

Seminar Higher Zariski Geometry (SS 2026)

Talk 11: Zariski Descent for 2-Rings

Ningyi Li

Ningyi.Li@stud.uni-regensburg.de

26 June 2026

Abstract

This note follows the Section 5.A–5.B of [ABC+25]. The first part proves the Mayer–Vietoris reduction criterion for Zariski descent, Proposition 5.8 of [ABC+25]. The second part explains why, for a rigid 2-ring \mathcal{K} , the canonical structure presheaf $\tilde{\mathcal{O}}_{\mathcal{K}}$ is already a sheaf. The key input is the stronger presentable-categorical descent theorem, Theorem 5.14 of [ABC+25]; after passing to dualizable objects one obtains Theorem 5.11. We finish with the full faithfulness of the absolute spectrum functor on rigid 2-rings, Corollary 5.21.

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I Introduction

Classical Zariski descent says that an object on a space can be reconstructed from its restrictions to an open cover, together with coherent compatibility on all intersections. For a

two-open cover $X = U \cup V$, the first visible piece of descent is the Mayer–Vietoris pullback square

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \times_{\mathcal{F}(U \cap V)} \mathcal{F}(V).$$

In a higher category this pullback already contains the coherent gluing isomorphism on $U \cap V$.

For a rigid 2-ring \mathcal{K} , a quasicompact open of $\mathrm{Spc}(\mathcal{K})$ is represented by a principal thick tensor ideal $\langle x \rangle$, and the canonical structure presheaf is

$$U(x) \longmapsto \mathcal{K}/\langle x \rangle.$$

The central binary descent statement is therefore

$$\mathrm{Ind}(\mathcal{K}) \simeq \mathrm{Ind}(\mathcal{K}/\langle x \rangle) \times_{\mathrm{Ind}(\mathcal{K}/\langle x \oplus y \rangle)} \mathrm{Ind}(\mathcal{K}/\langle y \rangle) \quad (x \otimes y \simeq 0). \quad (1)$$

The condition $x \otimes y \simeq 0$ is exactly the statement that the two corresponding basic opens cover $\mathrm{Spc}(\mathcal{K})$.

The proof has four conceptual steps:

- Step 1. A sheaf on a coherent frame is determined by the empty-open condition and binary Mayer–Vietoris squares on compact opens.
- Step 2. This translates sheaf descent on $\mathrm{Spec}\mathcal{K}$ into a test on binary Zariski covers of 2-rings (Proposition III.3).
- Step 3. For rigid 2-rings the Ind-completed localizations satisfy the pullback square (1) (Theorem IV.3).
- Step 4. The functor taking dualizable objects preserves limits, so big descent restricts to descent for the small rigid 2-rings themselves (Theorem IV.2).

II Sheaves on a frame

Definition II.1. A frame F is a complete lattice in which finite meets distribute over arbitrary joins:

$$a \wedge \left(\bigvee_i b_i \right) = \bigvee_i (a \wedge b_i).$$

Its least and greatest elements are denoted \perp and \top . A frame is coherent if its compact elements F^ω are closed under finite meets and every element of F is a directed join of compact elements.

A frame carries its canonical Grothendieck topology: a family $\{U_i \leq U\}$ covers U when $\bigvee_i U_i = U$. Thus a presheaf with values in an ∞ -category \mathcal{D} is a functor $\mathcal{F} : F^{\text{op}} \rightarrow \mathcal{D}$.

Theorem II.2 (Sheaves on a frame, [ABC+25], Theorem 5.1). *Let F be a frame and let \mathcal{D} admit all small limits. A presheaf $\mathcal{F} : F^{\text{op}} \rightarrow \mathcal{D}$ is a sheaf if and only if:*

- (a) $\mathcal{F}(\perp)$ is final;
- (b) for every $U, V \in F$, the square

$$\begin{array}{ccc} \mathcal{F}(U \vee V) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \wedge V) \end{array}$$

is Cartesian;

- (c) for every directed subset $D \subseteq F$, the canonical map

$$\mathcal{F}\left(\bigvee D\right) \longrightarrow \lim_{U \in D} \mathcal{F}(U)$$

is an equivalence.

