THE CELL TRADING LEMMA

GIOVANNI ROSSANIGO

Abstract. These are the notes for my talk in the seminar "Simple Homotopy Theory and Manifold Topology". The goal is to explain the cell trading lemma in simple-homotopy theory and its effect on the universal relative cellular chain complex of a finite CW-pair. In particular, for a finite acyclic CW-pair, repeated cell trading reduces the relative cell structure to two consecutive dimensions, yielding a two-term acyclic complex of free $\mathbb{Z}[\pi_1]$-modules.

1. CW-pairs

We introduce CW-pairs.

Definition 1.1. Let X be a CW-complex. A *CW-pair* (X, A) is the datum of a closed subspace $A \hookrightarrow X$ which is union of cells of X. A *relative n-cell* of (X, A) is an n-cell of X which is not contained in A. We will say that the pair is *finite* if X is a finite CW-complex.

In other terms, if (X, A) is a CW-pair then A is a closed subspace $A \hookrightarrow X$ with a CW-complex structure $\varnothing \subseteq A^0 \subseteq A^1 \subseteq \cdots \subseteq A$ in which the n-skeleton is given by $A^n = A \cap X^n$. The next result shows that the quotient of a CW-pair makes sense as a CW-complex and that its skeleton is rather simple.

Lemma 1.2. Let (X, A) be a CW-pair. For $n \geq 0$ let I_n be the set of relative n-cells. Then the quotient X/A has the structure of a CW-complex, in that there exists a filtration $* = A/A \subseteq X^0/A \subseteq X^1/A \subseteq \cdots \subseteq X/A$ in which $(X^n/A)/(X^{n-1}/A) \cong \bigvee_{I_n} S^n$.

Proof. For each $n \geq 0$, let

$$X^n/A := (X^n \cup A)/A \subseteq X/A.$$

Since A is a subcomplex of X, we have $A^n = A \cap X^n$, and therefore $X^n/A \cong X^n/A^n$. This gives an increasing filtration

$$* = A/A \subseteq X^0/A \subseteq X^1/A \subseteq \cdots \subseteq X/A.$$

Moreover,

$$(X^n/A)/(X^{n-1}/A) \cong X^n/(X^{n-1} \cup A^n).$$

Now X^n is obtained from $X^{n-1} \cup A^n$ by attaching exactly the relative n-cells of (X, A). Hence $(X^n/A)/(X^{n-1}/A) \cong \bigvee_{I_n} S^n$ and the claim follows. □

2. The relative chain complex

We now present an ∞-categorical construction of the relative chain complex of a CW-pair.

Notation 2.1. We let Spc be the ∞-category of spaces and Spc_* be the ∞-category of pointed spaces. We also denote by Sp the stable ∞-category of spectra, and we equip it with the smash symmetric monoidal structure. Recall that there exists an adjunction $\Sigma^\infty : \mathrm{Spc}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty$. Recall also that there exists a base change functor $H\mathbb{Z} \otimes - : \mathrm{Sp} \to \mathcal{D}(\mathbb{Z})$ which takes value in the stable derived ∞-category of abelian groups.

Construction 2.2. Let (X, A) be a CW-pair. Then the inclusion $A \hookrightarrow X$ is a cofibration and hence the pushout $* \leftarrow A \to X$ models the cofibre of $A \to X$ in the ∞-category Spc of spaces. Notice also that the cofibre $X/A \in \mathrm{Spc}_*$ is naturally a pointed space via the map $* \to X/A$. Furthermore, Lemma 1.2 produces a filtration

$$* \to X^0/A \to X^1/A \to \cdots \to X/A$$

in Spc_* whose successive cofibres are coproducts of n-spheres. Apply now the suspension functor $\Sigma^\infty : \mathrm{Spc}_* \to \mathrm{Sp}$ to get a filtered spectrum

$$\Sigma^\infty X^\bullet/A = (\mathbb{S} \to \Sigma^\infty X^0/A \to \Sigma^\infty X^1/A \to \cdots \to \Sigma^\infty X/A)$$

