Universality of Span $(\infty, 2)$-categories

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Abstract

These are the notes I have written to prepare my talk for the Workshop on Six-Functor Formalisms. The last talks of the workshop were based on the paper of Cnossen, Lenz, and Linskens [CLL25]. Although my talk focused on the construction of an $(\infty, 2)$-enhancement of the span $(\infty, 1)$-category $\mathrm{Span}(\mathcal{C}, E)$, these notes collect the surrounding material needed to explain its universal property.

1 Constructing 6-functor formalisms is hard (but still possible!)

As we have seen in the previous talks, 3-functor formalisms encode a wealth of structure in a highly coherent fashion, making them a powerful tool in algebra and geometry. However, constructing such formalisms is often a rather tedious process, if possible. In practice, it is usually much easier to construct a functor $\mathcal{D}_0 : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat})$ that encodes the pullback functors f^* and the tensor products $\otimes$, and it is therefore natural to look for conditions ensuring $\mathcal{D}_0$ extends in some preferred way to $\mathrm{Span}(\mathcal{C}, E)$. Let us have a look on what are the available options.

(1) One proposal for such an extension is due to Mann [Man22, Proposition A.5.10], building on the work of Liu–Zheng [LZ12a, LZ12b]. The construction requires specifying a *suitable decomposition* of E, that is, two classes of morphisms I (thought of as "local isomorphisms") and P (thought of as "proper maps"), satisfying certain axioms (see Definition 1.1). In particular, every morphism in E must factor as a composite $p \circ i$ with $i \in I$ and $p \in P$. Assuming moreover that each functor i^* for $i \in I$ admits a left adjoint $i_\sharp$, and that each p^* for $p \in P$ admits a right adjoint p_*, and that these adjoints satisfy suitable base-change equivalences (including a mixed condition $i_\sharp p'_* \simeq p_* i'_\sharp$) together with the expected projection formulæ, Mann shows that one can extend $\mathcal{D}_0$ to a 3-functor

$$\mathcal{D} : \mathrm{Span}(\mathcal{C}, E) \to \mathrm{Cat},$$

such that the exceptional pushforward $f_!$ is given by the left adjoint to f^* when $f \in I$, and by the right adjoint when $f \in P$. Despite all of that, the Mann-Liu-Zheng approach has an enormous drawback: it relies on some heavy $(\infty, 1)$-categorical machinery, deeply rooted in Lurie's language of quasi-categories.

(2) There second approach is due to Gaitsgory and Rozenblyum, and it focuses on the universal properties of certain $(\infty, 2)$-categories built from spans. Instead of working with Cat, Gaitsgory and Rozenblyum work with an arbitrary symmetric monoidal $(\infty, 2)$-category $\mathbb{D}$ and 2-functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbb{D}$. This is possible since the conditions on $\mathcal{D}_0$ considered previously may be formulated in this generality; for example, we say that F is left I-adjointable if the pullback functor $i^* := F(i)$ for $i \in I$

admits a left adjoint $i_\#$ satisfying the Beck–Chevalley condition, and dually we say that F is right P-adjointable if the functor p^* for $p \in P$ admits a right adjoint p_* satisfying the Beck–Chevalley condition. Gaitsgory and Rozenblyum conjectured that the universal target for left I-adjointable functors is a certain $(\infty, 1)$-categorical enhancement of $\mathrm{Span}(\mathcal{C}, I)$ whose 2-morphisms are spans between spans, and similarly for P. Rigorous proofs of this universal property were later provided independently by Macpherson [Mac22] and Stefanich [Ste20]. Gaitsgory and Rozenblyum also outlined two further conjectures that would allow the construction of suitable $(\infty, 2)$-functors more generally in the presence of a decomposition $E = P \circ I$ satisfying similar, but more restrictive assumptions than in Mann's result [GR17].

The contribution of Cnossen, Lenz and Linskens, the topic of these notes, is the proof of a $(\infty, 2)$-categorical universal property of span $(\infty, 2)$-categories which work in Mann's assumptions. In order to state their result, we need to introduce some basic language. Let's start with Mann's assumptions.

Definition 1.1. Let $\mathcal{C}$ be an $(\infty, 1)$-category and let $E \subseteq \mathcal{C}$ be a wide subcategory. A *suitable decomposition of* $(\mathcal{C}, E)$ is a pair $I, P \subseteq E$ of wide subcategories such that:

(1) The subcategories I and P are closed under base change and are left cancellable[1].

(2) Every morphism in E factors as $p \circ i$ for some $i \in I$ and $p \in P$.

(3) Every map in $I \cap P$ is truncated.

In the following, we will equivalently say that E admits a suitable decomposition (I, P). We will also denote by $\rightarrowtail$ maps in I and by $\twoheadrightarrow$ maps in P.

To formalize (I, P)-biadjointable functors we need some nomenclature. Let $\mathbb{D}$ be an $(\infty, 2)$-category and let $\sigma = \begin{tikzcd}[cramped, sep=small] \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ be a commutative square. We say that σ is *horizontally left adjointable* if its horizontal morphisms admit left adjoints and the resulting Beck–Chevalley map is an equivalence. Similarly, we say that σ is *vertically right adjointable* if its vertical morphisms admit right adjoints and the resulting Beck–Chevalley map is an equivalence. Finally, we say that σ is *biadjointable* if it is horizontally left adjointable and the resulting square of left adjoints is vertically right adjointable.

Definition 1.2. Let $\mathcal{C}$ be an $(\infty, 1)$-category with two wide subcategories $I, P \subseteq \mathcal{C}$ closed under pullbacks along arbitrary morphisms of $\mathcal{C}$. Let $\mathbb{D}$ be an $(\infty, 2)$-category and let $F : \mathcal{C}^{\mathrm{op}} \to \mathbb{D}$ be a functor.

(1) We will say that F is *left* I-*adjointable* if every pullback square $\begin{tikzcd}[cramped, sep=small] \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$ in $\mathcal{C}$ is sent to an horizontally left adjointable square in $\mathbb{D}$.

(2) We will say that F is *right* P-*adjointable* if every pullback square $\begin{tikzcd}[cramped, sep=small] \cdot \arrow[r] \arrow[d, two heads] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ in $\mathcal{C}$ is sent to an vertically right adjointable square in $\mathbb{D}$.

(3) We will say that F is (I, P)-*biadjointable* if it is both left I-adjointable and right P-adjointable, every pullback square $\begin{tikzcd}[cramped, sep=small] \cdot \arrow[r, tail] \arrow[d, two heads] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$ in $\mathcal{C}$ is sent to a biadjointable square.

