QUASI-AFFINE SCHEMES AND SINGLY COMPACTLY GENERATED t-STRUCTURES

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ABSTRACT. We show that for a quasi-compact quasi-separated scheme X with an ample family of line bundles, the connective half $\mathrm{QCoh}(X)_{\geq 0}$ of the standard t-structure on the derived ∞-category of quasi-coherent sheaves is compactly generated by a connective perfect object if and only if X is quasi-affine.

1. INTRODUCTION

In recent years there has been considerable attention to presentable stable ∞-categories equipped with a single compact generator. For such categories, [AJS02, Theorem A.1] says that there exists a t-structure whose connective half is compactly generated by the given single compact generator, in the sense that the connective half is the smallest full subcategory closed under extensions and small colimits containing the generator. A well-known result of Bondal and Van den Bergh [BdB02, Theorem 3.1.1] shows the derived stable ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ on a quasi-compact quasi-separated scheme X admits such compact generator. For applications, it suffices to know that this t-structure is *equivalent* (in the sense of [Nee25, Definition 0.18]) to the standard t-structure, a result proved in [Nee24, Theorem 3.2]. In any case, it is interesting to understand if the two t-structures agree, or, in equivalent terms, if the standard t-structure is compactly generated by a single object. Our main result shows that, under a mild assumption, this occurs precisely when the scheme is quasi-affine.

Theorem 1.1 (Main theorem). Let X be a quasi-compact quasi-separated scheme admitting an ample family of line bundles. Then X is quasi-affine if and only if the connective half $\mathrm{QCoh}(X)_{\geq 0}$ is compactly generated by a connective perfect object.

The key input on the "geometric" side is the 1-*resolution property*. Under the assumption that X admits an ample family of line bundles, [DHM20, Proposition 3.4] shows that the 1-resolution property forces X to be quasi-affine. On the "categorical" side, starting from a connective perfect generator $G \in \mathrm{QCoh}(X)_{\geq 0}$, the ample-family hypothesis implies that G can be chosen *strict*, that is, represented by a bounded complex of vector bundles. From such a representative we extract an associated vector bundle $E_G = \bigoplus_i E_i$. We then show that E_G generates the heart $\mathrm{QCoh}(X)^\heartsuit$, and a simple argument implies that every object of finite-type of $\mathrm{QCoh}(X)^\heartsuit$ is a quotient of a finite sum of copies of E_G. In other words, E_G is a *special* vector bundle and X satisfies the 1-resolution property, so the conclusion follows from [DHM20].

Finally, our result is compatible with recent nonexistence results: for instance, [BDH+25, Theorem A.1] proves that if X is a smooth projective curve over the complex numbers, then the standard connective half $\mathrm{QCoh}(X)_{\geq 0}$ is not singly compactly generated. Since such schemes admit ample line bundles but are not quasi-affine, this phenomenon is predicted by our characterization.

Organization. In section 2 we recall basic facts on Grothendieck prestable ∞-categories and recall the notion of finite type objects in the heart. In section 3 we relate strict compact generators to the

1-resolution property and prove the main theorem. In section 4 we discuss the optimality of the main result.

Acknowledgments. I got interested in the problem addressed here during various trips between Milan and Turin, all posthumous to February 5th, the day my uncle *Andrea* lost his battle against cancer. I would like to dedicate this work to his memory and to all the people fighting such terrible disease.

Spero che Andrea, posto sotto un cielo d'aria nuova, possa aprire gli occhi e sentire pace nell'aurora. Che sia un soffio o un sogno caldo che va, basta che sia libertà. Se poi c'è anche tanta musica...

Conventions. We freely use the language of stable ∞-categories and t-structures on them, which we will grade homologically. Given a scheme X, we denote by $\mathrm{QCoh}(X)$ the derived stable ∞-category of quasi-coherent sheaves on X. We denote by $\hom_{\mathcal{C}}(-,-)$ the mapping spectrum of a stable ∞-category $\mathcal{C}$. We will also denote by $\mathrm{QCoh}(X)_{\geq 0}$ the connective half of the standard t-structure, and by $\mathrm{QCoh}(X)^{\heartsuit}$ the Grothendieck abelian 1-category of discrete quasi-coherent sheaves on X. If A is a ring, we will denote by Mod_A the derived stable ∞-category of modules over A.

2. Grothendieck prestable ∞-categories

We start with some reminders of Grothendieck prestable categories that the reader can find in [Lur18, Appendix C].