Condition (b) is the Mayer–Vietoris condition. Condition (c) says that the value on an arbitrary open is recovered from a directed exhaustion by smaller opens.

Lemma II.3 (Reduction to compact opens, [ABC+25], Lemma 5.2). *Let F be coherent and let \mathcal{D} admit all small limits. Restriction to compact elements is fully faithful*

$$\text{Shv}(F; \mathcal{D}) \longrightarrow \text{Fun}((F^\omega)^{\text{op}}, \mathcal{D}).$$

Its essential image consists of the functors satisfying the empty-open and binary Mayer–Vietoris conditions on compact elements.

Proof. Every $U \in F$ is a directed join of compact elements. By Theorem II.2(c), a sheaf is therefore uniquely determined by its restriction to F^ω , proving full faithfulness.

Conversely, extend a functor on F^ω to all of F by the right Kan extension formula

$$\mathcal{F}(U) = \lim_{V \in F^\omega, V \leq U} \mathcal{F}(V).$$

This automatically gives directed continuity. If U and V are arbitrary, choose directed compact exhaustions $U = \bigvee_i U_i$ and $V = \bigvee_i V_i$. Coherence ensures that $U_i \wedge V_i$ and $U_i \vee V_i$

remain compact. Taking the limit of the Cartesian Mayer–Vietoris squares for (U_i, V_i) gives the required square for (U, V) . The empty-open condition is inherited from the compact basis. \square

The frame attached to a 2-ring

For a 2-ring \mathcal{K} , write $\text{Rad}(\mathcal{K})$ for its coherent frame of radical thick tensor ideals. The frame of opens of the Balmer spectrum is the Hochster dual

$$\Omega_{\mathcal{K}} := \text{Rad}(\mathcal{K})^{\vee}.$$

Consequently

$$\Omega_{\mathcal{K}}^{\omega} \simeq (\text{Rad}(\mathcal{K})^{\omega})^{\text{op}}.$$

A presheaf on the compact opens of $\text{Spc}(\mathcal{K})$ can therefore be written covariantly as a functor

$$\text{Rad}(\mathcal{K})^{\omega} \longrightarrow \mathcal{D}.$$

If $\mathcal{I}', \mathcal{I}'' \in \text{Rad}(\mathcal{K})^{\omega}$, the binary sheaf square becomes

$$\begin{array}{ccc} P(\mathcal{I}' \wedge \mathcal{I}'') & \longrightarrow & P(\mathcal{I}'') \\ \downarrow & & \downarrow \\ P(\mathcal{I}') & \longrightarrow & P(\mathcal{I}' \vee_{\text{Rad}} \mathcal{I}''). \end{array} \tag{2}$$

Here \wedge is intersection, while the join in $\text{Rad}(\mathcal{K})$ is the radical of the join of ordinary thick tensor ideals.

III Zariski covers and the reduction criterion

We use the following standard notation. A thick tensor ideal of \mathcal{K} is denoted $\mathcal{I} \subseteq \mathcal{K}$, its radical by $\sqrt{\mathcal{I}}$, and the Karoubi quotient by \mathcal{K}/\mathcal{I} . For a morphism $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{I}$, ideals in the quotient correspond to ideals of \mathcal{K} containing \mathcal{I} .

Definition III.1 (Zariski cover, [ABC+25], Definition 5.4). *A map of 2-rings*

$$f : \mathcal{K} \longrightarrow \mathcal{K}'$$

is a Zariski cover if \mathcal{K}' is a finite product $\prod_i \mathcal{K}_i$, each map $\mathcal{K} \rightarrow \mathcal{K}_i$ is a principal Karoubi

quotient with kernel \mathcal{I}_i , and

$$\bigcap_i \mathcal{I}_i \subseteq \sqrt{0}.$$

The Grothendieck topology generated by these maps is the Zariski topology on 2CAlg^{op} .