for which the n-graded piece is given by

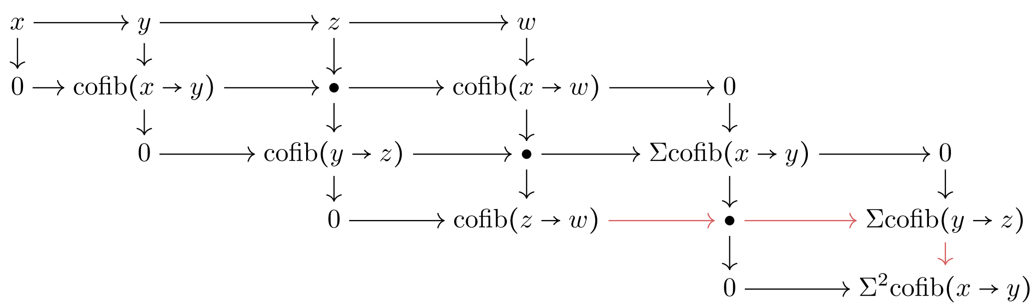
$$\begin{aligned} \mathrm{gr}_n(\Sigma^\infty X^\bullet/A) &= \mathrm{cofib}(\Sigma^\infty X^{n-1}/A \to \Sigma^\infty X^n/A) \\ &\simeq \Sigma^\infty \mathrm{cofib}(X^{n-1}/A \to X^n/A) \\ &\simeq \Sigma^\infty (\bigvee_{I_n} S^n) \\ &\simeq \bigoplus_{I_n} \Sigma^\infty S^n \\ &\simeq \bigoplus_{I_n} \Sigma^n \mathbb{S}. \end{aligned}$$

Here we have used the fact that Σ^∞ commutes with colimits (being a left adjoint) in the first and third equivalence, Lemma 1.2 in the second equivalence. The last equivalence follows since $S^n \simeq \Sigma^n S^0$ and since Σ^∞ commutes with colimits. Apply now the base change functor $H\mathbb{Z} : \mathrm{Sp} \to \mathcal{D}(\mathbb{Z})$ to get a filtration on $H\mathbb{Z} \otimes \Sigma^\infty X/A$.

On can now use the stable Dold-Kan correspondence [Lur17, Theorem 1.2.4.1 and Remark 1.2.4.3] to associate to this filtration a chain complex of abelian groups. Our next goal is to spell out the details. We need the following technical result.

Lemma 2.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ be a stable ∞-category. Given a sequence $x \to y \to z \to w$ in $\mathcal{C}$ there exists a sequence $\mathrm{cofib}(z \to w) \to \Sigma\mathrm{cofib}(y \to z) \to \Sigma^2\mathrm{cofib}(x \to y)$ whose composite is zero.

Proof. It follows by staring at the red arrows in the diagram



obtained by pasting pushouts. We have not depicted the irrelevant terms. □

We continue Construction 2.2.

Construction 2.4. Let (X, A) be a CW-pair and let $H\mathbb{Z} \otimes \Sigma^\infty X/A \in \mathcal{D}(\mathbb{Z})$ be the above constructed spectrum. Equip it with the filtration obtained by base change, and notice that the n-graded piece is equivalent to $\bigoplus_{I_n} \Sigma^n H\mathbb{Z}$ where I_n is the set of relative n-cells. For $n \geq 0$ we let

$$C_n^{\mathrm{rel}}(X, A; \mathbb{Z}) \coloneqq \pi_0 \Sigma^{-n} \mathrm{gr}_n(H\mathbb{Z} \otimes \Sigma^\infty X^\bullet/A) \simeq \bigoplus_{I_n} \mathbb{Z}$$

and notice that $C^{\mathrm{rel}}(X,A;\mathbb{Z})_n$ is a free abelian group. Apply now Lemma 2.3 to the composite

$$H\mathbb{Z}\otimes\Sigma^\infty X^n/A \to H\mathbb{Z}\otimes\Sigma^\infty X^{n+1}/A \to H\mathbb{Z}\otimes\Sigma^\infty X^{n+2}/A \to H\mathbb{Z}\otimes\Sigma^\infty X^{n+3}/A$$

to get a null-sequence between the graded pieces

$$\mathrm{gr}_n(\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A)) \to \Sigma\mathrm{gr}_{n-1}(\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A)) \to \Sigma^2\mathrm{gr}_{n-2}(\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A)).$$