Many names for very simple things... In order to remember all these definitions, it is useful to always depict maps in I as horizontal arrows and maps in P as vertical ones. Since maps in I are required to have left adjoints and maps in P are required to have right adjoints, Definition 1.2 then becomes straightforward.

[1]That is, if $gf, g \in I$, then $f \in I$, and similarly for P.

In this picture we are also assuming that F, being contravariant, exchanges vertical arrows with vertical arrows and horizontal arrows with horizontal arrows.

In any case, we also need to restrict the natural transformations that we allow between such functors. Indeed, given two functors $F, G : \mathcal{C}^{op} \to \mathbb{D}$ of one of the types above and a natural transformation $\alpha : F \Rightarrow G$ between them, we will say that α is of the same type if the obvious naturality squares are horizontally left adjointable, and vertically right adjointable respectively. For α being (I, P)-*biadjointable* will be the same of asking that it is both left I-adjointable and right P-adjointable. In particular, we can consider the locally full sub-$(\infty, 2)$-category

$$\mathrm{FUN}_{(I,P)\text{-biadj}}(\mathcal{C}^{op}, \mathbb{D}) \subseteq \mathrm{FUN}(\mathcal{C}^{op}, \mathbb{D})$$

of (I, P)-biadjointable functors and (I, P)-biadjointable natural transformations between them. This allows us to make the result of Cnossen, Lenz, and Linskens more precise.

Theorem 1.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category and E a class of maps admitting a suitable decomposition (I, P). Then there exists a universal (I, P)-biadjointable functor $h : \mathcal{C}^{op} \hookrightarrow \mathrm{SPAN}(\mathcal{C}, E)_{P,I}$, where $\mathrm{SPAN}(\mathcal{C}, E)_{P,I}$ is an $(\infty, 2)$-enhancement of the $(\infty, 1)$-category $\mathrm{Span}(\mathcal{C}, E)$ of spans in $\mathcal{C}$ [CLL25, Theorem A].

That is, for every $(\infty, 2)$-category $\mathbb{D}$ restriction along h induces an equivalence

$$h^* : \mathrm{FUN}(\mathrm{SPAN}(\mathcal{C}, E)_{P,I}, \mathbb{D}) \to \mathrm{FUN}_{(I,P)\text{-badj}}(\mathcal{C}^{op}, \mathbb{D})$$

of $(\infty, 2)$-categories. Moreover, the underlying $(\infty, 1)$-category satisfies

$$i_1 \mathrm{SPAN}(\mathcal{C}, E)_{P,I} \simeq \mathrm{Span}(\mathcal{C}, E).$$

From this description, it is clear that this result has an extremely practical consequence and addresses the difficulties discussed at the beginning of this section.

Remark 1.4. When constructing a six functor formalism, we usually start with an (I, P)-biadjointable functor $\mathcal{D}_0 : \mathcal{C}^{op} \to \mathbb{D}$. The universal property of the span $(\infty, 2)$-category gives a unique extension

$$\mathcal{D} : \mathrm{SPAN}(\mathcal{C}, E)_{P,I} \longrightarrow \mathbb{D}.$$

This 2-functor is fairly explicit. On objects, it is simply given by $\mathcal{D}(x) = \mathcal{D}_0(x)$.

A span

$$x \xleftarrow{f} u \xrightarrow{e} y$$

is sent to the composite

$$e_! f^* : F(x) \longrightarrow F(y),$$

where $f^* := \mathcal{D}_0(f)$. The *exceptional pushforward* $e_!$ is obtained by choosing a decomposition $e = p \circ i$ into a map $i \in I$ followed by a map $p \in P$ and setting

$$e_! = p_* i_\sharp,$$

where $i_\sharp$ is a left adjoint to i^* and p_* is a right adjoint to p^*. The effect of $\mathcal{D}$ on 2-morphisms is constructed using the units and counits for the adjunctions $i_\sharp \dashv i^*$ and $p^* \dashv p_*$.

Remark 1.5 (Our program)**.** The remaining goal of these pages is to understand how to prove Theorem 1.3. It's clear that there are many things that we have to do. Indeed:

(1) First of all, we have to construct the $(\infty,2)$-category of spans $\mathrm{SPAN}(\mathcal{C},\mathrm{E})_{\mathrm{P},\mathrm{I}}$ in such a way that there exists a functor $h : \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}(\mathcal{C},\mathrm{E})_{\mathrm{P},\mathrm{I}}$ which is (I,P)-biadjointable. Moreover, taking the underlying $(\infty,1)$-category of this $(\infty,2)$-category should give us the usual category of spans. This will be done in section 2.

(2) Secondly, it will be useful to prove a *recognition principle* that allows us to deduce when a certain (I,P)-biadjointable functor $h : \mathcal{C}^{\mathrm{op}} \to \mathbb{B}_{\mathrm{I},\mathrm{P}}(\mathrm{C})$ satisfies the conclusion of Theorem 1.3. This will be achieved with Theorem 3.3, which, essentially, tells us that a functor an (I,P)-biadjointable functor $\mathrm{F} : \mathcal{C}^{\mathrm{op}} \to \mathbb{D}$ is universal if and only if it is essentially surjective and for every object $a \in \mathcal{C}$ the composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{F}} \mathbb{B} \xrightarrow{\mathrm{HOM}_{\mathbb{B}}(\mathrm{F}(a),-)} \mathrm{Cat}$$

is a *free* (I,P)*-biadjointable functor on* $id_{\mathrm{F}(a)}$.

(3) We will then try to apply this result in the case where $\mathbb{B}$ is our $(\infty,2)$-category of spans and F is our h. This will bring us two problems: firstly, we have to identify the HOM-categories in this span $(\infty,2)$-category and, secondly, we have to understand when a functor is "freely generated". This last step will require the use of *parametrized category theory*; section 4 will help us understand what to do.

And with all of that we will prove Theorem 1.3.

2 The $(\infty,2)$-enhancement of Span

We start our program by constructing the $(\infty,2)$-categorical enhancement of the span $(\infty,1)$-category. Let us first recall how it is constructed.

Remark 2.1. Let $\mathcal{C}$ an $(\infty,1)$-category. An *adequate triple* on $\mathcal{C}$ is a pair of two wide subcategories $(\mathcal{C}_{\mathrm{L}},\mathcal{C}_{\mathrm{R}})$ of $\mathcal{C}$ such that morphisms of $\mathcal{C}_{\mathrm{L}}$ admit pullbacks along morphisms of $\mathcal{C}_{\mathrm{R}}$ and the resulting maps are again in $\mathcal{C}_{\mathrm{L}}$ and $\mathcal{C}_{\mathrm{R}}$. Similarly, a functor between adequate triples is a functor at the level of categories, which sends left (respectively, right) morphisms to left (respectively, right) morphisms and preserves the relevant pullback square. These data can be organized into an $(\infty,1)$-category AdTrip. The span category is then a functor

$$\mathrm{Span}_{\mathrm{L},\mathrm{R}}(-) : \mathrm{AdTrip} \to \mathrm{Cat}$$

whose value at a triple is constructed as a complete Segal space, which we now explain.