For a rigid 2-ring every thick tensor ideal is radical. Thus a binary cover has the form

$$\mathcal{L} \longrightarrow \mathcal{L}/\langle x \rangle \times \mathcal{L}/\langle y \rangle, \quad x \otimes y \simeq 0.$$

This is the categorified analogue of a cover $\text{Spec}R = D(f) \cup D(g)$ with $(f, g) = R$.

Lemma III.2 (Ideals after quotient, [ABC+25], Lemma 5.6). *Let $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{I}$ be a Karoubi quotient. Preimage gives an order isomorphism*

$$q^{-1} : \text{Idl}(\mathcal{K}/\mathcal{I}) \xrightarrow{\sim} \{\mathcal{J} \in \text{Idl}(\mathcal{K}) \mid \mathcal{I} \subseteq \mathcal{J}\}.$$

It restricts to radical ideals. Under the relevant compactness hypotheses it also restricts to compact radical ideals, and, when \mathcal{I} is principal, to principal ideals.

Proof. The inverse sends an ideal $\mathcal{J} \supseteq \mathcal{I}$ to the ideal generated by its image in \mathcal{K}/\mathcal{I} . The quotient-of-a-quotient equivalence

$$(\mathcal{K}/\mathcal{I})/(\mathcal{J}/\mathcal{I}) \simeq \mathcal{K}/\mathcal{J}$$

shows that these two operations are inverse. Radicality is preserved because

$$q^{-1}(\sqrt{\mathcal{J}}) = \sqrt{q^{-1}(\mathcal{J})}.$$

The assertions about compact radical and principal ideals follow from the characterization of compact localizations by principal kernels. \square

Proposition III.3 (Reduction schema, [ABC+25], Proposition 5.8). *Let $\mathcal{C} \subseteq 2\text{CAlg}$ be a full subcategory closed under finite products and Zariski covers. Let \mathcal{D} admit all small limits and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. For every $\mathcal{K} \in \mathcal{C}$, the functor*

$$\text{Rad}(\mathcal{K})^\omega \xrightarrow{\mathcal{K}/(-)} \mathcal{C}_{\mathcal{K}/} \xrightarrow{F} \mathcal{D}$$

extends to a sheaf on $\text{Spec}\mathcal{K} \simeq \text{Shv}(\text{Rad}(\mathcal{K})^\vee)$ if and only if, for every binary Zariski cover

$\mathcal{L} \rightarrow \mathcal{L}_1 \times \mathcal{L}_2$ in \mathcal{C} , the square

$$\begin{array}{ccc}
 F(\mathcal{L}) & \longrightarrow & F(\mathcal{L}_1) \\
 \downarrow & & \downarrow \\
 F(\mathcal{L}_2) & \longrightarrow & F((\mathcal{L}_1 \otimes_{\mathcal{L}} \mathcal{L}_2)/\sqrt{0})
 \end{array} \tag{3}$$

is Cartesian.

Proof. By Lemma II.3, descent is equivalent to the Mayer–Vietoris condition (2) for compact radical ideals. Evaluate the presheaf at an ideal by

$$P(\mathcal{I}) = F(\mathcal{K}/\mathcal{I}).$$

Thus the required square is

$$\begin{array}{ccc}
 F(\mathcal{K}/(\mathcal{I}' \wedge \mathcal{I}'')) & \longrightarrow & F(\mathcal{K}/\mathcal{I}'') \\
 \downarrow & & \downarrow \\
 F(\mathcal{K}/\mathcal{I}') & \longrightarrow & F(\mathcal{K}/(\mathcal{I}' \vee_{\text{Rad}} \mathcal{I}'')).
 \end{array} \tag{4}$$

Set $\mathcal{A} = \mathcal{I}' \wedge \mathcal{I}''$ and $\mathcal{L} = \mathcal{K}/\mathcal{A}$. By Lemma III.2, the images of \mathcal{I}' and \mathcal{I}'' in \mathcal{L} define the two basic localizations

$$\mathcal{L} \longrightarrow \mathcal{K}/\mathcal{I}', \quad \mathcal{L} \longrightarrow \mathcal{K}/\mathcal{I}''.$$

