

SEMINAR SS 2026  
HIGHER ZARISKI GEOMETRY

ORGANIZERS: Denis-Charles Cisinski, Giovanni Rossanigo

TIME AND PLACE: Tuesday, 12:15–13:45, M101

OVERVIEW

Algebraic geometry teaches us that commutative rings are best understood through the geometry of their spectra: to a ring  $R$  one attaches a topological space  $\mathrm{Spec}(R)$  together with a sheaf of rings  $\mathcal{O}_{\mathrm{Spec}(R)}$ , and much of the algebra of  $R$  is reflected in the geometry of this locally ringed space. Over the last two decades a parallel story has emerged in homotopy theory and representation theory. Small triangulated symmetric monoidal categories behave like “rings up to homotopy”, and tensor-triangular geometry, through the work of Balmer, Krause, Neeman, Stevenson and many others, provides the necessary tools to construct topological spaces out of these “2-rings”. Despite the numerous successes (reconstruction theorems and so on), the theory lacks a uniform description of structure sheaves on such spectra. The goal of this seminar is to learn a further categorification of this picture which solves many issues of the tensor triangulated approach and organizes coherently many well-known results. The main reference is [ABC+25].

ORGANIZATION

The seminar is divided into four parts. The first, corresponding to talks 2-3 and 6-7, discusses some complementary topics in category theory. In particular, talks 2-3 provide the higher categorical foundations for homological algebra, whereas talks 6-7 complement higher topos theory with the notion of geometry. The second part, talks 4-5, constitute the classic approach to tensor-triangulated geometry and Balmer’s spectrum theory. The approach is decidedly more down-to-earth. The third part, that is, talks 8-9-10, introduce Zariski geometry on compact 2-rings, deduce the formal properties and compare the spectrum of a 2-ring with the Balmer spectrum of the underlying homotopy category. In the final part, amounting to talks 11-14, discusses Zariski descent, local-to-global principles, support data, and the telescope conjecture for rigid 2-rings.

The audience is expected to have some familiarity with algebraic geometry and  $\infty$ -categories. In any case, there will be talks that will fill the gaps, making the talks accessible to everyone willing to treated higher categories as a black box. Please contact the organizers if you have any questions regarding the material.

TALKS

**Talk 1: Introduction and distribution of talks (Giovanni Rossanigo, 14.04.2026).** Overview of the seminar and distribution of the talks. Please attend if you are considering giving a talk, otherwise contact the organizers.

**Talk 2: Stable and symmetric monoidal categories (Matteo Munafò, 21.04.2026).** The goal of this talk is to setup the basic language needed in the rest of the seminar. Introduce stable  $\infty$ -categories following [Lur17, Section 1.1.1] or [Cno, Section 3.1]. Define exact functors and prove some of their properties. Define  $\mathrm{Cat}^{\mathrm{ex}}$ . Sketch the proof of [Lur17, Theorem 1.1.2.14], which shows that the homotopy category of a stable  $\infty$ -category is triangulated. Note also that not every triangulated category arises in this way.

Introduce symmetric monoidal  $\infty$ -categories as commutative monoids in  $\mathrm{Cat}$  and discuss symmetric monoidal functors. You should follow [Cno, Section 4.3, Section 4.5], but [Lur17, Chapter 2] can be

also useful. Say also some words on commutative algebra objects in symmetric monoidal  $\infty$ -categories. Conclude by noting that the homotopy category of a symmetric monoidal  $\infty$ -category is symmetric monoidal.

**Talk 3: 2-rings (Liva Diler, 28.04.2026).** Start by introducing the “big” version of 2-rings. That is, define presentable  $\infty$ -categories as in [Lur09b, Section 5.5]. Introduce  $\mathrm{Pr}^L$  and its symmetric monoidal structure. Consider also the stable version  $\mathrm{Pr}_{\mathrm{st}}^L$ . References for this are [Lur09b, Section 5.5] and [Lur17, Section 4.8.1], but I will provide you simpler reference. Introduce very briefly idempotent-complete categories as in [Lur09b, Section 4.4.5].

Introduce  $\mathrm{Cat}^{\mathrm{perf}}$  and explain how we can always assume to work internally to  $\mathrm{Cat}^{\mathrm{perf}}$ , by saying some words about the idempotent completion functor  $(-)^{\mathrm{h}} : \mathrm{Cat}^{\mathrm{ex}} \rightarrow \mathrm{Cat}^{\mathrm{perf}}$ . Explain then how idempotent completion and compact objects furnish a relation between  $\mathrm{Cat}^{\mathrm{perf}}$  and  $\mathrm{Pr}_{\mathrm{st}}^L$ . Define then commutative 2-rings as small, idempotent-complete, stable, symmetric monoidal  $\infty$ -categories with biexact tensor product following [ABC<sup>+</sup>25, Section 2.A]. Conclude giving the the main examples:  $\mathrm{Perf}(R)$  for a commutative ring (spectrum)  $R$  and  $\mathrm{Perf}(X)$  for a scheme  $X$ .