Unwinding the definition of produces a sequence of

$$\dots \to C^{\mathrm{rel}}_{n+1}(X,A;\mathbb{Z}) \to C^{\mathrm{rel}}_n(X,A;\mathbb{Z}) \to C^{\mathrm{rel}}_{n-1}(X,A;\mathbb{Z}) \to \dots$$

whose successive composites are zero, and hence a non-negative homologically graded chain complex of free abelian groups.

Definition 2.5. Let (X,A) be a CW-pair. We will refer to the chain complex $C^{\mathrm{rel}}_\bullet(X,A;\mathbb{Z})$ as the *relative chain complex of* (X,A).

With a little effort we can also compute the homology of this chain complex.

Proposition 2.6. Let (X,A) be a CW-pair. Then the relative chain complex $C^{\mathrm{rel}}_*(X,A;\mathbb{Z})$ computes the relative singular homology of the pair. More precisely, for every $p\geq 0$ there is a canonical isomorphism

$$H_p(C^{\mathrm{rel}}_*(X,A;\mathbb{Z})) \simeq H_p(X,A;\mathbb{Z}).$$

Proof. Consider the skeletal filtration $*\subseteq X^0/A\subseteq X^1/A\subseteq\dots\subseteq X/A$ of Lemma 1.2 and apply $H\mathbb{Z}\otimes\Sigma^\infty(-)$ to get a filtered object in $\mathcal{D}(\mathbb{Z})$ whose n-th graded piece is $\bigoplus_{I_n}\Sigma^n H\mathbb{Z}$ as in Construction 2.2. Then by [Lur17, Section 1.2.2] there exists a convergent spectral sequence

$$E^1_{p,q} = \pi_{p+q}\mathrm{gr}_p(H\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A) \Rightarrow \pi_{p+q}(\mathrm{colim}(H\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A)).$$

Now

$$\mathrm{colim}(H\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A) \simeq H\mathbb{Z}\otimes\Sigma^\infty\,\mathrm{colim}\,X^\bullet/A \simeq H\mathbb{Z}\otimes\Sigma^\infty X/A.$$

This follows since both tensoring with $H\mathbb{Z}$ and taking Σ^∞ commute with colimits and since the filtration of a CW-complex is always exhaustive. Therefore the spectral sequence converges to

$$\pi_{p+q}(\mathrm{colim}(H\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A)) \simeq \pi_{p+q}(H\mathbb{Z}\otimes\Sigma^\infty X/A) \simeq H_{p+q}(X/A;\mathbb{Z}) \simeq H_{p+q}(X,A;\mathbb{Z}).$$

Here we have used the fact that a CW-pair is good in the sense of [Hat00]. On the other side,

$$E^1_{p,q} = \pi_{p+q}\mathrm{gr}_p(H\mathbb{Z}\otimes\Sigma^\infty X^\bullet/A) \simeq \pi_{p+q}(\bigoplus_{I_p}\Sigma^p H\mathbb{Z}) \simeq \pi_q(\bigoplus_{I_p} H\mathbb{Z})$$

is equal to $\bigoplus_{I_p}\mathbb{Z}$ for $q=0$ and zero for $q\neq 0$. Therefore the E^1-page is concentrated in degree $q=0$, and its differential is precisely the differential of the relative cellular chain complex. It follows that

$$E^2_{p,0} \cong H_p(C^{\mathrm{rel}}_*(X,A;\mathbb{Z}))$$

are the only non-trivial terms (in particular, all the differentials are trivial) so that the spectral sequence collapses at E^2, thus providing equivalences

$$H_p(C^{\mathrm{rel}}_*(X,A;\mathbb{Z})) \simeq H_p(X,A;\mathbb{Z}).$$

□

Remark 2.7. Let (X,A) be a CW-pair. Notice that if $A\hookrightarrow X$ is a homotopy equivalence, then the cofibre $X/A\simeq *$ is trivial, and hence the relative chain complex of (X,A) is trivial.