Let $\mathrm{Tw}^{\mathrm{ct}}[n] \to [n]^{\mathrm{op}} \times [n]$ be the cartesian unstraightening of the Hom-functor $\mathrm{Hom}_{[n]}(-,-) : [n]^{\mathrm{op}} \times [n] \to \mathrm{Spc} \subseteq \mathrm{Cat}$. In these notes we refer to $\mathrm{Tw}^{\mathrm{ct}}[n]$ as the *twisted arrow category*. For example, $\mathrm{Tw}^{\mathrm{ct}}[2]$ is the poset

$$\begin{tikzcd}[column sep=small]
 & & (0\le 2) \arrow[dl] \arrow[dr] & & \\
 & (0\le 1) \arrow[dl] \arrow[dr] & & (1\le 2) \arrow[dl] \arrow[dr] & \\
(0\le 0) & & (1\le 1) & & (2\le 2)
\end{tikzcd}$$

and $\mathrm{Tw}^{\mathrm{ct}}[n]$ is the evident generalization. Every square is a pullback square. By defining left and right maps of $\mathrm{Tw}^{\mathrm{ct}}[n]$ as those pointing left and right, respectively, one obtains an adequate triple and hence the usual span construction [HHLN23].

To construct the $(\infty,2)$-categorical enhancement of span categories, we proceed in a similar fashion. The main difference is at the start: instead of working with simplicial spaces, we work we simplicial simplicial spaces.

Definition 2.2. A *double category* is a complete Segal object in complete Segal spaces.

In other terms, a double category is a functor $\mathbb{C} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Spc}$ such that $\mathbb{C}(n,-), \mathbb{C}(-,n) : \Delta^{\mathrm{op}} \to \mathrm{Spc}$ are complete Segal spaces for every $[n] \in \Delta$. The intuition is that $\mathbb{C}(0,0)$ is the space of objects, $\mathbb{C}(1,0)$ is the space of horizontal arrows, $\mathbb{C}(0,1)$ is the space of vertical arrows, and $\mathbb{C}(1,1)$ is the space of squares. The two Segal conditions encode horizontal and vertical composition, while completeness ensures that equivalences are detected correctly. In particular, it is clear that an $(\infty,2)$-category should be a double category $\mathbb{C}$ such that the functor $\mathbb{C}(0,-) : \Delta^{\mathrm{op}} \to \mathrm{Spc}$ is constant.

Remark 2.3. We can now proceed with the construction. First of all, we define Λ_n to be the full subcategory of $\mathrm{Tw}^{\mathrm{ct}}[n]$ spanned by those objects $(i \leq j)$ such that $j \leq i+1$. Then, if $\mathcal{C}$ is any $(\infty,1)$-category with pullbacks, we define

$$\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}) \subseteq \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Tw}^{\mathrm{ct}}(-) \times \mathrm{Tw}^{\mathrm{ct}}(-), \mathcal{C})$$

to be the subobject given in degree (m,n) by those functors $\mathrm{Tw}^{\mathrm{ct}}[m] \times \mathrm{Tw}^{\mathrm{ct}}[n] \to \mathcal{C}$ that are right Kan extended from $\Lambda_m \times \Lambda_n$. In particular, the resulting squares in $\mathcal{C}$ are pullbacks. Haugseng proves that this is a double category [Hau18, Proposition 5.14].

Definition 2.4. Let $\mathcal{C}$ be an $(\infty,1)$-category with pullbacks. We define $\mathrm{SPAN}(\mathcal{C})$ as the horizontal fragment of the double category $\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C})$.

Remark 2.5. In particular, by unravelling the definitions, we see that:

(1) The groupoid $\mathrm{SPAN}(\mathcal{C})(0,0)$ of objects of $\mathrm{SPAN}(\mathcal{C})$ agrees with the underlying groupoid of $\mathcal{C}$.

(2) The hom space between two objects x, y is given by the space of diagrams $x \leftarrow z \to y$.

(3) The space of 2-morphisms between two spans $x \leftarrow z \to y$ and $x \leftarrow z' \to y$ is given by the space of diagrams

$$\begin{tikzcd}
x \arrow[d, equal] & z \arrow[l] \arrow[r] & y \arrow[d, equal] \\
x \arrow[d, equal] & z'' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & y \arrow[d, equal] \\
x & z' \arrow[l] \arrow[r] & y.
\end{tikzcd}$$

Thus a 2-morphism is a span of spans, as expected.

In the following we will of course need a more elaborate version of the span $(\infty,2)$-category.

Remark 2.6. Assume we are given pullback-preserving embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$ into a category with all pullbacks. We then define $\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}, E)_{P,I}$ as the sub-double category of $\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}')$ where the subspace at

$(1,1)$ is given by the subspace of diagrams

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \arrow[r, "E"] \arrow[dr, phantom, "\text{P-pb}"] & \cdot \\
\cdot \arrow[u] \arrow[d, "E"'] \arrow[dr, phantom, "\text{I-pb}"] & \cdot \arrow[l] \arrow[r, "E"] \arrow[u] \arrow[d, "E"] & \cdot \arrow[u] \arrow[d, "E"] \\
\cdot & \cdot \arrow[l] \arrow[r, "E"'] & \cdot
\end{tikzcd}$$

where all objects are in $\mathcal{C}$, various maps belong to the subcategory E as indicated, the notation P-pb means that the induced map into the pullback belongs to P, and similarly for I-pb.

A direct diagram chase shows that these diagrams are closed under horizontal and vertical pasting, so this is a sub-double category. We define $\mathrm{SPAN}(\mathcal{C}, E)_{P,I}$ as its horizontal fragment. In other words, its objects are the objects of $\mathcal{C}$, its 1-morphisms are spans $x \leftarrow z \xrightarrow{e} y$ with $e \in E$, and its 2-cells are diagrams of the form

$$\begin{tikzcd}
& z \arrow[dl] \arrow[dr, "E"] & \\
x & z'' \arrow[l] \arrow[r, "E"] \arrow[u, "P"] \arrow[d, "I"'] & y, \\
& z' \arrow[ul] \arrow[ur, "E"'] &
\end{tikzcd}$$

Finally, this definition is independent of the choice of the embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$: equivalently, it can be written as a subfunctor of $\mathrm{Hom}(\mathrm{Tw}^{\mathrm{ct}}(-) \times \mathrm{Tw}^{\mathrm{ct}}(-), \mathcal{C})$ [CLL25, Section 4.1].

