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Results on cloaking by transformation optics and anomalous localized resonance in elliptic geometry

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Contents

Introduction

The ability to hide objects from electromagnetic, to make them "*cloaked*", has recently become a topic of both physical and mathematical interest [1, 22]. In recent years, great advances in materials science have opened the door to the creation of materials with unique optical properties, the so-called *metamaterials*. Metamaterials are characterized by customizable properties, such as the refractive index $n(x)$ for geometric optics, the electrical permittivity $\varepsilon(x)$, magnetic permeability $\mu(x)$ for vector optics and the conductivity $\sigma(x)$, which appears in the static limit of Maxwell's equations. These metamaterials are made via macroscopic cellular structures that are not present in nature [17].

In the last twenty years several proposals on cloaking have appeared in the physical and mathematical literature, taking advantage of the properties offered by these metamaterials. Among these, two appear to be particularly promising: cloaking by transformation optics, and cloaking by anomalous localized resonance (CALR).

The first proposal uses metamaterials to build particular structures, the *cloaking devices*.
Transformation optics exploit the transformation laws of an elliptic differential equation and the electrical conductive σ to create structures that appear equivalent when viewed from the outside, but that are different inside. Cloaking by transformation optics is closely related to electrical impedance tomography, which seeks to determine the conductivity σ through the knowledge of the Dirichlet-to-Neumann (or voltage-to-current) map Λ_σ.

The point is the following. The information on the conductivity σ of a bounded domain Ω can be determined from Λ_σ only up to a change of variables F. We therefore have two possibilities. We can change the coordinates in Ω, write σ in the new coordinates, and observe that the physical measurements at boundary $\partial\Omega$ do not change. We can also leave the coordinates fixed in Ω, change the conductivity σ, interpreting it as physically present in the system, and observe that the physical measurements at the boundary $\partial\Omega$ do not change. Following this second possibility we can modify the inside of Ω, for example by inserting an object that we want to hide. In principle, not only is the object hidden, but also the structure that hosts the object is not noticeable. We therefore have a cloaking device. This approach is the most used in the applications.

The second proposal, cloaking by anomalous localized resonance, exploits the mathematical phenomenon called anomalous localized resonance (ALR). The existence of ALR is linked to the fact that certain elliptic PDEs can exhibit localization effects near the boundary of ellipticity, causing the energy of the solution to diverge although the solution remains bounded outside some compact set [2]. Indeed, if an electromagnetic source produces a potential whose associated energy diverges in a specific region, the *shell* of the cloaking device, the system reacts by causing the potential to vanish outside a compact set. Since it is not possible to measure an appreciable potential from the outside, then the source is not detectable, making it cloaked.

This work started with the idea of studying a physically interesting phenomenon such as electromagnetic cloaking through mathematical methods accessible to an under-graduated physics student.

The present work aims to extend some results on transformation optics and CALR, namely [19] and [20], which provide theoretical results on cloaking and explicit estimates in circular geometry systems. More precisely, our goal is both to provide a theoretical introduction to cloaking and to present explicit results in elliptical geometry, where methods for representing solutions are still available.

Outline

This thesis is organized as follows. Further preliminary details are given at the beginning of each chapter.

Chapter 1 provides some preliminary information necessary for what follows. In particular, we introduce some basic definitions and ideas on partial differential equations (PDEs), such as the definition of a weak solution. We then introduce an explicit method for solving PDE, the method of separation of variables. Finally we construct the elliptical coordinate system in the plane, showing how it is related to the polar coordinate system.

In Chapter 2 we discuss cloaking by transformation optics. In presenting our arguments we mainly follow [19]. We first introduce electrical impedance tomography and the Dirichlet-to-Neumann map, then discuss how these two topics are related to electromagnetic cloaking. Our main results are Theorem 2.3.1 and Theorem 2.3.2, where we study the problem in an elliptical setting.

In Chapter 3 we discuss cloaking by anomalous localized resonance. We will follow the ideas in [20]. After introducing the differential problem under consideration, we analyze two ways to show ALR: via spectral theory and via variational principles, [2] and [20], respectively. We will follow the second method. The study of this problem in an elliptical setting has already been done in [9] by following the spectral approach in [2]. In Theorem 3.4.1 and Theorem 3.4.2 we study the problem via the variational approach presented in [20].

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Let me now say a few words from a broader viewpoint. I want to take the opportunity offered me by the occasion for some thanks, in memory of these three years of university. To all those teachers who have kept my interest in studying alive; to Alberto, Andrea, Cristiano, Davide, Edoardo and Nicole, for the many moments of those three years that will always remain good memories in

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I remained for a long time stuck after that comma, pondering over a possible sentence to put beyond it. Not a single one was enough to say *Thanks* to them.

List of symbols

Here is a list of the recurrent symbols in this work.

$\vert \cdot \vert$	The Euclidean norm.
$\langle \cdot, \cdot \rangle$	The Euclidean scalar product.
$\Vert \cdot \Vert_X$	The norm of the space X.
$B_r(x_0)$	The open ball with radius r and center x_0.
$\mathcal{E}_r(x_0)$	The open ellipse with elliptic radius r and center x_0.
$\partial\Omega$	The boundary of Ω.
Δ	The Laplacian.
∇	The gradient.
dx	The Lebesgue measure.
$\mathcal{H}^1 \lfloor \partial\Omega$	The 1-dimensional Hausdorff measure restricted to the set $\partial\Omega$.
$l^2(\mathbb{F})$	The space of square-summable sequences with values in $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$.
$C_c^\infty(\Omega)$	The space of infinite differentiable functions with compact support defined on Ω.
$L^1_{loc}(\Omega)$	The space of functions which are integrable on every compact subset of Ω.
$L^2(\Omega)$	The Lebesgue space of functions which are square-integrable on Ω.
$L^\infty(\Omega)$	The Lebesgue space of essentially bounded measurable functions Ω.
$H^1(\Omega)$	The space of functions which are square-integrable and whose weak gradient is L^2 in Ω.
$H^{-1}(\Omega)$	The dual space of $H^1(\Omega)$.

Chapter 1

Preliminaries

In this Chapter we introduce some notions which will be useful trough this thesis. In Section 1.1 we briefly recall some basic facts on the analysis of Partial Differential Equations (PDEs), such as the well posedness and the classical and weak formulation of solution as well as one of the methods to explicitly solve Laplace's equation: the method of separation of variables. In Section 1.2 we construct the elliptic coordinate system and we show that Laplace's equation is separable in this coordinates system. Section 1.3 is devoted to show how this coordinates system can be obtained through complex analysis. In particular we point out how the elliptic coordinates system is related to the polar one.

1.1 Some basic facts on PDE

In this section we recall some definitions, challenges and problems in the analysis of PDE. We first define what a PDE is. Let $\Omega \subseteq \mathbb{R}^n$ be open and fix $k \in \mathbb{N}$.

Definition 1.1.1. The equation

$$F(D^k u(x), D^{k-1}u(x), \cdots, Du(x), u(x), x) = 0 \qquad \text{with } x \in \Omega \tag{1.1}$$

is called a k^{th}-order partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is a given function, and $u : \Omega \to \mathbb{R}$ is the unknown and D^k is a derivation operator of order k with respect to one or more variables.

The definition of partial differential equation given in (1.1) is very general and opens the door to a very broad theory, and for this reason it is very unlikely that it can be fully explored. Our mathematical interest is addressed only to some PDEs: those that are important for applications, and, in particular, great attention is given to PDEs that arise from physical problems. PDEs coming from physics, in particular linear equations, represent a class that is already very rich of interesting mathematical phenomena.

In any case, given a PDE that describes a physical phenomenon, it is necessary to specify some conditions on the solution if we want to describe a real physical situation. For this reason, a real physical problem is described by the differential problem

$$\begin{cases} \text{PDE defined in } \Omega, \\ \text{boundary condition on } \partial\Omega. \end{cases} \tag{1.2}$$

Thus solving a PDE means finding all functions u that satisfy (1.1) together with some conditions imposed on the boundary $\partial\Omega$.

1.1.1 Well-posed problems, classical and weak solution

Given a differential problem (1.2) describing a physical situation, we are interested in quantitatively studying the behavior of the solution. In particular, since we are modeling a physical problem, we expect the existence and uniqueness of the solution. Furthermore, we expect that when the data at the boundary changes a little, the solution also changes a little. These three requirements are contained in the idea of a well-posed problem and it is due to Hadamard [15]. We say that a given problem for a partial differential equation is *well-posed* if

1. the problem has a solution;
2. the solution is unique;
3. the solution depends continuously on the data assigned by the problem.

