Results on cloaking by transformation optics and anomalous localized resonance in elliptic geometry

Giovanni Rossanigo

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Contents

Why study cloaking? Why in elliptical geometry?

Why study cloaking?

- Physical and engineering challenges and problems, such as the study and realization of metamaterials. *[Alù, Engheta, (2005).]*

Several suggestions based on metamaterials on how to achieve cloaking:

1. cloaking by transformation optics;
2. cloaking by anomalous localized resonance CALR.

Why in elliptical geometry?

- Explicit results known only for systems with circular geometries in $\mathbb{R}^2$.
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Cloaking by transformation optics - Electrical impedance tomography

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded domain.

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

$\sigma = (\sigma_{ij}) : \Omega \to \mathbb{R}^n$ is the unknown conductivity.

We use the Dirichlet to Neumann map

$$\Lambda_\sigma : u\big|_{\partial\Omega} \to (\sigma \nabla u) \cdot \bar{n}\big|_{\partial\Omega}.$$

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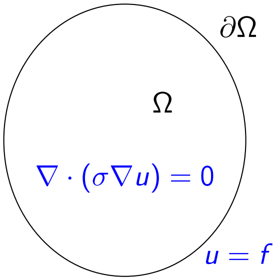

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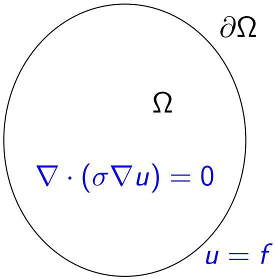

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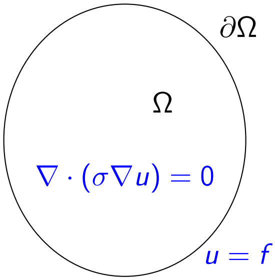

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Change of variables

Λ_σ allows us to determine σ at its best at less than a change of variables.

Proposition

Let $F : \Omega \to \Omega$ be such that $F(x) = x$ at $\partial\Omega$. Then the boundary measurements associated with σ and $F_*\sigma$ are identical, i.e. $\Lambda_\sigma(f) = \Lambda_{F_*\sigma}(f)$ for all f.

$F_*\sigma$ is the push-forward of σ by the change of variables F:

$$F_*\sigma(y) = \frac{1}{\det(DF(x))} DF(x)\sigma(x)(DF(x))^T.$$

$\to$ Cloaking is possible!

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There exists a transformation $F : \Omega \to \Omega$ such that

- F is continuous and piecewise smooth;
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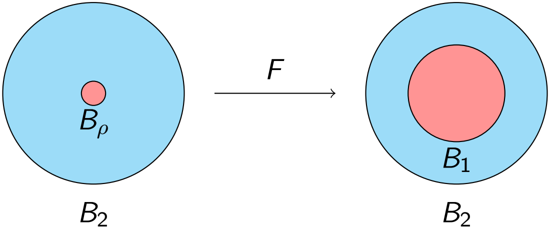

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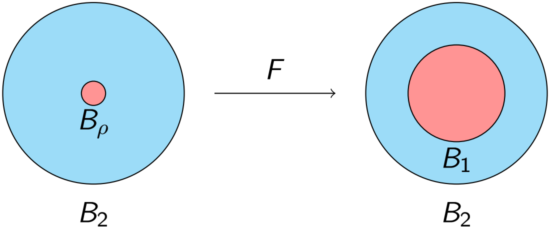

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Results in circular geometry

Theorem

Suppose that the shell $B_2(0) \setminus B_1(0)$ has conductivity F_*1. If ρ is small enough, then $B_1(0)$ is nearly cloaked, i.e. there exists some constant $C > 0$ such that

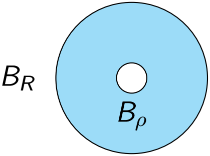
$$\|\Lambda_{\sigma_A} - \Lambda_1\| \leq C\rho^2.$$

The proof is based on problems with dielectric inclusions.