The following result summarizes the properties of $\mathrm{SPAN}(\mathcal{C}, E)_{P,I}$.

Lemma 2.7. Let $\mathcal{C}$ be an $(\infty,1)$-category and E a class of maps admitting suitable decomposition (I, P). Then:

(1) The underlying $(\infty,1)$-category of $\mathrm{SPAN}(\mathcal{C}, E)_{P,I}$ is $\mathrm{Span}(\mathcal{C}, E)$.

(2) The inclusion $h : \mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{SPAN}(\mathcal{C}, E)_{P,I}$ is (I, P)-biadjointable.

Proof. Restricting the defining bisimplicial space to $\Delta^{\mathrm{op}} \times \{[0]\}$ gives precisely the complete Segal space defining $\mathrm{Span}(\mathcal{C}, E)$, proving the first claim.

For the second claim, a map $i : x \to y$ in I is represented by the span $y \xleftarrow{i} x \xrightarrow{\mathrm{id}} x$. Its left adjoint is the reversed span $x \xleftarrow{\mathrm{id}} x \xrightarrow{i} y$; the diagonal $x \to x \times_y x$ and the evident identity span provide the unit and counit. Dually, the span associated to a map in P admits a right adjoint. For a pullback square, the corresponding Beck–Chevalley transformation is represented by the canonical comparison with the pullback and is therefore an equivalence. The same calculation in a square with horizontal maps in I and vertical maps in P gives the double Beck–Chevalley equivalence. Hence h is (I, P)-biadjointable. See [CLL25, Section 4.1] for the complete diagrammatic calculation. $\square$

3 A recognition principle

Definition 3.1. Let $\mathcal{C}$ be an $(\infty, 1)$-category and let $I, P \subseteq \mathcal{C}$ be wide subcategories closed under base change. Let $h : \mathcal{C}^{op} \to \mathbb{B}_{I,P}(\mathcal{C})$ be an (I, P)-biadjointable functor. We will say that h is *universal* if the restriction

$$h^* : \mathrm{FUN}(\mathbb{B}_{I,P}(\mathcal{C}), \mathbb{D}) \to \mathrm{FUN}_{(I,P)\text{-badj}}(\mathcal{C}^{op}, \mathbb{D})$$

is an equivalence of $(\infty, 2)$-categories for every $(\infty, 2)$-category $\mathbb{D}$.

Definition 3.2. Let $a \in \mathcal{C}$ be an object and let $F : \mathcal{C}^{op} \to \mathrm{Cat}$ be an (I, P)-biadjointable functor. We will say that F is *free on a class* $1_a \in F(a)$ if for every other (I, P)-biadjointable functor $G : \mathcal{C}^{op} \to \mathrm{Cat}$ evaluation at 1_a induces an equivalence of categories

$$\mathrm{HOM}_{\mathrm{FUN}_{(I,P)\text{-biadj}}(\mathcal{C}^{op}, \mathrm{Cat})}(F, G) \simeq G(a).$$

Theorem 3.3 (Recognition principle for biadjointable functors)**.** Let $F : \mathcal{C}^{op} \to \mathbb{B}$ be an (I, P)-biadjointable functor. Then F is universal if and only if it is essentially surjective and for every object $a \in \mathcal{C}$ the composite

$$\mathcal{C}^{op} \xrightarrow{F} \mathbb{B} \xrightarrow{\mathrm{HOM}_{\mathbb{B}}(F(a), -)} \mathrm{Cat}$$

is a free (I, P)-biadjointable functor on $\mathrm{id}_{F(a)}$.

Proof. If F is universal, the 2-categorical Yoneda lemma shows that each displayed Hom functor is free on the identity; universality also implies that F is essentially surjective.

Conversely, let $h : \mathcal{C}^{op} \to \mathbb{B}_{I,P}(\mathcal{C})$ be a universal (I, P)-biadjointable functor. Universality of h extends F uniquely to a 2-functor

$$\overline{F} : \mathbb{B}_{I,P}(\mathcal{C}) \longrightarrow \mathbb{B}.$$

Essential surjectivity of $F = \overline{F}h$ implies essential surjectivity of $\overline{F}$. To prove full faithfulness, essential surjectivity of h reduces the problem to the maps

$$\mathrm{HOM}_{\mathbb{B}_{I,P}(\mathcal{C})}(h(a), h(-)) \longrightarrow \mathrm{HOM}_{\mathbb{B}}(F(a), F(-)).$$

Both sides are free (I, P)-biadjointable functors on their identity classes, and the displayed map preserves those classes. Their universal properties therefore make it an equivalence. Thus $\overline{F}$ is an equivalence and F is universal [CLL25, Theorem 2.21]. □

4 Free completion of a parametrized $(\infty, 1)$-category

We are now in point (3) of Remark 1.5: we need to understand what it means for a functor to be "freely generated".

Let $\mathcal{C}$ be an $(\infty, 1)$-category. Suppose we are given a functor $F : \mathcal{C}^{op} \to \mathbb{D}$. What we have to do is to understand when, given an object $a \in \mathcal{C}$, the composite

$$\mathcal{C}^{op} \xrightarrow{F} \mathbb{B} \xrightarrow{\mathrm{HOM}_{\mathbb{B}}(F(a), -)} \mathrm{Cat}$$

is a free (I,P)-biadjointable functor on $\mathrm{id}_{\mathrm{F}(a)}$ in the sense of Definition 3.2. Since this composite is a contravariant functor from $\mathcal{C}$ into Cat, we are naturally lead to apply the methods of *parametrized category theory*.

Remark 4.1. Here it is a short dictionary.

(1) Functors $\mathcal{X} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ are called *$\mathcal{C}$-categories*, and natural transformations $\mathcal{X} \to \mathcal{Y}$ between $\mathcal{C}$-categories are called $\mathcal{C}$-functors.

(2) We will say that $\mathcal{X}$ is I-*cocomplete* if it is left I-adjointable, and dually that it is P-*complete* if it is right P-adjointable. We wish to think of the left adjoints to maps in I and right adjoints to maps in P as certain parametrized (co)limits; accordingly, we will refer to I-left adjointable natural transformations as I-*cocontinuous $\mathcal{C}$-functors*, while P-right adjointable transformations will be called P-*continuous $\mathcal{C}$-functors*.