**Talk 4: Tensor triangular geometry and the Balmer spectrum (Yiming Wang, 05.05.2026).**

The goal of this talk is to explain [ABC<sup>+</sup>25, Section 2.B and Section 4.B] and it should be divided into three parts. In the first one, discuss thick tensor ideals, radical ideals and principal ideals. Introduce lattices, complete lattices, frames and coherent frames ([ABC<sup>+</sup>25, Recollection 2.21 and Definition 4.5]) as well as their categories. Prove then the main result of the talk, that is [ABC<sup>+</sup>25, Proposition 2.23].

In the second part explain Karoubi quotients<sup>1</sup> [ABC<sup>+</sup>25, Definition 2.25], and how they fit into the lattice of localizations. You may give a rough sketch of [ABC<sup>+</sup>25, Proposition 2.27] and fully prove [ABC<sup>+</sup>25, Proposition 2.28]. There is no need to discuss the corollaries that follow.

Finally, the last part is dedicated to the introduction of Balmer spectrum [ABC<sup>+</sup>25, Definition 4.9]. For that, review the necessary Stone/Hochster duality background and the description of spectral spaces in terms of coherent frames. Notice how the Balmer spectrum depends only on the homotopy category.

**Talk 5: A reconstruction theorem (Antonin Milesi, 12.05.2026).** The goal of this talk is to show [Bal04, Theorem 6.3.(a)], that is, for a qcqs scheme  $X$ , the Balmer spectrum  $\mathrm{Spc}(\mathrm{Perf}(X))$  carries a sheaf of rings  $\mathcal{O}_{\mathrm{Perf}(X)}$  and there is an isomorphism of schemes  $X \simeq (\mathrm{Spc}(\mathrm{Perf}(X)), \mathcal{O}_{\mathrm{Perf}(X)})$ . The proof is done by first identifying the underlying topological spaces and then comparing the structure sheaves.

For the topological part, begin by introducing supports and support data [Bal04, Definition 3.1]. State and prove [Bal04, Theorem 3.2]. Deduce the functoriality of the Balmer spectrum at the triangulated level. Discuss then classifying support data and state [Bal04, Theorem 5.2]. Details of the proof can be omitted. Finally, state [Bal04, Theorem 5.5] without the “topologically noetherian” assumption (compare to the original reference [Tho97, Theorem 3.15]).

For identifying the structure sheaf, have a look at [Bal04, Remark 6.2] or [Bal02, Section 5]. There it is produced a presheaf of tt-categories. To deal with the case  $\mathrm{Spc}(\mathrm{Perf}(X))$ , you may use [Bal02, Theorem 2.13 and Theorem 7.8], without proof. Now to get a presheaf of commutative rings take endomorphism rings and use [Bal02, Lemma 9.6]. Deduce then the main theorem.

In all the references the scheme  $X$  is assumed to be topologically noetherian. This is not necessary; all the proofs work verbatim.

**Talk 6: Lurie’s geometries and  $\mathcal{G}$ -structured  $\infty$ -topoi I (Bastiaan Cnossen, 19.05.2026).**

The goal is to give a concise introduction to Lurie’s notion of *geometry* and  $\mathcal{G}$ -structured  $\infty$ -topoi.

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<sup>1</sup>You may want to note the distinction with Verdier sequences: Verdier sequences become Karoubi after idempotent completion.

The introduction of [Lur09a, Chapter 1] can be used as motivation. Divide the talk in two parts. In the first part, define  $\infty$ -topoi as in [Lur09b, Chapter 6] and say some words on why sheaves on a site are examples of  $\infty$ -topoi. Define morphisms of  $\infty$ -topoi and their category  $L\text{Top}$ , together with  $R\text{Top}$ . Say some words about points.

In the second one, explain [ABC<sup>+</sup>25, Section 3.A]. In particular, introduce admissible structures, geometries and morphisms of geometries. Explain also [ABC<sup>+</sup>25, Section 3.B]: introduce sheaves with values in any category with finite limits,  $\mathcal{G}$ -structures and local maps. Define  $L\text{Top}(\mathcal{G})$ .

Use the example of classical Zariski geometry as a motivation for all definitions.