Remark 2.1. Recall that an ∞-category is *prestable* if it is pointed, it admits finite colimits, the suspension functor is fully-faithful and every span $0 \to \Sigma x \leftarrow y$ can be completed to a pullback, and the resulting square is also a pushout. Among prestable categories, the one admitting finite limits may be characterized as the connective half of a non-canonical t-structure on certain stable ∞-categories (usually taken to be the Spanier-Whitehead stabilization or the spectrum objects stabilization). Given a prestable ∞-category $\mathcal{C}$, there is a notion of *heart* $\mathcal{C}^{\heartsuit}$, defined to be the full subcategory of discrete objects. In the case where $\mathcal{C}$ admits finite limits, the heart $\mathcal{C}^{\heartsuit}$ may be identified with the heart of the non-canonical t-structure.

Remark 2.2. Recall that a prestable ∞-category $\mathcal{C}$ is *Grothendieck prestable* if it is presentable and filtered colimits are left exact. In this case, the heart $\mathcal{C}^{\heartsuit}$ is a Grothendieck abelian 1-category. Furthermore, every Grothendieck abelian 1-category arises as the heart of a Grothendieck prestable ∞-category. A great number of examples of Grothendieck prestable ∞-categories is given by prestable compactly generated ∞-categories.

The main example for us is the following.

Example 2.3. Let X be a quasi-compact quasi-separated scheme and let $\mathrm{QCoh}(X)_{\geq 0}$ be the connective half of the standard t-structure on the derived ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves. Then $\mathrm{QCoh}(X)_{\geq 0}$ is Grothendieck prestable, being compactly-generated by the connective perfect objects.

The problem of compact generation of t-structure is well understood.

Remark 2.4. Let $\mathcal{C}$ be a presentable stable ∞-category and let E be an object. Then [Lur17, Proposition 1.4.4.11] states that there exists an accessible t-structure whose connective half $\langle E \rangle_{\geq 0}$ is the smallest full subcategory which contains E and is closed under extensions and small colimits. If E is furthermore compact, then $\langle E \rangle_{\geq 0}$ is Grothendieck prestable, since it is compactly generated by the closure under retracts of the smallest full subcategory which contains E and is closed under extensions and finite colimits. In other terms, E is a compact generator for $\langle E \rangle_{\geq 0}$ and the t-structure is *singly compactly generated*.

In the case of compact generators, the previous observation can be improved.

Remark 2.5. Let $\mathcal{C}$ be a presentable stable ∞-category with a single compact generator G. Then [Lur18, Proposition C.6.3.1] states that there exists an accessible, compatible with filtered colimits, right-complete t-structure whose connective half $\langle G \rangle_{\geq 0}$ is the smallest full subcategory which contains G and is closed under extensions and small colimits. This connective half is Grothendieck prestable (being compactly generated by the closure under retracts of the smallest full subcategory which contains G and is closed under extensions and finite colimits). Thus G is a compact generator for $\langle G \rangle_{\geq 0}$. The negative half is instead given by those objects $x \in \mathcal{C}$ such that $\pi_n \hom_{\mathcal{C}}(G, x) \simeq 0$ for $n \geq 1$.

Remark 2.6. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Recall that a generating subcategory is a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ such that for every object $x \in \mathcal{C}$, there exists a collection of maps $p_i : C_i \to x$, where each C_i belongs to $\mathcal{C}_0$ and the induced map $\oplus_{i \in I} C_i \to x$ induces a π_0-epimorphism in $\mathcal{C}^\heartsuit$. Notice that if $\mathcal{C}_0 \subseteq \mathcal{C}$ generates under colimits and extensions, then it is generating. Particular importance is the case where $\mathcal{C}_0$ consists of a single object G (which is not necessarily compact); this is the case of Remark 2.4 and Remark 2.5. In this case the truncation $\pi_0 G$ is a generator for the Grothendieck abelian 1-category $\mathcal{C}^\heartsuit$.

We conclude this section with an observation.

Definition 2.7. Let $\mathcal{C}^\heartsuit$ be a Grothendieck abelian 1-category. An object $x \in \mathcal{C}^\heartsuit$ is called *of finite type* if it satisfies the following equivalent conditions:
  (1) For every filtered diagram $(y_i)_{i \in I}$ in $\mathcal{C}^\heartsuit$ such that each transition map $y_i \to y_j$ is a monomorphism, the canonical map $\mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathcal{C}^\heartsuit}(x, y_i) \to \mathrm{Hom}_{\mathcal{C}^\heartsuit}(x, \mathrm{colim}_{i \in I} y_i)$ is an isomorphism.
  (2) If $(x_i)_{i \in I}$ is a filtered system of subobjects $x_i \subseteq x$ whose union is x, then $x = x_{i_0}$ for some $i_0 \in I$.
The equivalence of these conditions is shown in [Gar99, Theorem 1.8].