There are two strategies to achieve $1)-3)$. The first one consists in considering *classical* solution.

Definition 1.1.2. A classical solution of (1.1) defined in an open set $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^k(\Omega)$ such that (1.1) is satisfied pointwise in Ω.

Thus solving a PDE in the classical sense means finding a solution belonging to the space $C^k(\Omega)$. This fact, although desirable, turns out to be actually a big limitation since our proofs must check that the solution is indeed smooth. And indeed the success of this strategy depends on the PDE that we are investigating and, in general, this strategy turns out to be ineffective. Consider for example the scalar conservation law

$$u_t + (F(u))_x = 0. \tag{1.3}$$

In fluid dynamics, equation (1.3) guarantees the conservation of a mass of fluid in some domain. It can be shown that (1.3) does not admit classical solutions [10]. However, the physical phenomena described by (1.3) present shock waves. A shock wave is a propagating disturbance that moves faster than the local speed of sound in the medium and it is characterized by a discontinuous change in characteristic of the medium. Thus, these waves are not represented by functions $u \in C^0$. Therefore, if we want to recover the underlying physics, these waves must be solution in some *weak* sense of (1.2). These new solutions are called *weak solutions* and do not possess a priori the regularity of classical solutions.

The weak solution idea paves the way for the second strategy. Solving a PDE now means finding a solution in a larger class of functions, since the solution is no longer restricted to belonging to the space $C^k(\Omega)$. Although this is only necessary for PDE like (1.3), PDE that are classically solvable can also be treated weakly. Indeed this is a huge advantage in proofs: we don't have to check the regularity of the solution when we show i) -ii); regularity can be recovered later on through theory of regularity.

1.1.2 Weak formulation of the Laplace's equation

Although the idea of a weak solution is a huge advantage in the study of PDE, the precise definition of a weak solution depends on the problem that we are investigating. Since the cloaking phenomenon arises from equations from equations that involve the Laplace operator, we briefly discuss the weak formulations of these.

We now formulate existence and uniqueness results for three problems: Laplace's equation with inhomogeneous Dirichlet boundary condition, Laplace equation with mixed boundary condition and a more general second-order differential problem.

In Chapter 2 we will study the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is open with $\partial\Omega \in C^1$ and $f : \partial\Omega \to \mathbb{R}$ is given. Problem (1.4) is the Laplace equation with inhomogeneous Dirichlet boundary condition.

We derive a weak formulation for (1.4) by proceeding formally. Let $v \in C_c^\infty(\Omega)$ be an arbitrary *test function*. Multiply the PDE in (1.4) and integrate over Ω to obtain

$$0 = \int_\Omega \Delta uv \, dx, \tag{1.5}$$

for any $v \in C_c^\infty(\Omega)$. Equation (1.5) is equivalent to the PDE in (1.4) if Δu is smooth enough. Indeed, the following lemma holds.

Lemma 1.1.1 (Fundamental lemma of the calculus of variations)**.** *If a function $g \in L^1_{loc}(\Omega)$ satisfies $\int_\Omega gv \, dx = 0$ for all $v \in C_c^\infty(\Omega)$ then $g = 0$ almost everywhere.*

Using Green's formula we find

$$0 = \int_\Omega \Delta uv \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v\bar{n} \cdot \nabla u \, d\sigma$$

where $\bar{n}$ is the unit normal vector to $\partial\Omega$. Since v vanishes at the boundary, we have

$$\int_\Omega \nabla u \cdot \nabla v \, dx = 0 \qquad \forall v \in C_c^\infty(\Omega). \tag{1.6}$$

We can now recast the problem (1.6) into a functional framework, which is more suitable for the weak formulation. A natural solution space is the Sobolev space $H^1(\Omega)$. Without going into detail, the Sobolev space $H^1(\Omega)$ is the space of functions which are square-integrable and whose *weak gradient* is square-integrable

$$H^1(\Omega) = \{u : \Omega \to \mathbb{R} \, : \, u \in L^2(\Omega), \, \nabla u \in L^2(\Omega; \mathbb{R}^n)\}.$$

By introducing the inner product

$$\langle u, v\rangle_{H^1(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$$

defined for all $u, v \in H^1(\Omega)$, the space $H^1(\Omega)$ becomes a separable Hilbert space.

Suppose now that $f \in H^1(\Omega)$. We now have to account for the boundary condition in (1.4). We have to be very careful if we look for solutions in $H^1(\Omega)$. Indeed, as it happens for $L^p(\Omega)$ spaces, the elements of $H^1(\Omega)$ are not functions, but equivalence classes of functions that agree almost everywhere. Since $\partial\Omega$ has measure zero in $\mathbb{R}^n$ we cannot ask $u = f$ on $\partial\Omega$, and even worse we cannot define u on $\partial\Omega$. To solve this problem, we use the test functions once again. One way to prescribe that $u = f$ on the boundary $\partial\Omega$ is to require that there exists a sequence of test functions $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ which approximates the difference $u-f$ in the $H^1(\Omega)$-norm. Indeed, this request is equivalent to asking that $u - f = 0$ near the boundary $\partial\Omega$. The functional space that is created in this process is the Sobolev space $H_0^1(\Omega)$. In a more formal way, $H_0^1(\Omega)$ is the closure of the space $C_c^\infty(\Omega)$ of test functions with respect to the $H^1(\Omega)$-norm

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}} = \{v : \Omega \to \mathbb{R} : \exists\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega), \|v - \varphi_k\|_{H^1(\Omega)} \xrightarrow{k\to+\infty} 0\}.$$

Definition 1.1.3. A function $u \in H^1(\Omega)$ is called a weak solution of (1.4) if

1. $\int_\Omega \nabla u \cdot \nabla v \, dx = 0$ for all $v \in C_c^\infty(\Omega)$;
2. $u - f \in H_0^1(\Omega)$.

The most important result is stated in the next theorem.

Theorem 1.1.1 (Dirichlet's principle)**.** *For all $f \in H^1(\Omega)$ there exists a unique weak solution $u \in H^1(\Omega)$ of (1.4).*

Without going into the details of the proof, we mention that existence is based on showing that the energy functional $\mathcal{E} : H^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(v) = \int_\Omega |\nabla v|^2 \, dx$$

admits minimum in the subspace $\{v \in H^1(\Omega) : v - f \in H_0^1(\Omega)\}$. Indeed, the minimum of this functional turns out to be a weak solution of (1.4). The uniqueness of the solution is then given by Poincare's inequality [5]. Via regularity theory it is then possible to show that a weak solution of (1.4) is also a classical solution under appropriate assumptions on f. For example, if $f \in C^\infty$ then $u \in C^\infty$; but then using lemma (1.1.1) and the fact that $u \in C^2$ we have $\Delta u = 0$ *everywhere* in Ω and $u = f$ on $\partial\Omega$. Hence the weak solution is actually a classic solution.

Later on in Chapter 2 we will study the Laplace's equation with mixed boundary conditions. The problem we are referring to is the following. Let Ω_1, Ω_2 be bounded and open domains in $\mathbb{R}^n$ with boundary of class C^1. Assume that $\overline{\Omega_2} \subset \Omega_1$ and set $\Omega = \Omega_2 \setminus \overline{\Omega_1}$. For $f \in H^1(\Omega)$ we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega_1, \\ \frac{\partial u}{\partial \bar{n}} = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{1.7}$$

With techniques similar to those used previously, we can give a weak formulation of (1.7).

Definition 1.1.4. A function $u \in H^1(\Omega)$ is called a weak solution of (1.7) if

1. $\int_\Omega \nabla u \cdot \nabla v \, dx = 0$ for all $v \in H^1_0(\Omega \cup \partial\Omega_2)$;
2. $u - f \in H^1_0(\Omega \cup \partial\Omega_2)$.

Even for problem (1.7) it is shown that there exists only one weak solution.

Theorem 1.1.2. *For all $f \in H^1(\Omega)$ there exists a unique weak solution $u \in H^1(\Omega)$ of (1.7).*

Although the results of Theorems 1.1.1 and 1.1.2 are very similar, their proofs use different techniques. Indeed Theorem 1.1.2 proves the existence of the solution through the Fredholm alternative, while uniqueness is proved through maximum principle. See [12] and reference therein for further details on the weak generalization of problem (1.7).