Dielectric inclusions

Perfect insulation problem:

$$\begin{cases} \Delta u_0^\rho = 0 & \text{in } B_R(0) \setminus \overline{B_\rho(0)}, \\ u_0^\rho = f & \text{on } \partial B_R(0), \\ \frac{\partial u_0^\rho}{\partial \bar{n}} = 0 & \text{on } \partial B_\rho(0), \end{cases}$$



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Calculate $\Lambda_0^\rho, \Lambda_\infty^\rho$ and estimate:

$$\|\Lambda_1 - \Lambda_0^\rho\| \leq C\rho^2 \qquad \|\Lambda_1 - \Lambda_\infty^\rho\| \leq C\rho^2$$

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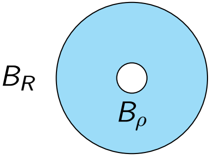

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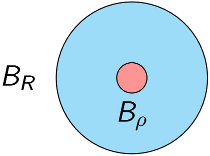

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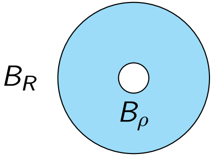

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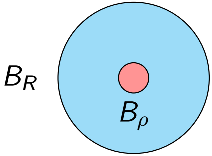

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Results in elliptic geometry I

Theorem

Assume that $f = \sum_{k\in\mathbb{Z}} f_k e^{ik\nu}$ *with* $f_k = 0$ *for all* $|k| < k_0$*, and* $f_{k_0} \neq 0$*. Then there exists* $C > 0$ *depending only on* R *such that*

$$\|\Lambda_1 - \Lambda_0^{\rho}\| \geq C k_0 |f_{k_0}| e^{2k_0\rho}$$

$$\|\Lambda_1 - \Lambda_{\infty}^{\rho}\| \geq C k_0 |f_{k_0}| e^{2k_0\rho}.$$

With this procedure there is no cloaking in elliptic geometry!

Why? The size of the inclusions matters!

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Results in elliptic geometry II

Theorem

Suppose the source $f : \partial\mathcal{E}_R(0) \to \mathbb{R}$, $f \in C^l$, *is high-frequency monochromatic, that is, there is a large* $k_0 \in \mathbb{N}$ *such that* $f = f_{k_0} e^{ik_0\nu}$. *Then there exists some constant* $C > 0$ *such that*

$$\left\| \frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_0^\rho}{\partial \bar{n}} \right\|_{L^2(\partial\mathcal{E}_R(0))} \leq \frac{C}{k_0} \quad \text{and} \quad \left\| \frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_\infty^\rho}{\partial \bar{n}} \right\|_{L^2(\partial\mathcal{E}_R(0))} \leq \frac{C}{k_0}.$$

Cloaking by anomalous localized resonance - The problem

$$\begin{cases} \nabla \cdot (a_\eta \nabla u_\eta) = f & \text{in } \mathbb{R}^2, \\ u_\eta \to 0 & \text{as } |x| \to \infty. \end{cases}$$

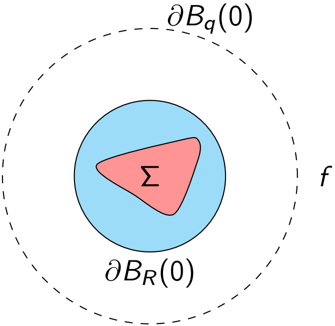
$a_\eta = A(x) + i\eta$ is the electric permittivity:

- $A(x)$ has a core-shell-matrix character:

$$A(x) = \begin{cases} +1 & \text{in the core } \Sigma, \\ -1 & \text{in the shell } B_R \setminus \Sigma, \\ +1 & \text{in the matrix } \mathbb{R}^2 \setminus B_R. \end{cases}$$

- $\eta > 0$ is a loss parameter;

The source f is supported on $\partial B_q(0)$, $q > R$.



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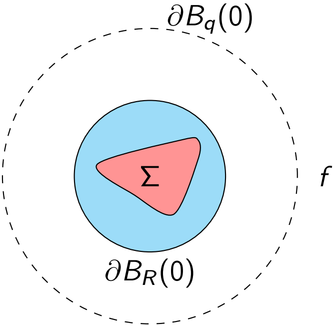

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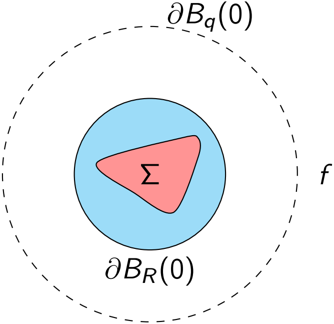

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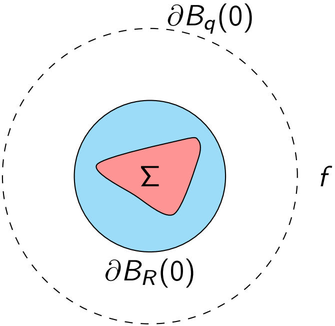

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Anomalous localized resonance

Energy of the solution:

$$\mathcal{E}_\eta = \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla u_\eta|^2 \, dx$$

When $\eta \to 0$

- Anomalous Localized Resonance occurs: $|\nabla u_\eta|$ diverges in a specific region while it converges smoothly outside this region. No dipedence from a_η.
- Normalize the problem by $\alpha_\eta \in \mathbb{R}$, with $\alpha_\eta \to 0$.
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Cloaking results in the litterature

Spectral theory techniques:

- Ammari, Ciraolo, Kang, Lee, Milton proved that in circular geometry (core $= B_r$, shell $= B_R$) cloaking happens only if $q < R^*$, where

$$R^* = r\left(\frac{R}{r}\right)^{3/2}.$$

[Ammari, Ciraolo, Kang, Lee, Milton, (2013).]