Their kernels have intersection contained in the nilradical of \mathcal{L} , hence they form a Zariski cover. Base change for Karoubi quotients gives

$$(\mathcal{K}/\mathcal{I}') \otimes_{\mathcal{K}/(\mathcal{I}' \wedge \mathcal{I}'')} (\mathcal{K}/\mathcal{I}'') \simeq \mathcal{K}/(\mathcal{I}' \vee \mathcal{I}''),$$

and quotienting this overlap by its nilradical gives

$$(\mathcal{K}/(\mathcal{I}' \vee \mathcal{I}''))/\sqrt{0} \simeq \mathcal{K}/(\mathcal{I}' \vee_{\text{Rad}} \mathcal{I}'').$$

Therefore (4) is precisely the square (3) for this Zariski cover.

Conversely, any binary Zariski cover is obtained in this way from its two kernels, after taking radicals and passing to the reduced overlap. Hence the family of squares (4) and the family of squares (3) are the same family. This proves both directions. \square

Corollary III.4 (Rigid form of the criterion). *Suppose $\mathcal{C} = 2\text{CAlg}^{\text{rig}}$ and $F(0)$ is final. Then it is enough to prove that for every rigid \mathcal{L} and every $x, y \in \mathcal{L}$ with $x \otimes y \simeq 0$, the square*

$$\begin{array}{ccc} F(\mathcal{L}) & \longrightarrow & F(\mathcal{L}/\langle x \rangle) \\ \downarrow & & \downarrow \\ F(\mathcal{L}/\langle y \rangle) & \longrightarrow & F(\mathcal{L}/\langle x \oplus y \rangle) \end{array}$$

is Cartesian.

Proof. For a rigid 2-ring, $\text{Rad}(\mathcal{L}) = \text{Idl}(\mathcal{L})$, and

$$\langle x \rangle \cap \langle y \rangle = \langle x \otimes y \rangle.$$

Thus the binary cover condition is exactly $x \otimes y \simeq 0$, and the join of the two principal ideals is $\langle x \oplus y \rangle$. \square

IV The canonical structure presheaf

Let \mathcal{K} be a rigid 2-ring. The comparison between the higher Zariski spectrum and the Balmer spectrum identifies

$$\text{Spec}\mathcal{K} \simeq \text{Shv}(\text{Spc}\mathcal{K}) \simeq \text{Shv}(\text{Rad}(\mathcal{K})^\vee).$$

The site of quasicompact opens can be identified with $\text{Rad}(\mathcal{K})^\omega = \text{Prin}(\mathcal{K})$.

Notation IV.1. *The canonical structure presheaf is denoted $\tilde{\mathcal{O}}_{\mathcal{K}}$, while its sheafification is denoted $\mathcal{O}_{\mathcal{K}}$. On compact opens it is*

$$\tilde{\mathcal{O}}_{\mathcal{K}} : \text{Rad}(\mathcal{K})^\omega \longrightarrow 2\text{CAlg}, \quad \mathcal{I} \longmapsto \mathcal{K}/\mathcal{I}. \quad (5)$$

For $\mathcal{I} = \langle a \rangle$, this is the localization attached to the basic open

$$U(a) = \{\mathfrak{p} \in \text{Spc}\mathcal{K} \mid a \in \mathfrak{p}\}.$$

After passage to homotopy categories, (5) agrees on quasicompact opens with Balmer's structure presheaf of tt-categories. The new point is that the enhanced presheaf satisfies genuine homotopy-coherent descent.

Theorem IV.2 (Theorem 5.11 of [ABC+25]). *For every rigid 2-ring \mathcal{K} , the canonical map*

from the structure presheaf to its sheafification is an equivalence:

$$\tilde{\mathcal{O}}_{\mathcal{K}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{K}}.$$

Equivalently, the assignment $\mathcal{I} \mapsto \mathcal{K}/\mathcal{I}$ is already a 2CAlg-valued sheaf on $\mathrm{Spec}\mathcal{K}$.

We deduce this from a stronger result after Ind-completion.