Lastly, in Chapter 3 we will study the problem

$$\begin{cases} Lu = f & \text{in } \mathbb{R}^n, \\ u \to 0 & \text{for } |x| \to +\infty. \end{cases} \tag{1.8}$$

where L is a linear second order differential operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u) = -\nabla \cdot (A(x) \cdot \nabla u),$$

and $f : \mathbb{R}^n \to \mathbb{R}$ is given and such that

$$\int_{\mathbb{R}^n} f(x)\, dx = 0.$$

This hypothesis will be justified in Chapter 3. Assume that $a_{ij}(x) \in L^\infty(\mathbb{R}^n)$ for $1 \le i, j \le n$ and that the matrix $A = (a_{ij})$ is symmetric. Assume also that L satisfies the *ellipticity* condition, i.e. there exists some constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \qquad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n.$$

Proceeding in the same way as seen above we derive the weak formulation of (1.8). Multiplying the PDE in (1.8) by a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ and and integrating on $\mathbb{R}^n$ we have

$$-\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u)\varphi \, dx = \int_{\mathbb{R}^n} f\varphi \, dx.$$

Integrating the first term by parts we find

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}\partial_i u \, \partial_j \varphi \, dx = \int_{\mathbb{R}^n} f\varphi \, dx. \tag{1.9}$$

Note that to make sense of the right-hand side of the above equation (1.9) it is necessary to require $f \in L^2(\mathbb{R}^n)$. Furthermore, the left-hand side of the equation (1.9) can be interpreted in functional terms. Indeed, if we look for solutions $u \in H^1(\mathbb{R}^n)$, we can use the density of $C_c^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$ [10] to rewrite the first term as a bilinear map on $H^1(\mathbb{R}^n)$.

Definition 1.1.5. The bilinear form associate with PDE of problem (1.8) is the bilinear map a : $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$ given by

$$a(u,v) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} \partial_i u \, \partial_j v \, dx \qquad \forall u, v \in H^1(\mathbb{R}^n).$$

Equation (1.8) can then be rewritten in a weak form.

Definition 1.1.6. A function $u \in H^1(\mathbb{R}^n)$ is called a weak solution of (1.8) if

$$a(u,v) = \langle f, v \rangle_{L^2(\mathbb{R}^n)}$$

for all $v \in H^1(\mathbb{R}^n)$.

The symmetry of the coefficients ensures that the bilinear form a is symmetric. A simple estimate shows also that a is continuous. Furthermore, the ellipticity hypothesis proves that the bilinear form is coercive. Hence we can apply Lax-Milgram's lemma and deduce the following theorem.

Theorem 1.1.3. *For any $f \in L^2(\mathbb{R}^n)$ there exists a unique weak solution $u \in H^1(\mathbb{R}^n)$ of (1.8). Moreover, u is obtained by*

$$\min_{v \in H^1(\mathbb{R}^n)} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} \partial_i u \, \partial_j v \, dx - \int_{\mathbb{R}^n} f v \, dx \right\}.$$

1.1.3 The method of separation of variables

Once we have proved existence and uniqueness of weak solutions for a given differential problem, we can look for the explicit form of the solution. As far we know, there are a few techniques to explicitly solve a PDE. For linear PDEs, like Laplace equation, an effective technique is the method of separation of variables. Let us now analyze the theoretical presuppositions behind this method.

Fix $p \in [1, +\infty)$. Let be $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ two measurable sets. Take two functions $v_E \in L^p(E)$, $w_F \in L^p(F)$ and consider the function $u : E \times F \to \mathbb{R}$ defined by $u(x,y) = v_E(x) w_F(y)$.

Since v_E and w_F are not defined pointwise, the function u is not defined pointwise as well. Fubini's theorem assures that if $\tilde{E} \subset E$ is with zero measure in $\mathbb{R}^n$, then the set $\tilde{E} \times F$ has zero measure in $\mathbb{R}^n \times \mathbb{R}^m$. A similar reasoning is valid for F. This proves that u is well defined as an equivalence class of functions equal almost everywhere. Fubini's theorem guarantees also that $u \in L^p(E \times F)$.

We call $L^p(E) \otimes L^p(F)$ the linear subspace of $L^p(E \times F)$ generated by the function built according to the previous process. This space is called the product tensor space. The following theorem holds.

Theorem 1.1.4. *The space $L^p(E) \otimes L^p(F)$ is dense in $L^p(E \times F)$.*

Proof. The proof is based on the following observation. It is known [23] that step functions, i.e. functions that belong to the vector space generated by the characteristic χ_E as E varies between the closed rectangles of $\mathbb{R}^n$, are dense in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$. A simple extension of the argument that led to this statement shows that step functions are dense also in the space $L(E \times F)$. To complete the proof of the thesis it is sufficient to observe that every rectangle $Q \subseteq E \times F$ of $E \times F$ is the Cartesian product of a rectangle Q_E contained in E and a Q_F contained in F. Writing $Q = Q_E \times Q_F$ we have $\chi_Q(x, y) = \chi_{Q_E}(x)\chi_{Q_F}(y)$ and this proves the thesis, since $\chi_{Q_E} \in L^p(E)$ and $\chi_{Q_F} \in L^p(F)$. □

For all $p \in [1, +\infty)$ the space $L^p(E)$ is separable, i.e. there exists a countable dense subset. In particular, the Hilbert space $L^2(E)$ is separable. However, this is equivalent to the existence of a complete orthonormal system. As a consequence, the following result holds.

Corollary 1.1.4.1. *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ two measurable sets. Let $\{\phi_h\}_{h\in\mathbb{N}}$ be an orthonormal basis of $L^2(E)$ and $\{\psi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(F)$. Then the family of functions $\{\theta_{h,k}\}_{h,k\in\mathbb{N}}$ defined by $\theta_{h,k}(x, y) = \phi_h(x)\psi_k(y)$ is an orthonormal basis of $L^2(E \times F)$.*

Proof. Starting from the definition and using Fubini's Theorem, it is immediate to verify that the $\theta_{h,k}$ are an orthonormal system. The completeness follows from the completeness of the single systems $\{\phi_h\}_{h\in\mathbb{N}}$, $\{\psi_k\}_{k\in\mathbb{N}}$ in the respective spaces, and from the previous theorem. □

We are now ready to discuss the method of separation of variables. We want to find an explicit form for weak solutions of Laplace equation

$$\Delta u = 0$$

in a rectangular domain $\Omega = E \times F$. Since $u \in H^1(\Omega)$ is a weak solution, we have $u \in L^2(\Omega)$. Then, the previous corollary (1.1.4.1) ensures that

$$u(x, y) = \sum_{h,\, k\in\mathbb{N}} C_{h,k}\theta_{h,k}(x, y) = \sum_{h,\, k\in\mathbb{N}} C_{h,k}\phi_h(x)\psi_k(y), \tag{1.10}$$

where $\{\phi_h\}_{h\in\mathbb{N}}$ is an orthonormal basis of $L^2(E)$ and $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(F)$, and $\{C_{h,k}\}_{h,k\in\mathbb{N}} \subset l^2(\mathbb{C})$. The method of separation of variables consists then in finding the solutions $\theta_{h,k}(x, y)$ of Laplace's equation which factorize in the product $\phi_h(x)\psi_k(y)$ of functions that depend on disjoint variables, and then writing the most general solution u as in (1.10). To be more precise we should verify that $\{\phi_h\}_{h\in\mathbb{N}}$ and $\{\psi_k\}_{k\in\mathbb{N}}$ constitute an orthonormal system of $L^2(E)$ and $L^2(F)$, respectively. This is indeed true, as stated in Theorem A.1, but the point is a bit technical. See Appendix A for more details.

1.2 Elliptic Coordinates

In this section we introduce the elliptic coordinate system on the plane $\mathbb{R}^2$. Since in Chapters 2 and 3 we study cloaking in elliptical geometries, we discuss the shape of ellipses in this coordinate system. Lastly we present calculus in the elliptic coordinate system.

1.2.1 Definitions

Elliptical coordinates are an orthogonal curvilinear system of coordinates for the plane $\mathbb{R}^2$. In this coordinate system, each point $(x, y) \in \mathbb{R}^2$ is identified by two numbers $\mu \in [0, +\infty)$, $\nu \in [0, 2\pi)$.