3. The π_1-action

We now show that the relative chain complex of a CW-pair carries an action of the fundamental group of the base. We recall the construction of the universal principal bundle of a group.

Remark 3.1. Let $G \in \mathrm{Grp}$ be a group. Then there are two topological spaces associated to G:

(1) Recall that its classifying space BG may be constructed as the geometric realization of the simplicial set $[n] \mapsto G^n$, namely the nerve of the one-object groupoid with endomorphism group G. Explicitly, the face maps are

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 0 < i < n, \\ (g_1, \dots, g_{n-1}) & i = n, \end{cases}$$

and the degeneracies are

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

One then sets

$$BG := \mathrm{colim}([n] \mapsto G^n).$$

(2) Similarly, EG may be constructed as the geometric realization of the simplicial set $[n] \mapsto G^{n+1}$, that is, of the two-sided bar construction $B_\bullet(*, G, G)$. Its face maps are

$$d_i(g_0, \dots, g_n) = (g_0, \dots, \widehat{g_i}, \dots, g_n),$$

and its degeneracies are

$$s_i(g_0, \dots, g_n) = (g_0, \dots, g_i, g_i, \dots, g_n).$$

There is a simplicial map $E_\bullet G \to B_\bullet G$ given by

$$(g_0, \dots, g_n) \longmapsto (g_0 g_1^{-1},\, g_1 g_2^{-1},\, \dots,\, g_{n-1} g_n^{-1}),$$

hence after realization a map $EG \to BG$. The simplicial set $E_\bullet G$ admits an extra degeneracy, so EG is contractible. Moreover, G acts freely on EG, for instance by right multiplication on the last coordinate, and the quotient is BG. Thus $EG \to BG$ is the universal principal G-bundle.

Now to universal covers:

Remark 3.2. Let X be a connected CW-complex. Since X is semilocal simply connected, there exists a universal cover $\tilde{X} \to X$ which has the structure of a CW-complex, and for which the projection map is cellular and each open cell of $\tilde{X}$ maps homoeomorphically to an open cell of X. Let $\pi_1(X)$ be the first homotopy group of X. Since $\tilde{X} \to X$ is a principal $\pi_1(X)$-bundle it is classified by (the homotopy class of) a map $X \to B\pi_1(X)$ into the classifying space of $\pi_1(X)$, in that it is given by a pullback square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & E\pi_1(X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\pi_1(X), \end{array}$$

in topological spaces. Notice that the map $E\pi_1(X) \to B\pi_1(X)$ is a fibration, so that the above square models the pullback in Spc.

Remark 3.3. Let (X, A) be a CW pair and assume that X is connected. Consider the universal cover $\tilde{X} \to X$. Then the inverse image along this covering of A produces a subspace $\tilde{A}$ which has the structure of a CW-complex and such that the pair $(\tilde{X}, \tilde{A})$ is a CW-pair. In particular, the results of the previous section apply and produce a non-negative chain complex $C^{\mathrm{rel}}_\bullet(\tilde{X},\tilde{A};\mathbb{Z})$ in which the n-th term is

$$C^{\mathrm{rel}}_n(\tilde{X},\tilde{A};\mathbb{Z}) \simeq \bigoplus_{\tilde{I}_n}\mathbb{Z}$$

a free abelian group. Here $\tilde{I}_n$ denotes the set of relative n-cells of $(\tilde{X},\tilde{A})$.

Universal covers have the advantage of having a natural action of the first homotopy group of the base. To induce this action on the above constructed chain complex it is useful to reinterpret everything in the ∞-category of spaces. This will have the effect of simplifying the sum appearing in the n-th group of the relative chain complex of (X,A), although non-canonically.