Theorem IV.3 (Strong descent, Theorem 5.14 of [ABC+25]). *For every rigid 2-ring \mathcal{K} , the assignment*

$$\tilde{\mathcal{O}}_{\mathcal{K}}^{\mathrm{Ind}} : \mathrm{Rad}(\mathcal{K})^{\omega} \longrightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}), \quad \mathcal{I} \longmapsto \mathrm{Ind}(\mathcal{K}/\mathcal{I})$$

is already a sheaf on $\mathrm{Spec}\mathcal{K}$.

The rest of the note is devoted to the mechanism behind this theorem.

V The Mayer–Vietoris theorem after Ind-completion

Smashing idempotents

Recollection V.1. *Let \mathcal{C} be a presentably symmetric monoidal stable ∞ -category. An object $A \in \mathrm{CAlg}(\mathcal{C})$ is an idempotent algebra if $A \otimes A \simeq A$. It determines a smashing localization*

$$\mathcal{C} \longrightarrow \mathrm{Mod}_A(\mathcal{C}), \quad c \longmapsto A \otimes c.$$

For a rigid 2-ring \mathcal{L} and a thick tensor ideal \mathcal{I} , the localization

$$\mathrm{Ind}(\mathcal{L}) \longrightarrow \mathrm{Ind}(\mathcal{L}/\mathcal{I})$$

is smashing. We denote its idempotent algebra by $\mathbb{L}_{\mathcal{I}}$, and write \mathbb{L}_x when $\mathcal{I} = \langle x \rangle$.

The fundamental properties are

$$\begin{aligned} \mathrm{Ind}(\mathcal{L}/\mathcal{I}) &\simeq \mathrm{Mod}_{\mathbb{L}_{\mathcal{I}}}(\mathrm{Ind}(\mathcal{L})), \\ \ker(\mathbb{L}_{\mathcal{I}} \otimes -) &= \mathrm{Ind}(\mathcal{I}), \\ \mathbb{L}_{\mathcal{I}} \otimes \mathbb{L}_{\mathcal{J}} &\simeq \mathbb{L}_{\mathcal{I} \vee \mathcal{J}}. \end{aligned}$$

Lemma V.2 (Intersection of the big kernels, [ABC+25], Lemma 5.17). *Let \mathcal{K} be rigid and let $x, y \in \mathcal{K}$. Then*

$$\mathrm{Ind}(\langle x \rangle) \cap \mathrm{Ind}(\langle y \rangle) = \mathrm{Ind}(\langle x \otimes y \rangle)$$

as localizing subcategories of $\text{Ind}(\mathcal{K})$.

Proof. The inclusion from right to left is immediate. For the reverse inclusion, let

$$\mathcal{E} = \text{Ind}(\langle x \rangle) \cap \text{Ind}(\langle y \rangle).$$

We claim that the objects $a \otimes b$, with $a \in \langle x \rangle$ and $b \in \langle y \rangle$, generate \mathcal{E} . Suppose $z \in \mathcal{E}$ and $\text{Hom}(a \otimes b, z) \simeq 0$ for every such pair. Since b is dualizable,

$$\text{Hom}(a \otimes b, z) \simeq \text{Hom}(a, b^\vee \otimes z).$$

The object $b^\vee \otimes z$ belongs to $\text{Ind}(\langle x \rangle)$, so the vanishing for all $a \in \langle x \rangle$ implies $b^\vee \otimes z \simeq 0$. Varying b and using that $z \in \text{Ind}(\langle y \rangle)$ now gives $z \simeq 0$. Hence these tensors are a set of generators for \mathcal{E} .