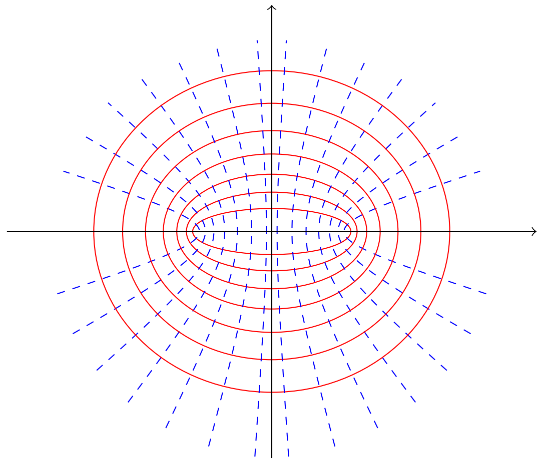

Figure 1.1: The figure shows several coordinated lines. In red we see the ellipses, corresponding to constant μ; in blue the hyperbolae, corresponding to constant ν.

Fix $a > 0$. The trasformation between the cartesian coordinates system (x, y) and the elliptic one (μ, ν) is given by

$$\begin{cases} x = a \cosh \mu \cos \nu \\ y = a \sinh \mu \sin \nu. \end{cases} \tag{1.11}$$

In order to give meaning to the variables (μ, ν), let's look at the coordinated lines. From the expression of the coordinates (1.11) we see that a correct linear combination of x^2 and y^2 turns out to be constant, and therefore represents a coordinate line. Indeed, if $\mu = \mu^* \in [0, +\infty)$ is fixed, we have that

$$\left.\frac{x^2}{a^2 \cosh^2 \mu^*} + \frac{y^2}{a^2 \sinh^2 \mu^*}\right|_{\substack{x = a \cosh \mu^* \cos \nu \\ y = a \sinh \mu^* \sin \nu}} = \cos^2 \nu + \sin^2 \nu = 1. \tag{1.12}$$

Consequently, the coordinate lines corresponding to constant μ are ellipses. Similarly, we have that

$$\left.\frac{x^2}{a^2 \cos^2 \nu^*} - \frac{y^2}{a^2 \sin^2 \nu^*}\right|_{\substack{x = a \cosh \mu \cos \nu^* \\ y = a \sinh \mu \sin \nu^*}} = \cosh^2 \mu - \sinh^2 \mu = 1 \tag{1.13}$$

for $\nu = \nu^* \in [0, 2\pi)$ fixed. Hence the curves corresponding to constant ν are hyperbolae. Therefore the coordinates (μ, ν) identify a point of $\mathbb{R}^2$ by the intersection of an ellipse with a hyperbola. Figure 1.1 shows some coordinated lines.

From equations (1.12) and (1.13) we deduce the meaning of the parameter $a > 0$: it represents the x-coordinate of the foci $F_\pm = (\pm a, 0)$ of the coordinate lines.

For $R > 0$ we define

$$\partial \mathcal{E}_R(0) = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2 \cosh^2 R} + \frac{y^2}{a^2 \sinh^2 R} = 1 \right\}$$

to be the ellipse with foci in $(\pm a, 0)$ and intersection with the x-axis in $(a \cosh R, 0)$. We will call the number R elliptical radius. The set $\partial\mathcal{E}_R$ is the boundary of the elliptic region

$$\mathcal{E}_R(0) = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2 \cosh^2 R} + \frac{y^2}{a^2 \sinh^2 R} < 1\right\}.$$

In elliptical coordinates, the region $\mathcal{E}_R(0)$ is given by

$$\mathcal{E}_R(0) = \{(\mu, \nu) \in [0, R) \times [0, 2\pi)\}.$$

1.2.2 Derivatives and Integrals in Elliptic Coordinates

We now present a useful formulary to carry out explicit calculations in elliptical coordinates. The information on the coordinate system (ξ_1, ξ_2) chosen to study a problem on $\mathbb{R}^2$ is contained in scale factors $h_x(\xi_1, \xi_2)$ and $h_y(\xi_1, \xi_2)$. For orthogonal coordinate systems, such as elliptical, scale factors are the equal $h_x = h_y = h$. For the elliptic coordinate system we have

$$h(\mu, \nu) = a\sqrt{\sinh^2 \mu + \sin^2 \nu}$$

as it can be deduced through differential geometry's tools.

In subsequent chapters we will make a lot of use of the instruments of mathematical analysis in systems with elliptical geometries. Therefore we give the expression of some of them.

Let us start with the gradient. Let $u : \mathbb{R}^2 \to \mathbb{R}$ with $u \in C^1(\mathbb{R}^2)$. The gradient of u in elliptical coordinates is given by

$$\nabla u(\mu, \nu) = \frac{1}{h(\mu, \nu)}\left(\frac{\partial u}{\partial \mu}, \frac{\partial u}{\partial \mu}\right).$$

If $u \in C^1(\overline{\mathcal{E}_R})$, the normal derivative of u in $(R, \nu) \in \partial\mathcal{E}_R$ is

$$\frac{\partial u}{\partial \bar{n}}(R, \nu) = \nabla u(R, \nu) \cdot \bar{n} = \frac{1}{h(R, \nu)} \frac{\partial u}{\partial \mu}(R, \nu) \qquad \nu \in [0, 2\pi).$$

where $\bar{n}$ is the unit normal pointing out of $\mathcal{E}_R$.

We will also need to calculate some integrals in system with elliptic geometry. If $u \in L^1(\mathbb{R}^2)$ its integral over $\mathbb{R}^2$ is given by

$$\int_{\mathbb{R}^2} u\, dx = \int_0^{2\pi} \int_0^{+\infty} u(\mu, \nu) h^2(\mu, \nu)\, d\mu d\nu.$$

Note that if $u \in C^1(\mathbb{R}^2)$ and $\nabla u \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ then

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u|^2\, dx &= \int_0^{2\pi} \int_0^{+\infty} |\nabla u(\mu, \nu)|^2 \delta^2(\mu, \nu)\, d\mu d\nu \\ &= \int_0^{2\pi} \int_0^{+\infty} \left(\left(\frac{\partial u}{\partial \mu}\right)^2 + \left(\frac{\partial u}{\partial \nu}\right)^2\right) d\mu d\nu.\end{aligned}$$

Moreover, in the elliptic coordinates system the Laplacian of $u \in C^2$ is given by

$$\Delta u(\mu,\nu) = \frac{1}{h^2(\mu,\nu)}\left(\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \nu^2}\right)$$

and the Laplace equation reads

$$\Delta u(\mu,\nu) = \frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \nu^2} = 0.$$

1.3 Conformal maps

We now discuss how the elliptical coordinate system was deduced and how it is possible to relate it to the polar coordinate system.

First we observe that the coordinate system (μ, ν) is orthogonal, as can be seen from the expression of the line element

$$ds^2 = a^2(\sinh^2\mu + \sin^2\nu)(d\mu^2 + d\nu^2).$$

The orthogonality of a coordinate system can also be interpreted in terms of the coordinate lines. Consider the Cartesian coordinate system (x, y). For this system the coordinate lines are given by the curves $\gamma_{x_0} : t \mapsto (x_0, t)$ and $\gamma_{y_0} : \tau \mapsto (\tau, y_0)$ for $(x_0, y_0) \in \mathbb{R}^2$ and $t, \tau \in \mathbb{R}$. These curves form an infinite grid of orthogonal lines, meaning that at every point $(x_0, y_0) \in \mathbb{R}^2$ where the two curves intersect, the velocity vectors $\dot{\gamma}_{x_0}$ and $\dot{\gamma}_{y_0}$ are orthogonal, i.e.

$$ds^2(\dot{\gamma}_{x_0}, \dot{\gamma}_{y_0}) = 0.$$

Consequently, the map $(\mu, \nu) \mapsto (x, y)$ must preserve the orthogonality between the vectors tangent to the coordinate curves. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) = x + iy = z$, this property is exactly expressed by conformal maps.

Definition 1.3.1. Let $U, V \subseteq \mathbb{C}$. A function $z : U \to V$ is called conformal at a point $u_0 \in U$ if it preserves angles between directed curves through u_0.

A sufficient and necessary condition for a map to be conformal is the following [18].