We recall the following well known result.

Lemma 2.8. Let $\mathcal{C}^\heartsuit$ be a Grothendieck abelian 1-category and let $E \in \mathcal{C}^\heartsuit$ be a generator. Then every object $x \in \mathcal{C}^\heartsuit$ of finite type admits an epimorphism $\oplus_{j \in J_0} E \to x$ from a finite direct sum.

Proof. Since E is a generator, there exists a set I and an epimorphism $p : \oplus_{i \in I} E \twoheadrightarrow x$. Define $x_J := \mathrm{im}(\oplus_{j \in J} E \xrightarrow{p} x) \subseteq x$ for every finite subset $J \subseteq I$. If $J \subseteq J'$, then $x_J \subseteq x_{J'}$, so the poset of finite subsets of I indexes a filtered system of subobjects (x_J) of x. Since p is an epimorphism, the object x is generated by the images of the individual summands $E \to \oplus_{i \in I} E \xrightarrow{p} x$, and any finite sum of such images is precisely some x_J. Thus $\mathrm{colim}_J x_J \simeq x$, that is, the union of the x_J is all of x. Now, since x is of finite type, it follows that $x \simeq x_{J_0}$ for some finite $J_0 \subset I$. By construction, the map $\oplus_{j \in J_0} E \to x$ has image $x_{J_0} \simeq x$, hence it is an epimorphism. $\square$

3. THE 1-RESOLUTION PROPERTY

Recall that a scheme X has the *resolution property* if every discrete quasi-coherent module of finite type is the quotient of a vector bundle.

Example 3.1. Let X be a scheme. If X has an ample family of line bundles, then [Aut18, Lemma 0GMM] implies that X has the resolution property.

Remark 3.2. For us [Aut18, Lemma 0F8F] is the main result: it states that, for a quasi-compact quasi-separated scheme with the resolution property, every perfect complex can be represented by a bounded complex of vector bundles. In other terms, every perfect complex is *strict*.

We now need a stronger resolution property, which resolves every quasi-coherent module of finite type by a single vector bundle.

Definition 3.3. Let X be a quasi-compact quasi-separated scheme. We will say that X has the 1-*resolution property* if there exists a vector bundle E such that every discrete quasi-coherent sheaf of finite type on X is a quotient of $\oplus_{i \in I} E$ for some finite set I. In this case, E is called a *special vector bundle*.

Remark 3.4. Let X be a quasi-compact scheme admitting an ample family of line bundles. If X has the 1-resolution property then it is quasi-affine. This is [DHM20, Proposition 3.4].

Construction 3.5. Let X be a quasi-compact quasi-separated scheme and let $G \in \mathrm{QCoh}(X)_{\geq 0}$ be a strict compact generator. Pick an equivalence

$$G \simeq (0 \to E_d \to \cdots \to E_1 \to E_0 \to 0)$$

with a bounded complex of vector bundles. Let $E_G = \bigoplus_{i=0}^{d} E_i \in \mathrm{QCoh}(X)^\heartsuit$ and notice that E_G depends on the choice of the above equivalence. We will refer to E_G as a *vector bundle associated to* G.

Lemma 3.6. Let X be a quasi-compact quasi-separated scheme. Let $G \in \mathrm{QCoh}(X)_{\geq 0}$ be a strict compact generator for the connective half and let E_G be an associated vector bundle. Then:
  (1) There is an equality $\mathrm{QCoh}(X)_{\geq 0} = \langle E_G \rangle_{\geq 0}$.
  (2) The object E_G is a generator of $\mathrm{QCoh}(X)^\heartsuit$.