Every $a \otimes b$ lies in $\langle x \otimes y \rangle$, so $\mathcal{E} \subseteq \text{Ind}(\langle x \otimes y \rangle)$. □

The idempotent fracture square

Let \mathcal{L} be rigid and suppose $x \otimes y \simeq 0$. Consider

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbb{L}_x \\ \downarrow & & \downarrow \\ \mathbb{L}_y & \longrightarrow & \mathbb{L}_x \otimes \mathbb{L}_y \simeq \mathbb{L}_{x \oplus y}. \end{array} \tag{6}$$

Proposition V.3. *The square (6) is Cartesian in $\text{Ind}(\mathcal{L})$.*

Proof. Because $\text{Ind}(\mathcal{L})$ is stable, it is enough to compare horizontal fibers. Set $C_x = \text{fib}(\mathbf{1} \rightarrow \mathbb{L}_x)$. The induced map is

$$f : C_x \longrightarrow \mathbb{L}_y \otimes C_x.$$

Let $Z = \text{fib}(f)$. Tensoring with \mathbb{L}_x kills C_x , hence kills Z . Tensoring f with \mathbb{L}_y gives an equivalence because \mathbb{L}_y is idempotent, hence $\mathbb{L}_y \otimes Z \simeq 0$. Therefore

$$Z \in \ker(\mathbb{L}_x \otimes -) \cap \ker(\mathbb{L}_y \otimes -) = \text{Ind}(\langle x \rangle) \cap \text{Ind}(\langle y \rangle).$$

By Lemma V.2 and $x \otimes y \simeq 0$, this intersection is zero. Hence $Z \simeq 0$, so f is an equivalence and the square is Cartesian. □

Tensoring (6) with an arbitrary object a gives the objectwise fracture square

$$\begin{array}{ccc}
 a & \longrightarrow & \mathbb{L}_x \otimes a \\
 \downarrow & & \downarrow \\
 \mathbb{L}_y \otimes a & \longrightarrow & \mathbb{L}_{x \oplus y} \otimes a.
 \end{array} \tag{7}$$

Gluing localized categories

There is a standard adjunction (adjoint descent, [HY17, Theorem B])

$$\mathrm{Ind}(\mathcal{L}) \rightleftarrows \mathrm{Ind}(\mathcal{L}/\langle x \rangle) \times_{\mathrm{Ind}(\mathcal{L}/\langle x \oplus y \rangle)} \mathrm{Ind}(\mathcal{L}/\langle y \rangle). \tag{8}$$

The left adjoint restricts an object to the two opens. The right adjoint takes a compatible triple (c_x, c_y, α) to the pullback

$$c_x \times_{\mathbb{L}_{x \oplus y} \otimes c_x} c_y$$

in $\mathrm{Ind}(\mathcal{L})$, using α to identify the two restrictions on the overlap.

Proposition V.4. *Under the assumption $x \otimes y \simeq 0$, the adjunction (8) is an equivalence. Equivalently, the square*

$$\begin{array}{ccc}
 \mathrm{Ind}(\mathcal{L}) & \longrightarrow & \mathrm{Ind}(\mathcal{L}/\langle x \rangle) \\
 \downarrow & & \downarrow \\
 \mathrm{Ind}(\mathcal{L}/\langle y \rangle) & \longrightarrow & \mathrm{Ind}(\mathcal{L}/\langle x \oplus y \rangle)
 \end{array} \tag{9}$$

is Cartesian.

Proof sketch. For $a \in \mathrm{Ind}(\mathcal{L})$, the unit map into the object obtained by restricting and gluing again is exactly

$$a \longrightarrow (\mathbb{L}_x \otimes a) \times_{\mathbb{L}_{x \oplus y} \otimes a} (\mathbb{L}_y \otimes a),$$

which is an equivalence by (7). Thus the restriction functor is fully faithful.

It remains to see that the gluing functor is conservative. If a morphism between compatible triples becomes an equivalence after gluing, tensoring the glued morphism with \mathbb{L}_x recovers its x -component; tensoring with \mathbb{L}_y recovers its y -component. Both components are therefore equivalences, and hence so is the original morphism of triples. A fully faithful left adjoint whose right adjoint is conservative is an equivalence. \square

Proof of Theorem IV.3. Apply the rigid reduction criterion, Corollary III.4, to

$$F = \text{Ind} : 2\text{CAlg}^{\text{rig}} \longrightarrow \text{CAlg}(\widehat{\text{Cat}}_\infty).$$

The category of large symmetric monoidal ∞ -categories admits all limits. Proposition V.4 verifies the required Mayer–Vietoris square for every binary Zariski cover. Therefore $\tilde{\mathcal{O}}_{\mathcal{K}}^{\text{Ind}}$ is a sheaf. \square

VI From big descent to the structure sheaf

Lemma VI.1. *The functor taking dualizable objects*

$$(-)^{\text{dbl}} : \text{CAlg}(\widehat{\text{Cat}}_\infty) \longrightarrow 2\text{CAlg}$$

preserves all limits on diagrams whose dualizable subcategories are small.