Theorem 1.3.1. *A function $z : U \subseteq \mathbb{C} \to \mathbb{C}$ is conformal if and only if it is holomorphic and $z'(w) \neq 0$ for all $w \in U$.*

Using the properties expressed by the necessary and sufficient condition we can show that the Laplace's operator is separable in an orthogonal coordinate system. This is due to the fact that the Laplace's operator is separable when expressed in Cartesian coordinates, and theorem (1.3.1) guarantees that all coordinate systems obtained from the Cartesian system through conformal maps behave exactly like the latter. Indeed, let us consider a conformal map $z : \mathbb{C} \to \mathbb{C}$, where we equip the domain with coordinates $w = \xi_1 + i\xi_2$ and the codomain with coordinates $z = x + iy$. We have

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

so

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right).$$

Then, the Laplacian is

$$\Delta u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y) = 4\frac{\partial^2}{\partial z \partial \bar{z}}u(z,\bar{z}).$$

Now $z = z(w)$ and hence

$$\frac{\partial}{\partial z} = \frac{dw}{dz}\frac{\partial}{\partial w}, \qquad \frac{\partial}{\partial \bar{z}} = \frac{d\bar{w}}{d\bar{z}}\frac{\partial}{\partial \bar{w}};$$

since the Laplacian in terms of the (ξ_1, ξ_2) coordinates is

$$4\left|\frac{dw}{dz}\right|^2 \frac{\partial^2}{\partial w \partial \bar{w}}u(w,\bar{w}) = 4\left|\frac{dw}{dz}\right|^2 \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}\right)u(\xi_1,\xi_2),$$

then the Laplace's equation takes the form

$$\left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}\right)u(\xi_1,\xi_2) = 0,$$

where we used that $|dw/dz|^2 \neq 0$. This last property of conformal maps can be found in [18].

1.3.1 The elliptic and polar coordinates system via conformal maps

We seek now for conditions on the map $z = z(w)$. Since the function $z = z(w)$ is also conformal, it satisfies the Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial x}{\partial \xi_1} = \dfrac{\partial y}{\partial \xi_2} \\ \dfrac{\partial x}{\partial \xi_2} = -\dfrac{\partial y}{\partial \xi_1}. \end{cases} \tag{1.14}$$

Cauchy-Riemann equations ensure also that if $z' \neq 0$ then the scale factors

$$h_x = \sqrt{\left(\frac{\partial x}{\partial \xi_1}\right)^2 + \left(\frac{\partial x}{\partial \xi_2}\right)^2} \qquad \text{and} \qquad h_y = \sqrt{\left(\frac{\partial y}{\partial \xi_1}\right)^2 + \left(\frac{\partial y}{\partial \xi_2}\right)^2}$$

are equal and never vanish

$$h_x = h_y = \left|\frac{dz}{dw}\right| \neq 0.$$

The Jacobian of the trasformation $w \mapsto z(w)$ is given by

$$\det(Jz) = \det\begin{pmatrix} \dfrac{\partial x}{\partial \xi_1} & \dfrac{\partial x}{\partial \xi_2} \\ \dfrac{\partial y}{\partial \xi_1} & \dfrac{\partial y}{\partial \xi_2} \end{pmatrix} = \left(\frac{\partial x}{\partial \xi_1}\right)^2 + \left(\frac{\partial x}{\partial \xi_2}\right)^2 = h_x h_y = \left|\frac{dz}{dw}\right|^2$$

where we used (1.14). We now suppose that

$$\left|\frac{dz}{dw}\right|^2 = f_1(\xi_1) + f_2(\xi_2), \tag{1.15}$$

which implies

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}\left|\frac{dz}{dw}\right|^2 = 0. \tag{1.16}$$

Since $|dz/dw|$ is the scale factor of the trasformation, we have now an equation for the kind of scale factor a coordinate system must have to be separable for Laplace equation under the assumption (1.15). Now dz/dw is a function of w and $d\bar{z}/d\bar{w}$ is a function of $\bar{w}$ so we have to rewrite the differential operator in (1.16). We have

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2} = i\frac{\partial^2}{\partial w^2} - i\frac{\partial^2}{\partial\bar{w}^2}$$

Hence equation (1.16) is equivalent to

$$\left(\frac{d\bar{z}}{d\bar{w}}\right)\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) = \left(\frac{dz}{dw}\right)\frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right)$$

or

$$\frac{1}{dz/dw}\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) = \frac{1}{d\bar{z}/d\bar{w}}\frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right).$$

Since the left hand side is a function depending only on w and the right hand side is a function of z, they must equal the same constant λ. Therefore equation (1.16) is then equivalent to the system

$$\begin{cases} \dfrac{\partial^2}{\partial w^2}\left(\dfrac{dz}{dw}(w)\right) = \lambda\dfrac{dz}{dw}(w) \\ \dfrac{\partial^2}{\partial\bar{w}^2}\left(\dfrac{d\bar{z}}{d\bar{w}}(\bar{w})\right) = \lambda\dfrac{d\bar{z}}{d\bar{w}}(\bar{w}). \end{cases} \tag{1.17}$$

The system of equations (1.17) must be solved for different value of $\lambda \in \mathbb{R}$. In practice, it is sufficient to solve it for $\lambda = 0, 1$, since any different value is equivalent to a change of scale (or a change of orientation) and it is not relevant for the behavior of the solution.

The case in which we set $\lambda = 0$ gives the Cartesian coordinates. Indeed the solution for dz/dw is $\beta + \gamma w$, and if $\gamma = 0$ we have

$$z = \alpha + i\beta$$

where $\alpha, \beta \in \mathbb{C}$. If $\alpha = a + ib, \beta = c + id$ we have

$$x = a + c\xi_1 - d\xi_2, \qquad yb + c\xi_2 + d\xi_1$$

and the trasformation correspond to a rotation, change of scale and translation.

Set $\lambda = 1$ in (1.17). It is straightforward to check that

$$z(w) = ae^{w} + be^{-w} \tag{1.18}$$

is a solution of (1.17) for $a, b \in \mathbb{R}$. Letting $a = 1, b = 0$ we find

$$z = e^w = e^{\xi_1 + i\xi_2} = e^{\xi_1} e^{i\xi_2}$$

that is the polar coordinates

$$\begin{cases} x = e^{\xi_1} \cos(\xi_2) \\ y = e^{\xi_1} \sin(\xi_2), \end{cases} \qquad h_x = h_y = e^{\xi_1}.$$

Suppose now $b \neq 0$. If in (1.18) we set

$$\begin{cases} a = \frac{1}{2} d e^{-\beta} = \frac{1}{2} e^{\alpha - \beta} \\ b = \frac{1}{2} d e^{+\beta} = \frac{1}{2} e^{\alpha + \beta} \end{cases} \tag{1.19}$$

with $d = e^{\alpha} = \sqrt{4ab}$, $e^{\beta} = \sqrt{b/a}$ and $d, \alpha, \beta \in \mathbb{R}$, we obtain

$$z = \frac{1}{2} d e^{w - \beta} + \frac{1}{2} d e^{-(w - \beta)} = d \cosh(w - \beta).$$

Since $w = \xi_1 + i\xi_2$, from the definition of the hyperbolic cosine and Euler's formula we find

$$z = d \cosh(\xi_1 + i\xi_2 - \beta) = d\Big(\cosh(\xi_1 - \beta) \cos(\xi_2) + i \sinh(\xi_1 - \beta) \sin(\xi_2) \Big).$$

Comparing the latter equation with $z = x + iy$ we find the coordinate

$$\begin{cases} x = d \cosh(\xi_1 - \beta) \cos(\xi_2) \\ y = d \sinh(\xi_1 - \beta) \sin(\xi_2). \end{cases} \tag{1.20}$$

Moreover the scale factor is

$$\left| \frac{dz}{dw} \right| = h_x = h_y = d \sqrt{\sinh^2(\xi_1 - \beta) + \sin^2(\xi_2)}.$$

The coordinate system given by equations (1.20) is the elliptic coordinates system. Curves with constant ξ_1 describe confocal ellipses with foci in $(\pm d, 0)$ and intersection with the x-axis in $(d \cosh(R), 0)$.

Notice that if we set $\alpha = \beta + \ln 2$ in (1.19) we find

$$\begin{cases} a = \frac{1}{2} e^{\alpha - \beta} = 1 \\ b = \frac{1}{2} e^{\alpha + \beta} = e^{2\beta} \\ d = e^{\alpha} = 2e^{\beta} \end{cases}$$

and by taking the limit $\beta \to -\infty$ we find that $d \to 0$. In this limit the elliptic foci merge together into the origin: the elliptic coordinate system changes in to polar coordinates system.