Proof. Consider point (1). Since $\mathrm{QCoh}(X)_{\geq 0}$ is closed under extension and small colimits and contains E_G, it follows that $\langle E_G \rangle_{\geq 0} \subseteq \mathrm{QCoh}(X)_{\geq 0} = \langle G \rangle_{\geq 0}$. For the converse, it suffices to show that G lies in $\langle E_G \rangle_{\geq 0}$. The proof now goes by induction on $0 \leq i \leq d$, where d is the coconnectivity of G. Let $\sigma_{\leq i}$ be the stupid truncation and consider the cofibre sequence

$$\sigma_{\leq i-1} G \to \sigma_{\leq i} G \to \Sigma^i E_i.$$

If $i = 0$ then the connectivity of G implies that $\sigma_{\leq -1} G \simeq 0$, so that $\sigma_{\leq} G \simeq E_0$ belongs to $\langle E_G \rangle_{\geq 0}$, being E_0 a retract of E_G. The inductive hypothesis assumes now $\sigma_{\leq i-1} G \in \langle E_G \rangle_{\geq 0}$. Since the last term belongs to $\langle E_G \rangle_{\geq 0}$, being a suspension of a retract of E_G, it follows that $\sigma_{\leq i} G \in \langle E_G \rangle_{\geq 0}$, being $\langle E_G \rangle_{\geq 0}$ closed under extensions. Thus $G \simeq \sigma_{\leq d} G$ belongs to $\langle E_G \rangle_{\geq 0}$.

For (2), since point (1) shows that E_G is a generator for $\mathrm{QCoh}(X)_{\geq 0}$, it suffices to apply Remark 2.6 to deduce that $E_G \simeq \pi_0 E_G$ is a generator for $\mathrm{QCoh}(X)^\heartsuit$. $\square$

The following is the main result of these pages.

Lemma 3.7. Let X be a quasi-compact quasi-separated scheme. Let $G \in \mathrm{QCoh}(X)_{\geq 0}$ be a strict compact generator and let E_G be the associated vector bundle. Then E_G is a special vector bundle for $\mathrm{QCoh}(X)^\heartsuit$. In particular, X has the 1-resolution property.

Proof. Since Lemma 3.6 implies that the object $E_G \in \mathrm{QCoh}(X)^\heartsuit$ is a generator, an application of Lemma 2.8 shows that every discrete quasi-coherent sheaf of finite type admits a surjection from a finite sum of copies of E_G. Since E_G is a vector bundle by definition, the claim follows. $\square$

Corollary 3.8. Let X be a quasi-compact quasi-separated scheme. Let $G \in \mathrm{QCoh}(X)_{\geq 0}$ be a strict compact generator and let E_G be the associated vector bundle.
  (1) If X is a noetherian normal scheme which is J-2, then it is quasi-affine.
  (2) If X is a normal noetherian $\mathbb{Q}$-factorial scheme (for example, regular), then it is quasi-affine.

Proof. Standing Lemma 3.7, point (1) follows from [DHM20, Theorem 3.13] and point (2) from [DHM20, Corollary 3.8]. $\square$

We summarize our discussion with the following.

Theorem 3.9. Let X be a quasi-compact quasi-separated scheme with an ample family of line bundles. Then X is quasi-affine if and only if the connective half $\mathrm{QCoh}(X)_{\geq 0}$ is compactly generated by a connective perfect object.

Proof. Let X be quasi-affine and let $\Gamma(X, \mathcal{O}_X)$ denote the ordinary ring of global sections of the structure sheaf. Then the canonical map $j : X \hookrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is an open immersion. It follows that $j^* : \mathrm{Mod}_{\Gamma(X, \mathcal{O}_X)} \to \mathrm{QCoh}(X)$ is a t-exact localization, and hence $\mathcal{O}_X \simeq j^*(\mathbb{1}_{\Gamma(X, \mathcal{O}_X)})$ compactly generates $\mathrm{QCoh}(X)_{\geq 0}$.

Conversely, assume that $\mathrm{QCoh}(X)_{\geq 0}$ is compactly generated by a connective perfect object. Since X admits an ample family of line bundles, this connective perfect object is strict by Example 3.1 and Remark 3.2. By Lemma 3.7 the scheme X satisfies the 1-resolution property, and since it has an ample family of line bundles, Remark 3.4 implies that X is quasi-affine. $\qquad\square$

4. Final observations

The main result recovers an "horrible discovery".

Example 4.1. In [BDH$^+$25, Theorem A.1] it is stated that a smooth projective curve over the complex numbers has connective half which is not singly compactly generated. This is an example of a scheme with an ample family of line bundles (and actually, a single one), for which Theorem 3.9 applies. Notice furthermore that Theorem 3.9 provides a different proof (since [BDH$^+$25, Theorem A.1] relies on the Harder-Narasimhan filtration).

Notice that Theorem 3.9 is also far from being optimal: [BDH$^+$25, Proposition 3.7] shows that algebraic spaces proper over a field with singly compactly generated connective half are precisely the artinian affine schemes. It raises the question: are there quasi-compact quasi-separated schemes with singly compactly generated connective half which are not quasi-affine?

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