Remark 3.4. Let (X,A) be a CW-pair and assume that X is connected. Then the principal $\pi_1(X)$-bundle $E\pi_1(X)\to B\pi_1(X)$ furnish a map $*\to B\pi_1(X)$ in the ∞-category of spaces, and the map $X\to B\pi_1(X)$ allows us to regard X as an object of the slice $\mathrm{Spc}_{/B\pi_1(X)}\simeq \mathrm{Fun}(B\pi_1(X),\mathrm{Spc})$. In particular, X may be regarded as a $\pi_1(X)$-space and the pullback

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & E\pi_1(X)\simeq * \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\pi_1(X), \end{array}$$

in Spc defining the universal cover, induces the structure of a free $\pi_1(X)$-space on $\tilde{X}$, which identifies with the usual action by deck transformations. Extending the diagram to a pullback along $A\to X$, produces the same structure on $\tilde{A}$, and since $\tilde{A}$ is stable under this action, the pair $(\tilde{X},\tilde{A})$ is a pair of $\pi_1(X)$-spaces. Now, as CW-complexes, the pair $(\tilde{X},\tilde{A})$ produces a filtration in Spc_* of the cofibre X/A and the successive cofibres of this filtration are coproducts of spheres. In particular, it is clear that now this filtration refines to a filtration $\tilde{X}^\bullet/\tilde{A}$ of $\pi_1(X)$-spaces. Hence Construction 2.2 produces a filtered object in $\mathrm{Fun}(B\pi_1(X),\mathcal{D}(\mathbb{Z}))$. Taking the heart as in Construction 2.4 produces the chain complex $C^{\mathrm{rel}}_\bullet(\tilde{X},\tilde{A};\mathbb{Z})$ whose terms and differentials live in $\mathrm{Fun}(B\pi_1(X),\mathrm{Mod}_{\mathbb{Z}})$, thus giving to $C^{\mathrm{rel}}_\bullet(\tilde{X},\tilde{A};\mathbb{Z})$ a canonical structure of $\mathbb{Z}[\pi_1(X)]$-module.

Finally, picking lifts of relative n-cells specifies an isomorphism

$$I_n\times\pi_1(X)\to\tilde{I}_n$$

and hence a basis of $C^{\mathrm{rel}}_n(\tilde{X},\tilde{A};\mathbb{Z})$ as a free $\mathbb{Z}[\pi_1(X)]$-module, in that one has a non-canonical isomorphism

$$C^{\mathrm{rel}}_n(\tilde{X},\tilde{A};\mathbb{Z})\simeq\bigoplus_{I_n}\mathbb{Z}[\pi_1(X)].$$

With respect to these basis, the differential is given by a matrix with coefficients in $\mathbb{Z}[\pi_1(X)]$.

4. The cell trading lemma

The goal of this section is to explain how to reduce the relative chain complex of $(\tilde{X},\tilde{A})$ for a connected finite CW-pair (X,A) for which the inclusion is a homotopy equivalence to a two term acyclic complex of $\mathbb{Z}[\pi_1(X)]$-modules. Before stating the lemma we recall the notion of simple-homotopy equivalence relative to a subcomplex.

Definition 4.1. Let (X,A) be a CW-pair.

(1) Let $e^n, f^{n+1}\subseteq X\smallsetminus A$ be a n-cell and $(n+1)$-cell. We will say that $e^n\subseteq\overline{f^{n+1}}$ is a *free face* it it is contained only in ∂f^{n+1}.
(2) An *elementary collapse relative* A is a collapse of a pair of cells $e^n\subseteq\overline{f^{n+1}}$ in $X-A$, where e^n is a free face of f^{n+1}, leaving A fixed.
(3) An *elementary expansion relative* A is the gluing of a pair of cells $e^n\subseteq\overline{f^{n+1}}$ in $X-A$, where e^n is a free face of f^{n+1}, leaving A fixed.

Two finite CW-pairs (X, A) and (X', A) are *simple-homotopy equivalent relative* A if one can be obtained from the other by a finite sequence of elementary expansions and collapses relative A.

We shall also recall the notion of relative homotopy groups.