In particular, it is possible to map the ellipse with foci in $(\pm d, 0)$ and intersection with the x-axis in $(d\cosh(R), 0)$ given by

$$\partial\mathcal{E}_R(0) = \left\{(x,y) \in \mathbb{R}^2 \,:\, \frac{x^2}{d^2\cosh^2 R} + \frac{y^2}{d^2\sinh^2 R} = 1\right\} \tag{1.21}$$

in the open ball $\partial B_r(0) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = r^2\}$. To determine the radius r we operate the substitution

$$\begin{cases} d \longrightarrow d = 2e^{\beta} \\ R \longrightarrow R = R - \beta \end{cases}$$

in (1.21). By letting $\beta \to -\infty$ we find that

$$r = e^R. \tag{1.22}$$

Chapter 2

Cloaking by transformation optics

In this Chapter we introduce cloaking by transformation optics. We follow the ideas presented by Khon et al. in [19]. In Section 2.1 we introduce electrical impedance tomography, a technique used to obtain information on the inside of a body by measurements on its boundary. We then introduce cloaking based on the change of variables in circular geometry. In Section 2.2 we present a technical analysis of *near-cloaking* in circular geometry, and the main results are Theorem 2.2.2 and Theorem 2.2.3. Finally in Section 2.3 we study the problem in elliptic geometry and show that cloaking by transformation optics does not occur for generic sources, Theorem 2.3.1, while for high frequency sources cloaking can still occur, Theorem 2.3.2.

2.1 The main ideas

In this section we introduce some basic concepts and some definitions to make more precise what is meant by cloaking by transformation optic.

2.1.1 Electric impedance tomography

The main purpose of electrical impedance tomography is to obtain information on the interior of an object by using only the information that is available at its boundary. Electrical impedance tomography is currently used in medical screening techniques to infer the composition of body tissues.

In this procedure, electrodes are attached to the skin of the examined region. Electrostatic stimuli are transmitted from the electrodes to the skin tissue, and the electrostatic response is recorded through other electrodes. This process is then repeated for several initial stimuli. Physically, the electrodes apply a small current to the boundary, which for the laws of electrostatics, results in the application of a known elettrostatic potential to the boundary. The presence of this potential determines the potential within the examined region. Since each tissue has a different conductivity, by determining the current generated at the boundary we can try to determine the composition of the studied area. This idea is attributed to Webster and first appeared in [16].

Mathematically speaking, the problem of recovering conductivity from surface measurements of current and potential is a non-linear inverse problem and is severely ill-posed. It is precisely this ill-position of the problem that allows the phenomenon of cloaking [13]. This problem was posed and partiality addressed by Calderon in [7].

Let us now formulate the problem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $\partial\Omega \in C^1$, be the region we want to study. Suppose Ω is filled with a material with unknown electrical conductivity σ. Mathematically, σ is a non-negative symmetric matrix-valued function defined on Ω. The equation describing the electrostatic potential $u : \Omega \to \mathbb{R}$ is the PDE

$$\nabla \cdot (\sigma \nabla u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sigma_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \qquad \text{in } \Omega, \tag{2.1}$$

with the Dirichlet data $u = f$ on $\partial\Omega$. Equation (2.1) relates the electric field ∇u, or rather the voltage u, to the current $\sigma\nabla u$. For what has been said, electrical impedance tomography studies the Dirichlet-to-Neumann map Λ_σ associated with problem (2.1), seeking information about σ. The Dirichlet-to-Neumann map is the map which, given a voltage at the boundary, returns the corresponding current at the boundary

$$\Lambda_\sigma : u\big|_{\partial\Omega} \to (\sigma\nabla u)\cdot \bar{n}\big|_{\partial\Omega}$$

where $\bar{n}$ is the outward unit normal to $\partial\Omega$.

In this context, a subset $D \subset \Omega$ is cloaked if its contents, as well as its existence, are invisible to electrostatic measurements performed on the boundary $\partial\Omega$. However, this definition needs to be refined, since we have no way of translating the above sentence into mathematical terms.

Definition 2.1.1. Let $D \subset \Omega$ be a fixed domain and let $\sigma_c : \Omega \setminus D$ be a non-negative, matrix valued conductivity. We will say that σ_c cloaks the region D if any extension of it on D

$$\sigma_A(x) = \begin{cases} A(x) & \text{for } x \in D, \\ \sigma_c(x) & \text{for } x \in \Omega \setminus D \end{cases}$$

produces the same measurements at the boundary of a conductivity $\sigma = 1$ defined in all Ω, regardless the choice of A in D.

Note that this definition captures an important feature: if we place our structure Ω inside another object we would like the conductivity of definition (2.1.1) to still cloak D. Indeed, suppose σ_c cloaks $D \subset \Omega$, and let Ω' be any domain containing the structure Ω. Then the Dirichlet-to-Nemann map associated with the conductivity

$$\sigma(x) = \begin{cases} A(x) & \text{for } x \in D, \\ \sigma_c(x) & \text{for } x \in \Omega \setminus D, \\ 1 & \text{for } x \in \Omega' \setminus \Omega \end{cases}$$

is independent of $A(x)$, and it is identical to that of a conductivity $\sigma = 1$ defined on all Ω'.

2.1.2 Invariance by change of variables

Equation (2.1) has an interesting symmetry. Actually, the map Λ_σ allows us to determine σ at its best at less than a change of variables. This observation is pointed in [21] with an attribution to Tartar.

With ideas similar to those that led to Theorem 1.1.3, we can prove that there is a unique solution of (2.1) with Dirichlet data $u = f$ at $\partial\Omega$. Indeed, if σ is bounded, symmetric, positive definite (and therefore the corresponding operator L is elliptic), and $f \in H^1(\Omega)$, the solution of (2.1) is unique and can be obtained by minimizing the energy functional in the appropriate Sobolev space $H^1(\Omega)$

$$\mathcal{E}(u) = \int_\Omega \langle \nabla u, \sigma \nabla u \rangle \, dx \tag{2.2}$$

with the constraint $u - f \in H_0^1(\Omega)$ to take account of the boundary condition $u = f$ at $\partial\Omega$.

Proposition 2.1.1. *In the above setup, the knowledge of Λ_σ determines the minimum energy (and hence the solution u). Furthermore, the knowledge of the minimum for all Dirichlet data determines the boundary map Λ_σ.*

Proof. Let u be the minimum of the energy (2.2). Since $f = u$ at $\partial\Omega$ we have

$$\begin{aligned}
\int_{\partial\Omega} f \Lambda_\sigma(f) \, d\sigma(x) &= \int_{\partial\Omega} u \langle \sigma \nabla u, \bar{n} \rangle \, d\sigma(x) \\
&= \int_\Omega \langle \nabla, (u\sigma\nabla u) \rangle \, dx \\
&= \int_\Omega (\langle \nabla u, \sigma \nabla u \rangle + \langle u \nabla, (\sigma \nabla u) \rangle) \, dx \\
&= \int_\Omega \langle \nabla u, \sigma \nabla u \rangle \, dx = \mathcal{E}(u)
\end{aligned}$$

where we used (2.1). The converse follows from the polarization identity: for all f, g

$$4 \int_{\partial\Omega} f \Lambda_\sigma(g) \, dx = \int_{\partial\Omega} (f+g) \Lambda_\sigma(f+g) \, dx - \int_{\partial\Omega} (f-g) \Lambda_\sigma(f-g) \, dx.$$

□

Let us now discuss the change of variables. Let $F : \Omega \to \Omega$ be an invertible map which preserves orientation. Set $y = F(x)$. The energy functional becomes then

$$\mathcal{E}(u) = \int_\Omega \sum_{i,j=1}^n \sigma_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx = \int_\Omega \sum_{i,j,k,l=1}^n \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} \frac{\partial u}{\partial y_l} \frac{\partial y_l}{\partial x_j} \det\left(\frac{\partial x}{\partial y}\right) dy.$$

We can rewrite this more compactly as

$$\int_\Omega \langle \sigma(x) \nabla u(x), \nabla u(x) \rangle \, dx = \int_\Omega \langle (F_* \sigma(y)) \nabla u(y), \nabla u(y) \rangle \, dy$$

where

$$F_*\sigma(y) = \frac{1}{\det(DF)(x)} DF(x)\sigma(x)(DF(x))^T$$

is the *push-forward* of σ by the change of variables F. DF is the matrix with (i,j) element $\partial y_i/\partial x_j$ and the right hand side is evaluated at $x = F^{-1}(y)$.