**Talk 7: Lurie’s geometries and  $\mathcal{G}$ -structured  $\infty$ -topoi II (Bin Wu, 26.05.2026).** Continue talk 6 by discussing [ABC<sup>+</sup>25, Section 3.C]. Recall briefly all the necessary notions and state [ABC<sup>+</sup>25, Theorem 3.21]. There is no need to state [ABC<sup>+</sup>25, Lemma 3.22]. Proceed then to discuss the case where one of the geometries is discrete. Introduce the  $\mathcal{G}$ -structured global sections functor and prove [ABC<sup>+</sup>25, Lemma 3.23 and Lemma 3.24]. Identify then  $\text{Spec}_{\mathcal{G}_{\text{disc}}}^{\mathcal{G}}$  with the absolute  $\mathcal{G}$ -spectrum functor via [ABC<sup>+</sup>25, Theorem 3.28].

Finally, take us once again to the familiar setting of classical Zariski geometry [ABC<sup>+</sup>25, Section 3.D] and present the spectral Dirac geometry as a second example [ABC<sup>+</sup>25, Section 3.E].

**Talk 8: The Zariski geometry on  $2\text{CAlg}$  (Speaker, 02.06.2026).** The goal of this talk is to introduce the Zariski geometry on compact 2-rings and compare with the Balmer spectrum.

Show first that  $2\text{CAlg}$  is compactly generated [ABC<sup>+</sup>25, Proposition 2.5] and study its generators [ABC<sup>+</sup>25, Corollary 2.14]. Use then [ABC<sup>+</sup>25, Proposition 2.28], proved in talk 4, to deduce its corollaries. Introduce the Zariski geometry on 2-rings and prove [ABC<sup>+</sup>25, Theorem 4.2]. Prove also [ABC<sup>+</sup>25, Lemma 4.3] Prove [ABC<sup>+</sup>25, Theorem 4.11] and all the result needed. Coordinate the proof of this theorem with the speaker of talk 9.

**Talk 9: Points of  $\text{Spec}$  (Speaker, 09.06.2026).** Continue talk 8 by identifying the behavior of spectrum at the level of points. Recall the definition of points of a topos. Recall the comparison theorem [ABC<sup>+</sup>25, Theorem 4.11] and deduce [ABC<sup>+</sup>25, Corollary 4.14], thus identify points of the spectrum with prime ideals. Continue by explaining [ABC<sup>+</sup>25, Section 4.C]. State and prove all the results needed to deduce [ABC<sup>+</sup>25, Corollary 4.27]

**Talk 10: Comparison with classical geometries (Marco Volpe, 16.06.2026).** The goal of this talk is to state and prove [ABC<sup>+</sup>25, Theorem 4.48] relating the two notions of spectrum of a classical ring. Black box Theorem 5.11 in the proof. To prepare the proof, explain how taking endomorphisms of the unit furnishes a filtered colimit preserving right adjoint to taking perfect complexes. Recall the Dirac geometry and prove [ABC<sup>+</sup>25, Proposition 4.33]. Explain then [ABC<sup>+</sup>25, Proposition 4.41]. If time permits, tease us with Theorem 4.49.

**Talk 11: Zariski descent for 2-rings (Ou Liu, 23.06.2026).** The first goal of this talk is to state and prove the reduction schema for proving Zariski descent, that is [ABC<sup>+</sup>25, Proposition 5.8]. For that introduce Zariski covers and explain sheaves on a frame. The second goal of the talk is Theorem 5.11, which says that for a rigid 2-ring the canonical structure presheaf is already a sheaf. Deduce it from the stronger descent result [ABC<sup>+</sup>25, Theorem 5.14], and sketch the details of the proof. Conclude with [ABC<sup>+</sup>25, Corollary 5.21].

**Talk 12: Zariski descent for modules (Speaker, 30.06.2026).** The goal of this talk is to state and prove [ABC<sup>+</sup>25, Theorem F] which is contained in Section 5.C. The results are Theorem 5.28, Theorem 5.40 and Corollary 5.41. Since the intermediate steps needed to prove these are notationally technical, feel free to black box certain computations.

**Talk 13: Support data (Speaker, 07.07.2026).** The goal of this talk is to explain support data, that is [ABC<sup>+</sup>25, Appendix A]. Explain how the Balmer spectrum of a tensor-triangulated category

is characterized as the final support datum. Introduce then all the results needed to prove Theorem A.14.

**Talk 14: The telescope conjecture for 2-rings (Speaker, 14.07.2026).** The goal of this talk is to explain the “vibe” behind smashing localizations and the telescope conjecture. You are free to organize the talk as you want; just state [ABC<sup>+</sup>25, Theorem B.2] which explains that a rigid 2-ring satisfies the telescope conjecture if and only if all its stalks do. If time permits, say also some words on the proof.

#### REFERENCES

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