(3) We say that $\mathcal{X}$ is (I,P)-*semiadditive* if it is (I,P)-biadjointable; this may be interpreted as asking I-colimits to commute with P-limits. We will call (I,P)-biadjointable natural transformations (I,P)-*bicontinuous*.

(4) We denote by

$$\mathrm{Fun}_{\mathcal{C}}^{\mathrm{I}\amalg}(\mathcal{X},\mathcal{Y}), \qquad \mathrm{Fun}_{\mathcal{C}}^{\mathrm{P}\Pi}(\mathcal{X},\mathcal{Y}), \quad \mathrm{Fun}_{\mathcal{C}}^{\mathrm{I}\amalg,\mathrm{P}\Pi}(\mathcal{X},\mathcal{Y}),$$

the categories of I-cocontinuous, P-continuous, and (I,P)-bicontinuous functors. These are the Hom-categories in the corresponding 2-categories of adjointable functors: left I-adjointable in the first case, right P-adjointable in the second, and (I,P)-biadjointable in the third.

In this new language, our question is: given a $\mathcal{C}$-category $\mathcal{X}$, what is the free (I,P)-semiadditive $\mathcal{C}$-category generated by $\mathcal{X}$? Taking $\mathcal{X} = \mathrm{Hom}_{\mathcal{C}}(-,a) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc} \subseteq \mathrm{Cat}$ will then identify the free (I,P)-biadjointable functor generated by an object $a \in \mathcal{C}$. Here it is the answer:

Theorem 4.2 (Free semiadditive cocompletion)**.** Let $\mathcal{C}$ be an $(\infty,1)$-category and let $\mathrm{E} \subseteq \mathcal{C}$ be a class of arrows with a suitable decomposition (I,P). Let $\mathcal{X}$ be a $\mathcal{C}$-category. Then there exists a universal (I,P)-semiadditive $\mathcal{C}$-category $\underline{\mathrm{Span}}_{\mathrm{P}\text{-fold},\mathrm{I}}(\mathcal{X}^{\mathrm{E}\amalg})$ [CLL25, Theorem 3.6].

In other words, we can construct a functor $i : \mathcal{X} \to \underline{\mathrm{Span}}_{\mathrm{P}\text{-fold},\mathrm{I}}(\mathcal{X}^{\mathrm{E}\amalg})$ such that precomposition with i induces equivalence

$$i^* : \mathrm{Fun}_{\mathcal{C}}^{\mathrm{I}\amalg,\mathrm{P}\Pi}(\underline{\mathrm{Span}}_{\mathrm{P}\text{-fold},\mathrm{I}}(\mathcal{X}^{\mathrm{E}\amalg}),\mathcal{Y}) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$$

of $(\infty,1)$-categories for every (I,P)-semiadditive $\mathcal{C}$-category $\mathcal{Y}$. The proof of this theorem is in two steps.

(1) In the first one we construct the *free* E-*cocomplete $\mathcal{C}$-category* associated to $\mathcal{X}$. That is, a $\mathcal{C}$-category $\mathcal{X}^{\mathrm{E}\amalg}$ together with a functor $j : \mathcal{X} \to \mathcal{X}^{\mathrm{E}\amalg}$ which is universal among all functors from $\mathcal{X}$ to a E-cocomplete $\mathcal{C}$-category.

(2) In the second one we construct the $\mathcal{C}$-parametrized category $\underline{\mathrm{Span}}$ appearing in the statement. This is the hard part of Theorem 4.2, and is also the one that we will treat as a black box.

Before jumping directly into computations, proofs and examples, let us motivate why we should believe in such statement.

Example 4.3. Recall first that a category with finite products and coproducts is called *semiadditive* if it is pointed and finite product and finite coproducts coincide. Suppose that we have a 1-category $\mathcal{X}$. How do we construct the free semiadditive category on $\mathcal{X}$?

First of all, we add finite sums to $\mathcal{X}$. This is achieved by considering the free completion $\mathcal{X}^{\amalg}$ of $\mathcal{X}$ under finite coproducts. Then we force finite coproducts to be finite products. To do that, we consider the full subcategory $(\mathcal{X}^{\amalg})_{\mathrm{fold}}$ of $\mathcal{X}^{\amalg}$ spanned by the fold maps, that is by the codiagonal maps

$$\nabla : X \amalg \cdots \amalg X \to X$$

and by $X \amalg Y \to X$. Then one checks (by hand) that this 1-category is semiadditive and that the natural functor $i : \mathcal{X} \to \mathrm{Span}_{\mathrm{fold,all}}(\mathcal{X}^{\amalg})$ is fully-faithful and it induces an equivalence

$$i^* : \mathrm{Fun}^{\oplus}(\mathrm{Span}_{\mathrm{fold,all}}(\mathcal{X}^{\amalg}), \mathcal{Y}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{Y})$$

of 1-categories for semiadditive 1-category $\mathcal{Y}$. Here $\mathrm{Fun}^{\oplus}(-,-)$ is the category of additive functors.

So let us start with point (1).

Definition 4.4. Let $\mathcal{X} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ be a $\mathcal{C}$-parametrized $(\infty, 1)$-category. The free E-cocompletion of $\mathcal{X}$ is an E-cocomplete $\mathcal{C}$-parametrized $(\infty, 1)$-category $\mathcal{X}^{\mathrm{E}\amalg} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ together with a $\mathcal{C}$-functor $j : \mathcal{X} \to \mathcal{X}^{\mathrm{E}\amalg}$ such that, for every $\mathcal{C}$-parametrized E-cocomplete $(\infty, 1)$-category $\mathcal{Y}$, precomposition with j induces an equivalence

$$j^* : \mathrm{Fun}_{\mathcal{C}}^{\mathrm{E}\amalg}(\mathcal{X}^{\mathrm{E}\amalg}, \mathcal{Y}) \to \mathrm{Fun}_{\mathcal{C}}(X, \mathcal{Y})$$

of $(\infty, 1)$-categories.

That is, the free E-cocompletion of $\mathcal{X}$ is the analogue of the free cocompletion of a category under a chosen class of morphisms. In particular, such a $\mathcal{C}$-category exists and is unique up to equivalence by [MW24, Theorem 7.1.13]; the inverse to j^* is given by parametrized left Kan extension along j in the sense of [MW24, Definition 6.3.1].

Since believing it's not enough for us (we want to compute!), let us discuss another model provided by Macpherson.