Proof. Dualizability is expressed by the existence of an object together with coevaluation and evaluation maps satisfying two finite diagrams of identities. All of this data is computed componentwise in a limit. Equivalently, taking dualizable objects is a right adjoint to the inclusion of rigid symmetric monoidal categories, and therefore preserves limits. \square

Proof of Theorem IV.2. Every quotient \mathcal{K}/\mathcal{I} of a rigid 2-ring is rigid. Moreover,

$$\text{Ind}(\mathcal{K}/\mathcal{I})^{\text{dbl}} \simeq \mathcal{K}/\mathcal{I},$$

because in the Ind-completion of a rigid 2-ring the compact and dualizable objects coincide with the original small category. Hence, objectwise,

$$(\tilde{\mathcal{O}}_{\mathcal{K}}^{\text{Ind}})^{\text{dbl}} \simeq \tilde{\mathcal{O}}_{\mathcal{K}}.$$

Theorem IV.3 says that $\tilde{\mathcal{O}}_{\mathcal{K}}^{\text{Ind}}$ carries every covering diagram to a limit diagram. Applying the limit-preserving functor $(-)^{\text{dbl}}$ shows that $\tilde{\mathcal{O}}_{\mathcal{K}}$ does the same. Thus it is already a sheaf, and the sheafification map $\tilde{\mathcal{O}}_{\mathcal{K}} \rightarrow \mathcal{O}_{\mathcal{K}}$ is an equivalence. \square

VII Full faithfulness of the absolute spectrum

Recall the spectrum–global sections adjunction

$$\text{Spec} : 2\text{CAlg} \rightleftharpoons \text{LTop}(G_{\text{Zar}}) : \Gamma.$$

Corollary VII.1 (Corollary 5.21 of [ABC+25]). *The absolute spectrum functor*

$$\mathrm{Spec} : 2\mathrm{CAlg}^{\mathrm{rig}} \longrightarrow \mathrm{LTop}(G_{\mathrm{Zar}})$$

is fully faithful.

Proof. Let \mathcal{K} be rigid. By Theorem IV.2, the structure sheaf is already the presheaf $\mathcal{I} \mapsto \mathcal{K}/\mathcal{I}$. The global open corresponds to the zero ideal, so

$$\Gamma(\mathrm{Spec}\mathcal{K}, \mathcal{O}_{\mathcal{K}}) \simeq \mathcal{O}_{\mathcal{K}}(\mathrm{Spc}\mathcal{K}) \simeq \mathcal{K}/0 \simeq \mathcal{K}.$$

This equivalence is the unit of the adjunction. A left adjoint is fully faithful precisely when its unit is an equivalence, so Spec is fully faithful on rigid 2-rings. \square

VIII Classical model and conceptual summary

Example VIII.1 (Perfect complexes over a ring). *Let R be a commutative ring and $\mathcal{K} = \mathrm{Perf}(R)$. For $f \in R$, set $x_f = \mathrm{cofib}(f : R \rightarrow R)$. Then*

$$\mathrm{Perf}(R)/\langle x_f \rangle \simeq \mathrm{Perf}(R[f^{-1}]).$$

If f and g generate the unit ideal, then $x_f \otimes x_g \simeq 0$, and Theorem IV.3 specializes to

$$\mathrm{Mod}_R \simeq \mathrm{Mod}_{R[f^{-1}]} \times_{\mathrm{Mod}_{R[(fg)^{-1}]}} \mathrm{Mod}_{R[g^{-1}]}.$$

Taking dualizable objects recovers descent for perfect complexes.

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