mathcal{J}^{sA+1}$ is an epimorphism and since $\mathcal{J}^{sA}\pi_0(E) = 0$, it follows that $\mathcal{J}^{sA}(\mathcal{O}_X/\mathcal{J}^{sA+1}) = 0$. Equivalently, $\mathcal{J}^{sA} \subseteq \mathcal{J}^{sA+1}$, so that $\mathcal{J}^{sA} = \mathcal{J}^{sA+1}$. Therefore $\mathcal{J}^{sA}$ is a finitely generated idempotent quasi-coherent ideal. As before, [Aut18, Lemma 00EH] implies that $V(\mathcal{J}) = V(\mathcal{J}^{sA})$ is open and closed in X, making $|Z|$ open. □

References

[Aut18] The Stacks Project Authors. *Stacks Project*. 2018. Available online at The Stacks Project. 9, 10

[BdB02] Alexei Bondal and Michel Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry, 2002. 4

[BNP23] Jesse Burke, Amnon Neeman, and Bregje Pauwels. Gluing approximable triangulated categories. *Forum of Mathematics, Sigma*, 11, 2023. 9

[CDH+25] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian k-theory for stable ∞-categories ii: Cobordism categories and additivity, 2025. 8, 9

[DLR25] Timothy De Deyn, Pat Lank, and Kabeer Manali Rahul. Descending strong generation in algebraic geometry, 2025. 2, 4

[Nee22] Amnon Neeman. Finite approximations as a tool for studying triangulated categories, 2022. 4

[Nee24] Amnon Neeman. Bounded t-structures on the category of perfect complexes, 2024. 2, 4

[Nee25] Amnon Neeman. Triangulated categories with a single compact generator and two brown representability theorems, 2025. 2, 3, 4, 6, 7, 10