Remark 4.5. Let $\mathrm{Ar}_{\mathrm{E}}(\mathcal{C}) \subseteq \mathrm{Ar}(\mathcal{C})$ be the full subcategory of arrows in E. Given a $\mathcal{C}$-category $\mathcal{X} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, we can consider its cartesian unstraightening $\mathrm{Un}^{\mathrm{ct}}(\mathcal{X}) \to \mathcal{C}$ and therefore the pullback square

$$\begin{tikzcd}
\mathrm{Ar}_{\mathrm{E}}(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Un}^{\mathrm{ct}}(\mathcal{X}) \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Un}^{\mathrm{ct}}(\mathcal{X}) \arrow[d] \\
\mathrm{Ar}_{\mathrm{E}}(\mathcal{C}) \arrow[r, "\mathrm{ev}_0"'] & \mathcal{C}
\end{tikzcd}$$

where q is cartesian. Since ev_1 is cartesian, the composite

$$\mathrm{ev}_1 q : \mathrm{Ar}_{\mathrm{E}}(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Un}^{\mathrm{ct}}(\mathcal{X}) \to \mathcal{C}$$

is a cartesian fibration. Macpherson proved in [Mac22, Proposition 3.5.2] that the cartesian straightening of this cartesian fibration gives exactly the free E-cocompletion of $\mathcal{X}$. More precisely, there exists an equivalence of cartesian fibrations over $\mathcal{C}$

$$\mathrm{Un}^{\mathrm{ct}}(\mathcal{X}^{\mathrm{E}\amalg}) \simeq \mathrm{Ar}_{\mathrm{E}}(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Un}^{\mathrm{ct}}(\mathcal{X})$$

such that the cartesian straightening of $j : \mathcal{X} \to \mathcal{X}^{\mathrm{E}\amalg}$ is induced by the identity section $\mathcal{C} \hookrightarrow \mathrm{Ar}_{\mathrm{E}}(\mathcal{C}), x \mapsto \mathrm{id}_x$.

This is extremely useful, since it allows to deduce the value of $\mathcal{X}^{\mathrm{E}\amalg}$ at an object $a \in \mathcal{C}$. Indeed, $\mathcal{X}^{\mathrm{E}\amalg}(a)$ is exactly the fiber of $\mathrm{ev}_1 q$ at a, hence its objects are

$$\mathcal{X}^{\mathrm{E}\amalg}(a) = \big\{ (f : b \to a) \in \mathrm{E},\, x \in \mathcal{X}(b) \big\}$$

which we may think as a "formal E-colimit" $f_\#(j(x))$ of the "generator" $j(x) \in \mathcal{X}^{\mathrm{E}\amalg}(b)$. We can also describe morphisms in $\mathcal{X}^{\mathrm{E}\amalg}(a)$. These are pairs

$$\begin{tikzcd}
x \arrow[rr, dashed] & & x' \\
\\
b \arrow[rr, dashed] \arrow[dr] & & b' \arrow[dl] \\
& a &
\end{tikzcd}$$

of a morphism $b \to b'$ in $\mathrm{E}_{/a}$ and a morphism $x \to x'$ in $\mathrm{Un}^{\mathrm{ct}}(\mathcal{X})$ that maps to $b \to b'$.

Perfect! We can now proceed with the second step. Since we will construct $\underline{\mathrm{Span}}_{\text{P-fold},\mathrm{I}}(\mathcal{X}^{\mathrm{E}\amalg})$ as a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{AdTrip} \xrightarrow{\mathrm{Span}} \mathrm{Cat}$$

it will suffice to associate, in a functorial way, to every object of $\mathcal{C}$ an adequate triple. Let $a \in \mathcal{C}$ be an object. Since Remark 4.5 provided us an explicit description of the morphism in $\mathcal{X}^{\mathrm{E}\amalg}(a)$,

(1) We say that a morphism is P-*fold map* if the morphism $b \to b'$ is in P and the morphism $x \to x'$ is a cartesian lift of $b \to b'$. We will denote by $\mathcal{X}^{\mathrm{E}\amalg}_{\text{P-fold}}(a) = \mathrm{Un}^{\mathrm{ct}}(\mathcal{X})_{\mathrm{ct}} \times_{\mathcal{C}} \mathrm{E}_{/a} \times_{\mathrm{E}} \mathrm{P}$ the wide subcategory of $\mathcal{X}^{\mathrm{E}\amalg}(a)$ spanned by the P-fold morphisms.

(2) We say that a morphism *lies over* I if the map $b \to b'$ is in I. We will denote by $\mathcal{X}^{\mathrm{E}\amalg}_{\mathrm{I}}(a) = \mathrm{Un}^{\mathrm{ct}}(\mathcal{X}) \times_{\mathcal{C}} \mathrm{E}_{/a} \times_{\mathrm{E}} \mathrm{I}$ the wide subcategory of $\mathcal{X}^{\mathrm{E}\amalg}(a)$ spanned by the morphisms that lie over I.

In order to get a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{AdTrip}$ there are many things that one should check. First of all, one should use the description of the cartesian edges for $\mathrm{Un}^{\mathrm{ct}}(\mathcal{X}^{\mathrm{E}\amalg}) \to \mathcal{C}$ to check that these wide subcategories are closed under cartesian transport (so that resulting in wide $\mathcal{C}$-subcategories). Secondly, for fixed $a \in \mathcal{C}$, one should check that $(\mathcal{X}^{\mathrm{E}\amalg}(a), \mathcal{X}^{\mathrm{E}\amalg}_{\text{P-fold}}(a), \mathcal{X}^{\mathrm{E}\amalg}_{\mathrm{I}}(a))$ is adequate. This follows since $(\mathrm{E}, \mathrm{P}, \mathrm{I})$ is adequate plus the fact that AdTrip has pullbacks preserved by the inclusion into Cat...

We will stop here with our discussion of Theorem 4.2, so let us have a look at its consequences.

Corollary 4.6. Let $a \in \mathcal{C}$ be an object. Then the functor

$$\mathrm{Span}_{\mathrm{P},\mathrm{I}}(\mathrm{E}_{/-} \times_{\mathcal{C}} \mathcal{C}_{/a}) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$$

is the free (I, P)-biadjointable functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ on the class $(a = a = a)$ in $\mathrm{F}(a)$.

Proof. Let $\text{よ}(a) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc} \subseteq \mathrm{Cat}$ be the presheaf represented by $a \in \mathcal{C}$. By applying Theorem 4.2 to $\mathcal{X} = \text{よ}(a)$ we construct the free (I, P)-semiadditive $\mathcal{C}$-category

$$i : \text{よ}(a) \to \underline{\mathrm{Span}}_{\text{P-fold},\mathrm{I}}(\text{よ}(a)^{\mathrm{E}\amalg})$$

on $\text{よ}(a)$. Translated in the language of category theory, we have constructed a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ which is (I, P)-biadjointable. We now claim that this functor is free on the class $1_a \in \underline{\mathrm{Span}}_{\text{P-fold},\mathrm{I}}(\text{よ}(a)^{\mathrm{E}\amalg})(a)$. Indeed, the parametrized Yoneda lemma identifies the element 1_a, and the universal property gives the required equivalence

$$\mathrm{HOM}_{\mathrm{FUN}_{(I,P)\text{-biadj}}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat})}\big(\underline{\mathrm{Span}}_{P\text{-fold},I}(よ(a)^{E\amalg}), G\big) \simeq G(a)$$

for every (I,P)-biadjointable functor $G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$.