Remark 4.2. Let (X, A) be a CW-pair and assume that A is non-empty. Fix a basepoint $a \in A$. For $n \geq 1$, the *relative homotopy group* $\pi_n(X, A, a)$ is defined as the set of homotopy classes, relative to S^{n-1}, of maps of pairs $(D^n, S^{n-1}) \to (X, A)$ sending a chosen basepoint of S^{n-1} to a. Equivalently,

$$\pi_n(X, A, a) = \pi_0 \mathrm{Hom}_{\mathrm{Spc}_*}(D^n, S^{n-1}, *), (X, A, a).$$

One can show that for $n \geq 2$ this is naturally a group, and for $n \geq 3$ it is abelian. One can also show that there exist a long exact sequence

$$\dots \pi_{n+1}(X, A, a) \to \pi_n X, a \to \pi_n(A, a) \to \pi_n(X, A, a) \to \dots \pi_0(A) \to \pi_0(X)$$

of homotopy groups. Since $A \hookrightarrow X$ is a cofibration, the quotient X/A models the cofibre of $A \to X$ so that for every $n \geq 1$ there is a canonical identification $\pi_n(X, A, a) \cong \pi_n(X/A, *)$ where $* \in X/A$ denotes the image of A.

Now let $e^n \subseteq X \smallsetminus A$ be a relative n-cell, with attaching map

$$\chi_e \colon (D^n, S^{n-1}) \to (X^n, X^{n-1} \cup A) \to (X, A).$$

Its class in $\pi_n(X, A, a)$ is by definition the homotopy class of χ_e. Under the above identification $\pi_n(X, A, a) \cong \pi_n(X/A, *)$ this is precisely the class of the corresponding n-cell of the quotient X/A. Therefore, a class of e^n is trivial in $\pi_n(X, A, a)$ if and only the attaching map of e^n is null-homotopic as a map of pairs, or equivalently that the induced map $D^n/S^{n-1} \simeq S^n \to X/A$ is null-homotopic.

Then:

Lemma 4.3 (The cell trading lemma)**.** Let (X, A) be a CW-pair and let $e^n \subseteq X \smallsetminus A$ be a relative n-cell whose class is trivial in $\pi_n(X, A)$. Then there exists a finite CW-pair (X', A) such that:

(1) The CW-pair (X', A) is simply homotopy equivalent to (X, A) relative to A.
(2) The CW-complex $X' \smallsetminus A$ has one fewer n-cell and one more $(n+2)$-cell than $X \smallsetminus A$.

Proof. Let $\varphi : (D^n, S^{n-1}) \to (X, A)$ be the attaching map for the cell e^n, chosen so that $\varphi(\mathring{D}^n) = e^n$. Since the class of e^n is trivial in $\pi_n(X, A)$, there exists a homotopy of pairs $\Phi : (D^n, S^{n-1}) \times I \to (X, A)$ such that $\Phi(-, 0) = \varphi$. In particular, $\Phi(S^{n-1} \times I) \subseteq A$ and $\Phi(D^n \times \{1\}) \subseteq A$. Now regard $(D^n \times I)$ as an $(n+1)$-disk D^{n+1}, and embed this D^{n+1} as a face of the boundary of an $(n+2)$-ball D^{n+2}. Using Φ as attaching map on that face, form

$$\begin{array}{ccc} D^{n+1} \subseteq \partial D^{n+2} & \longrightarrow & D^{n+2} \\ {\scriptstyle\Phi}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Notice that this is an elementary expansion relative A. Let $C = \partial D^{n+2} \smallsetminus \mathring{D}^{n+1}$ be the complementary face. By construction, the part of ∂D^{n+2} meeting X is exactly D^{n+1}, while C is attached along A. Hence there is an elementary collapse $F : C \cup A \to A$ and therefore one may define

$$\begin{array}{ccc} C \cup A & \xrightarrow{F} & A \\ {\scriptstyle\Phi}\downarrow & & \downarrow \\ Y & \longrightarrow & X' \end{array}$$

The pair (X', A) has the required properties. First, the passage from X to Y adds one new $(n+2)$-cell. Second, inside the attached ball D^{n+2}, the original cell e^n appears as the bottom face $D^n \times \{0\}$ of the track of the homotopy Φ, and after reducing the complementary face C to A, this cell becomes paired with a unique new $(n+1)$-cell. Thus there is an elementary collapse removing this n-cell together with that $(n+1)$-cell. Therefore the net effect is to remove the relative cell e^n and to keep only the new $(n+2)$-cell. ☐

Applying iteratively the cell trading lemma yields the following algorithm.