Proposition 2.1.2. *In the above setup, let F be such that $F(x) = x$ at $\partial\Omega$. Then the boundary measurements associated with σ and $F_*\sigma$ are identical, i.e.*

$$\Lambda_\sigma(f) = \Lambda_{F_*\sigma}(f) \qquad \text{for all } f.$$

Proof. Since $F(x) = x$ at $\partial\Omega$, the change of variables does not affect the Dirichlet data f. But then for any f we have

$$\begin{aligned}\int_{\partial\Omega} f\Lambda_\sigma(f)\, d\sigma(x) &= \min_{u=f \text{ at } \partial\Omega} \int_\Omega \langle \sigma(x)\nabla_x u(x), \nabla_x u(x)\rangle\, dx \\ &= \min_{u=f \text{ at } \partial\Omega} \int_\Omega \langle (F_*\sigma(y))\nabla_y u(y), \nabla_y u(y)\rangle\, dy \\ &= \int_{\partial\Omega} f\Lambda_{F_*\sigma}(f)\, d\sigma(y).\end{aligned}$$

The thesis then follows from the Proposition 2.1.1. □

2.1.3 Cloaking via change of variables

The result expressed in the proposition opens the door to the *near-cloak* phenomenon. Cloaking in systems with circular symmetries has been extensively studied. See [11]. We briefly illustrate the idea in the case of circular geometry. For simplicity, let $\Omega = B_2(0) \subset \mathbb{R}^2$ and $D = B_1(0)$ the region to be cloaked. The case with generic radii is analogous. Fix $\rho > 0$ and consider the diffeomorphism

$$F(x) = \begin{cases} \frac{x}{\rho} & \text{for } |x| \leq \rho, \\ \left(\frac{2-2\rho}{2-\rho} + \frac{|x|}{2-\rho}\right)\frac{x}{|x|} & \text{for } \rho < |x| \leq 2. \end{cases} \tag{2.3}$$

The diffeomorphism F has the following properties. First of all, F is a continuous and piecewise smooth map. Then F maps $B_\rho(0)$ in $B_1(0)$, while mapping $B_2(0)$ into itself. Finally $F(x) = x$ at the boundary $\partial B_2(0)$.

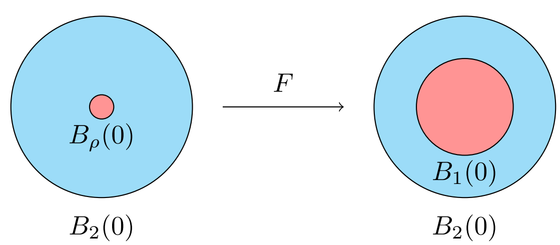

Figure 2.1: The map F expands $B_\rho(0)$ to $B_1(0)$.

F is an acceptable candidate for Proposition 2.1.2. Suppose now that the conductivity of the system is given by

$$\sigma_A(y) = \begin{cases} A(y) & \text{for } y \in B_1(0), \\ F_* 1 & \text{for } y \in B_2(0) \setminus B_1(0), \end{cases}$$

for some arbitrary conductivity $A(y)$. By Proposition 2.1.2, the measurements at the boundary of this system are the same as those of a system with conductivity

$$((F^{-1})_* \sigma_A)(x) = \begin{cases} ((F^{-1})_* A)(x) & \text{for } x \in B_\rho(0), \\ 1 & \text{for } x \in B_2(0) \setminus B_\rho(0). \end{cases}$$

In other words, the boundary data associated with the system with the inclusion $B_1(0)$, the interior of which can be filled with any material, are the same as for a ball perturbed by a small inclusion $B_\rho(0)$. The content of $B_1(0)$ is cloaked, but its presence is known. The system is *near-cloaked*.

2.2 Near-cloak

Our goal is now is to show that when ρ is small enough, the system described above is sufficient to cloak the unit disk. This result is contained in Theorem 2.2.3. To prove this result, we first construct the Dirichlet to Neumann map in Section 2.2.1. Then in Section 2.2.2 we study two systems with dielectric inclusions and show how these are close to a system with uniform conductivity when the radius of the inclusions is small (see Theorem 2.2.2).

2.2.1 The Dirichlet to Neumann map

We introduce a functional setting for the Dirichlet-to-Neumann map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with $\partial\Omega \in C^1$. Suppose that the conductivity is positive and uniformly bounded, in the sense that there exists two constants $M, m > 0$ such that

$$m|\xi|^2 \leq \langle \sigma(x)\xi, \xi \rangle \leq M|\xi|^2 \tag{2.4}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Consider again the PDE (2.1) with Dirichlet data f. In Section 1.1.2 we said that the solution must be found in the Sobolev space $H^1(\Omega)$. We noticed that in order for the condition $u = f$ on $\partial\Omega$ to be fulfilled, it is necessary to request $u - f \in H^1_0$. However, this second condition is not convenient for the definition of the Dirichlet to Neumann map, since it forces us to work with the entire Ω domain, while we want to use only the boundary $\partial\Omega$. Indeed, we can reformulate the condition $u - f \in H^1_0$ in equivalent terms using the trace operator, and get a more comfortable space on which to define Λ_σ. The trace operator tr allows us to assign values to functions in Sobolev spaces, as expressed by the following Theorem [10].

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be open with $\partial\Omega \in C^1$. Then there exists a linear and continuous operator $tr : H^1(\Omega) \to L^2(\partial\Omega)$ such that*

1. *$tr(u) = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C^0(\bar{\Omega})$;*
2. *there exists a constant $C > 0$ such that $\|tr(u)\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$.*

Through the trace operator we can define the fractional Sobolev space

$$H^{1/2}(\partial\Omega) = \{f \in L^2(\partial\Omega) \,:\, \exists u \in H^1(\Omega) : tr(u) = f\}.$$

An element $f \in H^{1/2}(\partial\Omega)$ is a function with *one-half derivative in* $L^2(\partial\Omega)$, see []. The space $H^{1/2}(\partial\Omega)$ is already a good space on which to define Λ_σ. Note that however, if f is constant, then the solution is also constant and Λ_σ is trivial, which would imply that the kernel of Λ_σ contains all the constants. It is therefore appropriate to define the Dirichlet to Neumann map on

$$H_*^{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega) \cap \left\{ \int_{\partial\Omega} f = 0 \right\}$$

On this new space we set the norm

$$\|f\|^2_{H_*^{1/2}(\partial\Omega)} = \min_{v=f \text{ on } \partial\Omega} \int_\Omega |\nabla v|^2 \, dx$$

The Dirichlet to Neumann map is then the operator

$$\begin{aligned} \Lambda_\sigma \;:\; H_*^{1/2}(\partial\Omega) &\longmapsto \quad H_*^{-1/2}(\partial\Omega) \\ u|_{\partial\Omega} &\longmapsto \quad (\sigma\nabla u)\cdot\bar{n}\big|_{\partial\Omega} \end{aligned}$$

where $H_*^{-1/2}(\partial\Omega)$ is the dual space of $H_*^{1/2}(\partial\Omega)$. We note that Λ_σ is a bounded and linear operator. Furthermore, since σ is symmetric and satisfies (2.4), Λ_σ turns out to be symmetric in the L^2-inner product, positive definite and invertible. Hence Λ_σ defines a positive definite quadratic form on $H_*^{1/2}(\partial\Omega)$, whose action can be written explicitly as

$$\langle \Lambda_\sigma f_1, f_2 \rangle = \int_{\partial\Omega} \Lambda_\sigma(f_1) f_2 \, d\sigma(x) = \int_\Omega \langle \sigma\nabla u_1, \nabla u_2 \rangle \, dx,$$

where u_1, u_2 solve the equation (2.1) with Dirichlet data f_1, f_2. The natural norm for measuring Λ_σ is therefore

$$\|\Lambda_\sigma\| = \sup\{|\langle \Lambda_\sigma f,\, f\rangle|,\ \|f\|_{H_*^{1/2}(\partial\Omega)} \le 1\}.$$

Proposition 2.2.1. *Suppose that σ and η are two ordered conductivities, i.e.*

$$\langle \sigma(x)\xi, \xi \rangle \le \langle \eta(x)\xi, \xi \rangle$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then $\Lambda_\sigma \le \Lambda_\eta$ in the sense that

$$\langle \Lambda_\sigma(f), f \rangle \le \langle \Lambda_\eta(f), f \rangle$$

for all $f \in H_*^{1/2}(\Omega)$.