It remains to identify the functor in the statement with $\underline{\mathrm{Span}}_{P\text{-fold},I}(よ(a)^{E\amalg})$. Recall that $よ(a)^{E\amalg}$ is the cartesian straightening of $\mathrm{ev}_1 q : \mathrm{Ar}_E(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Un}^{\mathrm{ct}}(よ(a)) \to \mathcal{C}$. But the cartesian straightening $\mathrm{Un}^{\mathrm{ct}}(よ(a)) \to \mathcal{C}$ of $よ(a)$ is given by the source functor $\mathcal{C}_{/a} \to \mathcal{C}$. So we compute

$$\begin{aligned} よ(a)^{E\amalg} &\simeq \mathrm{St}^{\mathrm{ct}}\big(\mathrm{Ar}_E(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Un}^{\mathrm{ct}}(よ(a)) \to \mathcal{C}\big) \\ &\simeq \mathrm{St}^{\mathrm{ct}}\big(\mathrm{Ar}_E(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/a} \to \mathcal{C}\big) \\ &\simeq E_{/-} \times_{\mathcal{C}} \mathcal{C}_{/a} \end{aligned}$$

as functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. Similarly, one computes the morphisms that lie over I and the p-fold maps, thus proving the claim. □

Since in the proof of Theorem 1.3 we wish to apply Theorem 3.3, we must compare the HOM-categories with free (I,P)-biadjointable functors on the identity. We do this by comparing their cocartesian unstraightenings.

Example 4.7. The cocartesian unstraightening of $\underline{\mathrm{Span}}_{P\text{-fold},I}(よ(a)^{E\amalg})$ is the target map

$$\mathrm{Span}(\mathrm{ev}_1) : \mathrm{Span}_{P\text{-pb},I\text{-fw}}\big(\mathrm{Ar}_E(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/a}\big) \longrightarrow \mathcal{C}^{\mathrm{op}},$$

where the pullback is taken over the source map $\mathrm{ev}_0 : \mathrm{Ar}_E(\mathcal{C}) \to \mathcal{C}$. This follows from the explicit cartesian unstraightening of $よ(a)$ and the description of parametrized span categories in [CLL25, Example 3.26].

5 Universality of span $(\infty,2)$-categories

Once again, we fix an $(\infty,1)$-category $\mathcal{C}$ and wide subcategories $I, P \subseteq E \subseteq \mathcal{C}$ such that (I,P) is a suitable decomposition of $(\mathcal{C},E)$. The final ingredient needed to prove Theorem 1.3 is identifying the HOM-categories of $\mathrm{SPAN}(\mathcal{C},E)_{P,I}$. This requires some knowledge on how the unstraightening of the functor of $(\infty,1)$-categories

$$\mathrm{HOM}_{\mathbb{C}}(-,-) : (i_1\mathbb{C})^{\mathrm{op}} \times (i_1\mathbb{C}) \to \mathrm{Cat}$$

associated to an $(\infty,2)$-category $\mathbb{C}$ works. We only need the case where $\mathbb{C}$ is a span $(\infty,2)$-category, but the following general observation is useful.

Remark 5.1. Let $\boxtimes$ denote the Gray tensor product. For an $(\infty,2)$-category $\mathbb{C}$ we define the $(\infty,1)$-category

$$\mathrm{Ar}^{\mathrm{oplax}}(\mathbb{C}) = \mathrm{HOM}_{\mathrm{Cat}_{(\infty,2)}}([1] \boxtimes -, \mathbb{C}) : \Delta^{\mathrm{op}} \to \mathrm{Spc}$$

as a complete Segal space. The source and target maps exhibit $\mathrm{Ar}^{\mathrm{oplax}}(\mathbb{C})$ as the orthofibration straightening the restricted HOM functor [HHLN23, Theorem 7.21].

Remark 5.2. Let $\mathcal{C}$ be an $(\infty,1)$-category and let $I, P \subseteq E$ be closed under base change. Then there exists a natural equivalence

$$\mathrm{Ar}^{\mathrm{oplax}}\mathrm{SPAN}(\mathcal{C}, E)_{P,I} \simeq \mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}, E)_{P,I}(1, -)$$

of categories over $\mathrm{Span}(\mathcal{C}, E) \times \mathrm{Span}(\mathcal{C}, E)$. In other terms, the unstraightening of the Hom functor

$$\mathrm{HOM}_{\mathrm{SPAN}(\mathcal{C},E)_{P,I}}(-,-) : (i_1\mathrm{SPAN}(\mathcal{C}, E)_{P,I})^{\mathrm{op}} \times (i_1\mathrm{SPAN}(\mathcal{C}, E)_{P,I}) \to \mathrm{Cat}$$

is given by $\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}, E)_{P,I}(1, -)$ with its two canonical maps to $\mathrm{Span}(\mathcal{C}, E)$ [CLL25, Section 4.2].

Theorem 5.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category and let E admit a suitable decomposition (I, P). Then the composition $h : \mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathcal{C}, E) \hookrightarrow \mathrm{SPAN}(\mathcal{C}, E)_{P,I}$ is the initial (I, P)-biadjointable functor. That is, for every $(\infty, 2)$-category $\mathbb{D}$ restriction along h induces an equivalence

$$h^* : \mathrm{FUN}(\mathrm{SPAN}(\mathcal{C}, E)_{P,I}, \mathbb{D}) \to \mathrm{FUN}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, \mathbb{D})$$

of $(\infty, 2)$-categories.

Proof. Since point (2) of Lemma 2.7 implies that h is (I, P)-biadjointable, we can apply Theorem 3.3. This leaves us to prove that h is essentially surjective (which is trivial, given the explicit description of the space of objects of Remark 2.5) and that for every $x \in \mathcal{C}$ the composite

$$\mathcal{C}^{\mathrm{op}} \overset{h}{\hookrightarrow} \mathrm{SPAN}(\mathcal{C}, E)_{P,I} \xrightarrow{\mathrm{HOM}(x,-)} \mathrm{Cat}$$

is the free (I, P)-biadjointable functor generated by the class id_x. We now put everything together.