Proposition 4.4. Let (X, A) be a finite CW-pair such that the inclusion $A \hookrightarrow X$ is a homotopy equivalence. Then there exists a finite CW-pair (X', A), simple-homotopy equivalent to (X, A) rel A, such that $X' \smallsetminus A$ has cells only in two consecutive dimensions.

Proof. Let

$$N := \max\{k : \text{there exists a relative } k\text{-cell in } (X, A)\}.$$

The proof goes is by finite iteration. Since $\pi_k(X, A) = 0$ for all k, as $A \hookrightarrow X$ is a homotopy equivalence, the cell-trading lemma applies and eliminates all relative k-cells for $k = 0, 1, \ldots, N-2$. Each trade removes one k-cell and creates one $(k+2)$-cell. Since $k+2 \le N$, no cells above dimension N are created. Moreover, once all k-cells are removed, no later operation can recreate them, since subsequent trades only involve cells of higher dimension. Thus after stage k, there are no cells in dimensions $\le k$. After completing all stages $k = 0, \ldots, N-2$, the only remaining relative cells lie in dimensions $N-1$ and N. This produces a CW-pair (X', A) with the required property, simple-homotopy equivalent relative A to (X, A). ☐

Corollary 4.5. Let (X, A) be a finite CW-pair such that the inclusion $A \hookrightarrow X$ is a homotopy equivalence. Then the universal relative cellular chain complex $C_*^{\mathrm{rel}}(\widetilde{X}, \widetilde{A}; \mathbb{Z})$ is chain homotopy equivalent to a 2-term acyclic complex of free $\mathbb{Z}[\pi_1(X)]$-modules

$$0 \to C_N \xrightarrow{\partial} C_{N-1} \to 0.$$

In particular, ∂ is an isomorphism.

Proof. In the notations of Proposition 4.4, there exists a finite CW-pair (X', A) that has relative cells only in dimensions $N-1$ and N and is simple homotopy equivalent to (X, A) relative to A. By Construction 2.2, Construction 2.4 and Remark 3.4, the relative cellular chain complex of $(\widetilde{X}', \widetilde{A})$ is concentrated in degrees $N-1$ and N. Since $A \hookrightarrow X'$ is a homotopy equivalence, the pair $(\widetilde{X}', \widetilde{A})$ is acyclic, hence the resulting 2-term complex is acyclic, and ∂ is an isomorphism. ☐

Remark 4.6. Let (X, A) be a finite CW-pair such that the inclusion $A \hookrightarrow X$ is a homotopy equivalence. By Corollary 4.5, after replacing (X, A) by a simple-homotopy equivalent finite CW-pair relative to A, the universal relative cellular chain complex takes the form

$$0 \to C_N \xrightarrow{\partial} C_{N-1} \to 0,$$

where C_N and C_{N-1} are free $\mathbb{Z}[\pi_1(X)]$-modules of the same finite rank, and ∂ is an isomorphism. Choosing lifts of the relative cells to the universal cover and orientations of these lifts identifies both C_N and C_{N-1} with $\mathbb{Z}[\pi_1(X)]^r$. With respect to these bases, the differential is represented by a matrix

$$\partial \in \mathrm{GL}_r(\mathbb{Z}[\pi_1(X)]).$$

Thus the homotopical information carried by the acyclic relative CW-pair is encoded by an invertible matrix over the group ring.

Notice, however, that these choices are not canonical. Changing the lift of a cell multiplies the corresponding basis vector by an element of $\pi_1(X)$, while reversing its orientation multiplies it by -1. Permuting the cells permutes the basis vectors. It follows that different choices of lifts, orientations, and orderings change the matrix of ∂ by left and right multiplication by monomial matrices whose non-zero entries are of the form $\pm g$, with $g \in \pi_1(X)$. Equivalently, ∂ is well-defined only up to the subgroup generated by elementary changes of basis and by the trivial units $\pm g \in \mathbb{Z}[\pi_1(X)]^\times$.

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