Milton and Nicorovici performed numerical simulations which confirm R^*.

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$$R^* = \begin{cases} (3R - r)/2 & \text{for } R \le 3r, \\ 2(R - r) & \text{for } R > 3r, \end{cases}$$

if core $= \mathcal{E}_r$, shell $= \mathcal{E}_R$, where

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Khon, Lu, Schweizer, Weinsten approach based on two dual variational principles. *[Khon, Lu, Schweizer, Weinsten, (2012).]*

- Primal variational principles: $\mathcal{E}_\eta \leq \mathcal{I}_\eta$, used to prove that cloaking does not happen.
- Dual variational principles: $\mathcal{E}_\eta \geq \mathcal{J}_\eta$, used to prove that cloaking happens.

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Results in elliptic geometry I

Theorem (No core implies resonance for sources at any distance)

Assume that the configuration has no core (i.e. $\Sigma = \emptyset$). Let $f = F\mathcal{H}^1\lfloor\partial\mathcal{E}_R(0)$ with $0 \neq F : \partial\mathcal{E}_R(0) \to \mathbb{R}$ be a source at a distance $q > R$. Then

$$\mathcal{E}_\eta(u_\eta) \to +\infty \qquad \text{as } \eta \to 0.$$

$\rightsquigarrow$ Cloaking always happens

$\rightsquigarrow$ Dual variational principle

Results in elliptic geometry II

Theorem (Non-resonance beyond R^*)

Let $\Sigma = \mathcal{E}_r(0) \subset \mathcal{E}_R(0)$ and let $A(x) = +1$ in Σ and $\mathbb{R}^2 \setminus \mathcal{E}_R(0)$, $A(x) = -1$ in $\mathcal{E}_R(0) \setminus \Sigma$.
Let $f = F\mathcal{H}^1 \lfloor \partial\mathcal{E}_q(0)$, $0 \neq F : \partial\mathcal{E}_q(0) \to \mathbb{R}$, be a source at a distance $q > R$ with zero average and $F \in L^2(\partial\mathcal{E}_q(0))$. Then the configuration is non-resonant if $q > R^*$ where

$$R^* = (3R - r)/2$$

Thank you for the attention

Elliptic coordinates

Definition

The elliptical coordinates $(\mu, \nu) \in [0, +\infty) \times [0, 2\pi)$ on $\mathbb{R}^2$ are defined via

$$\begin{cases} x = a \cosh \mu \cos \nu \\ y = a \sinh \mu \sin \nu \end{cases}$$

where $a > 0$.

- The coordinate μ is called the elliptic radius.
- The coordinate lines are hyperbolae and ellipses.
- We define the elliptical region $\mathcal{E}_r(0) = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{\cosh^2 r} + \frac{y^2}{\sinh^2 r} < a^2\right\}$.

The dual variational principles

Set $u_\eta = v_\eta + i/\eta w_\eta$, then

$$\nabla \cdot (a_\eta \nabla u_\eta) = f \iff \begin{cases} \nabla \cdot (A \nabla v_\eta) - \Delta w_\eta & = f, \\ \nabla \cdot (A \nabla w_\eta) + \eta^2 \Delta v_\eta & = 0 \end{cases}$$

The energy becomes

$$\mathcal{E}_\eta = \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla u_\eta|^2 \, dx \Rightarrow \mathcal{I}_\eta = \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla v_\eta|^2 \, dx + \frac{1}{2\eta} \int_{\mathbb{R}^2} |\nabla w_\eta|^2 \, dx$$

- PVP: the solution of the original problem is obtained by minimizing $\mathcal{E}_\eta$, so we minimize $\mathcal{I}_\eta$ with the constraint $\nabla \cdot (A \nabla v) - \Delta w = f$.
- DVP: we take the Legendre transform of $\mathcal{I}_\eta$

$$\mathcal{J}_\eta = \int_{\mathbb{R}^2} f \psi \, dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx - \frac{1}{2\eta} \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx$$

then we maximize $\mathcal{J}_\eta$ with the constraint $\nabla \cdot (A \nabla \psi) + \eta \Delta v = 0$.