(1) Let us forget for a moment about h. Remark 5.2 tells us that the cocartesian unstraightening of the 1-functor $\mathrm{HOM}(x, -)$ is given by $\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}, E)_{P,I}(1, -)$ with its projection to $\mathrm{Span}(\mathcal{C}, E) \times \mathrm{Span}(\mathcal{C}, E)$. Now Remark 2.3 tells us that

$$\mathrm{SPAN}_{\mathrm{dbl}}(\mathcal{C}, E)_{P,I}(1, -) \subseteq \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Tw}^{\mathrm{ct}}[1] \times \mathrm{Tw}^{\mathrm{ct}}(-), \mathcal{C})$$

and identifies it with the category of spans in a subcategory of $\mathrm{Fun}(\mathrm{Tw}^{\mathrm{ct}}[1], \mathcal{C})$ whose morphisms are of the form

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r, "E"] \arrow[dr, phantom, "\mathrm{I}\text{-pb}"] & \cdot \\
\cdot \arrow[u] \arrow[d, "E"'] \arrow[dr, phantom, "\mathrm{P}\text{-pb}"] & \cdot \arrow[l] \arrow[u] \arrow[r, dashed, "E"] \arrow[d, dashed, "E"] & \cdot \arrow[u] \arrow[d, "E"] \\
\cdot & \cdot \arrow[l] \arrow[r, "E"'] & \cdot
\end{tikzcd}$$

This is obtained by currying a functor $\mathrm{Tw}^{\mathrm{ct}}[1] \times \mathrm{Tw}^{\mathrm{ct}}[n] \to \mathcal{C}$ and translating the defining right Kan extension conditions into the indicated I- and P-pullback conditions.

In this description, the structure maps to $\mathrm{Span}(\mathcal{C}, E) \times \mathrm{Span}(\mathcal{C}, E)$ are given by projecting to the top and bottom row. In particular, restricting to $\{x\} \times \mathrm{Span}(\mathcal{C}, E)$ we arrive at the category of spans of the form

$$\begin{tikzcd}
x \arrow[r, equal] & x \arrow[r, equal] & x \\
\cdot \arrow[u] \arrow[d, "E"'] \arrow[dr, phantom, "\text{P-pb}"] & \cdot \arrow[l] \arrow[u] \arrow[r, dashed, "I"] \arrow[d, dashed, "E"] & \cdot \arrow[u] \arrow[d, "E"] \\
\cdot & \cdot \arrow[l] \arrow[r, "E"'] & \cdot
\end{tikzcd}$$

in $\mathcal{C}_{/x} \times_{\mathcal{C}} \mathrm{Ar}_E(\mathcal{C})$. If we now precompose with h, then we recover the description of the unstraightening of the free (I, P)-biadjointable functor generated by $x = x = x$ from Example 4.7.

(2) Until now we have shown that the free (I, P)-biadjointable functor generated by $x = x = x$ and the restricted $\mathrm{HOM}(x, h(-))$ are equivalent. To conclude, we only need to show that the class $x = x = x$ is sent to id_x.

In the cocartesian unstraightening of $i_1 \mathrm{HOM}(x, -)$, namely the slice $\mathrm{Span}(\mathcal{C}, E)_{x/}$, the identity of x corresponds to the initial object. Equivalences preserve initial objects, so it suffices to identify the initial object in the subfibration above spanned by the cocartesian edges. For a span $x \leftarrow z \xrightarrow{e} y$, the unique map from $x = x = x$ is represented by

$$\begin{tikzcd}
x \arrow[r, equal] & x \arrow[r, equal] & x \\
x \arrow[u, equal] \arrow[d, equal] & z \arrow[l] \arrow[u] \arrow[r, equal] \arrow[d, equal] & z \arrow[u] \arrow[d, "e"] \\
x & z \arrow[l] \arrow[r, "e"'] & y.
\end{tikzcd}$$

The relevant pullback condition forces the middle vertical map to be an equivalence, so this map is unique. Hence $x = x = x$ is initial and corresponds to id_x.

□

We conclude these notes with a monoidal version of the main theorem.

Theorem 5.4. Let $\mathcal{C}$ be an $(\infty, 1)$-category with pullbacks and E a class of maps admitting a suitable decomposition (I, P). Assume also that $\mathcal{C}$ has finite products. Then the restriction

$$h^* : \mathrm{FUN}^{\text{lax-}\otimes}(\mathrm{SPAN}(\mathcal{C}, E)^{\otimes}_{P,I}, \mathbb{D}^{\otimes}) \longrightarrow \mathrm{FUN}^{\text{lax-}\otimes}((\mathcal{C}^{\times})^{\mathrm{op}}, \mathbb{D}^{\otimes})$$

is a locally full inclusion for any symmetric monoidal $(\infty, 2)$-category $\mathbb{D}^{\otimes}$. Moreover, the objects in its image are precisely the lax symmetric monoidal (I, P)-biadjointable functors, and its 1-morphisms are precisely those natural transformations of lax symmetric monoidal 2-functors whose underlying natural transformations of functors $\mathcal{C}^{\mathrm{op}} \Rightarrow \mathbb{D}$ are (I, P)-biadjointable.

References

[CLL25] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. Universality of span 2-categories and the construction of 6-functor formalisms, 2025. arXiv:2505.19192. 1, 3, 6, 7, 8, 11, 12

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry. Volume I: Correspondences and Duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2017. 2

[Hau18] Rune Haugseng. Iterated spans and classical topological field theories. *Mathematische Zeitschrift*, 289(3–4):1427–1488, 2018. 5

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. Two-variable fibrations, factorisation systems and ∞-categories of spans. *Forum of Mathematics, Sigma*, 11:e70, 2023. 5, 11

[LZ12a] Yifeng Liu and Weizhe Zheng. Enhanced six operations and base change theorem for higher Artin stacks, 2012. arXiv:1211.5948. 1

[LZ12b] Yifeng Liu and Weizhe Zheng. Gluing restricted nerves of ∞-categories, 2012. arXiv:1211.5294. 1

[Mac22] Andrew W. Macpherson. A bivariant Yoneda lemma and $(\infty, 2)$-categories of correspondences. *Algebraic & Geometric Topology*, 22(6):2689–2774, 2022. 2, 9

[Man22] Lucas Mann. A p-adic 6-functor formalism in rigid-analytic geometry, 2022. arXiv:2206.02022. 1

[MW24] Louis Martini and Sebastian Wolf. Colimits and cocompletions in internal higher category theory. *Higher Structures*, 8(1):97–192, 2024. 9

[Ste20] Germán Stefanich. Higher sheaf theory i: Correspondences, 2020. arXiv:2011.03027. 2