Proof. We use the variational principle with energy (2.2). Indeed if u, v are such that $\nabla\cdot(\sigma\nabla u) = 0$ and $\nabla\cdot(\eta\nabla v) = 0$ in Ω with $u = v = f$ on $\partial\Omega$, then

$$\begin{aligned} \langle \Lambda_\sigma(f), f \rangle &= \int_\Omega \langle \sigma\nabla u, \nabla u \rangle \, dx \\ &\le \int_\Omega \langle \sigma\nabla v, \nabla v \rangle \, dx \\ &\le \int_\Omega \langle \eta\nabla v, \nabla v \rangle \, dx = \langle \Lambda_\eta(f), f \rangle. \end{aligned}$$

□

2.2.2 Dielectric inclusions

Let $\Omega = B_R(0)$ and $D = B_\rho(0)$ with $0 < \rho < R$. Let us consider the conductivity

$$\sigma_{\alpha,\rho}(x) = \begin{cases} \alpha & \text{for } x \in B_\rho(0), \\ 1 & \text{for } x \in B_R(0) \setminus B_\rho(0). \end{cases}$$

As it follows from Proposition 2.2.1, the effects of the inclusion $B_\rho(0)$ depend monotonically on the value of the conductivity α. It is therefore natural to study the limits $\alpha \to 0$ and $\alpha \to +\infty$, which respectively correspond to the case of perfect insulator and perfect conductor.

Let us therefore consider the differential problems

$$\begin{cases} \Delta u_0^\rho = 0 & \text{in } B_R(0) \setminus \overline{B_\rho(0)}, \\ u_0^\rho = f & \text{on } \partial B_R(0), \\ \frac{\partial u_0^\rho}{\partial \bar{n}} = 0 & \text{on } \partial B_\rho(0), \end{cases} \tag{2.5}$$

and

$$\begin{cases} \Delta u_\infty^\rho = 0 & \text{in } B_R(0) \setminus \overline{B_\rho(0)}, \\ u_\infty^\rho = f & \text{on } \partial B_R(0), \\ u_\infty^\rho = c_\infty & \text{on } \partial B_\rho(0), \end{cases} \tag{2.6}$$

where the constant $c_\infty \in \mathbb{R}$ is uniquely determined by the condition

$$\int_{\partial B_\rho(0)} \frac{\partial u_\infty^\rho}{\partial \bar{n}} \, d\sigma(x) = 0.$$

Physically, we expect the u_α^ρ solution to converge to u_0^ρ and u_∞^ρ in the limits $\alpha \to 0$ and $\alpha \to +\infty$. This is indeed true, as shown by the following result.

Proposition 2.2.2. *In the above setting, we have $u_\alpha^\rho \to u_0^\rho$ when $\alpha \to 0$ and $u_\alpha^\rho \to u_\infty^\rho$ when $\alpha \to +\infty$ weakly in $H^1(B_R(0) \setminus \overline{B_\rho(0)})$.*

Proof. For any $\alpha > 0$, let u_α^ρ be the only solution of

$$\begin{cases} \Delta u_\alpha^\rho = 0 & \text{in } B_R(0) \setminus \overline{B_\rho(0)}, \\ u_\alpha^\rho = f & \text{on } \partial B_R(0), \\ \Delta u_\alpha^\rho = 0 & \text{in } B_\rho(0), \\ \alpha \frac{\partial u_\alpha^\rho}{\partial \bar{n}}\big|_- = \frac{\partial u_\alpha^\rho}{\partial \bar{n}}\big|_+ & \text{on } \partial B_\rho(0). \end{cases} \tag{2.7}$$

Case $\alpha \to 0$. First, we notice that the energy of the solution u_α^ρ is uniformly bounded in α, i.e. there exists a constant $C > 0$ such that

$$\|\nabla u_\alpha^\rho\|_{L^2(B_R(0)\setminus\overline{B_\rho(0)})} \leq C \qquad \forall \alpha > 0.$$

Moreover, u_α^ρ is uniformly bound in α in the L^2-norm

$$\|u_\alpha^\rho\|_{L^2(B_R(0)\setminus\overline{B_\rho(0)})} \leq C \qquad \forall\alpha > 0.$$

This follows from the fact that u_α^ρ is bounded on a bounded set. The boundedness of u_α^ρ follows from the maximum principle. Let $M = \max_{\partial B_R(0)} f$ and suppose that the maximum of u_α^ρ is assumed in the interior, i.e. $\max_{B_R(0)} u_\alpha^\rho \geq M$. Let $\varphi = (u_\alpha^\rho - M)^+$. Clearly $\varphi \in H_0^1(B_R(0))$, so the weak formulation states that

$$0 = \int_{B_R(0)} \sigma\nabla u_\alpha^\rho \cdot \nabla\varphi\, dx = \int_{\{u\geq M\}} \sigma|\nabla u|^2\, dx$$

that is, the measure of the set $\{u_\alpha^\rho \geq M\}$ is zero, that is, $u_\alpha^\rho \leq M$ for all $\alpha > 0$. Similarly, it is proved that $u_\alpha^\rho \geq m$, where $m = \min_{\partial B_R(0)} f$.

We have then

$$\left\|u_\alpha^{\rho+}\right\|_{H^1(B_R(0)\setminus\overline{B_\rho(0)})} \leq C \qquad \forall\alpha > 0.$$

Then, by Banach-Alaoglu Theorem the solution u_α^ρ converges weakly to u_α^ρ in $H^1(B_R(0) \setminus \overline{B_\rho(0)})$ when $\alpha \to 0$ up to a subsequence. Also, from the third equation in (2.7) we have $\|\alpha\nabla u_\alpha^\rho\|_{L^2(B_\rho(0))} \leq C$ for any $\alpha > 0$, and analogously to above we have $\alpha u_\alpha^\rho \to 0$ weakly in $H^1(B_\rho(0))$ when $\alpha \to 0$.

Regarding the conditions at the boundary, we have clearly that $u_\alpha^\rho = f$ on $\partial B_R(0)$. About the condition on $\partial B_\rho(0)$, we can take the weak limit in

$$\left.\frac{\partial u_\alpha^\rho}{\partial\bar{n}}\right|_+ = \alpha\left.\frac{\partial u_\alpha^\rho}{\partial\bar{n}}\right|_-$$

and get that

$$\frac{\partial u_\alpha^\rho}{\partial\bar{n}} = 0 \qquad \text{for } x \in \partial B_\rho(0).$$

Case $\alpha \to +\infty$. Again $u_\alpha^\rho \to u_\infty^\rho$ weakly in $H^1(B_R(0) \setminus \overline{B_\rho(0)})$. From the second equation of (2.7) we have that $\|\nabla u_\alpha^\rho\|_{L^2(B_\rho(0))} \to 0$ when $\alpha \to +\infty$, since the energy of the solution must be bounded. The boundary condition on $\partial B_R(0)$ is clearly satisfied. Moreover, on $\partial B_\rho(0)$ we have

$$\int_{\partial B_\rho(0)} \left.\frac{\partial u_\alpha^\rho}{\partial\bar{n}}\right|_+ = \int_{\partial B_\rho(0)} \alpha\left.\frac{\partial u_\alpha^\rho}{\partial\bar{n}}\right|_-$$

so by taking the weak limit we have

$$\int_{\partial B_\rho(0)} \frac{\partial u_\infty^\rho}{\partial\bar{n}} = 0.$$

□

The presence of the inclusion $B_\rho(0)$ can be studied in the limit $\rho \to 0$.

Theorem 2.2.2. *In the above setting, let Λ_1 be the Dirichlet-to-Neumann map when $\sigma = 1$, and let $\Lambda_0^\rho, \Lambda_\infty^\rho$ be the Dirichlet-to-Neumann maps associated with the problems (2.5) and (2.6) respectively. Then there exists some constant $C > 0$ such that*

$$\|\Lambda_1 - \Lambda_0^\rho\| \leq C\rho^2 \qquad \textit{and} \qquad \|\Lambda_1 - \Lambda_\infty^\rho\| \leq C\rho^2.$$

Without going into too much details, the proof is based on the method of separation of variables. Indeed, once we find the solutions u_1, u_0^ρ and u_∞^ρ, the normal derivatives at boundary $\partial B_R(0)$ are computed and compared in the $L^2(\partial B_R(0))$-norm. Using Parseval's identity and a simple estimate we obtain the thesis.

2.2.3 The regular near-cloak is almost invisible

Let us once again consider the near-cloak of Section 2.1. Suppose that $\Omega = B_2(0)$ with conductivity

$$\sigma_A(y) = \begin{cases} A(y) & \text{for } y \in B_1(0), \\ F_*1(y) & \text{for } y \in B_2(0) \setminus B_1(0), \end{cases}$$

where F is given by (2.3). We assume that the conductivity $A(x)$ of the region being cloaked is positive definite and finite, that is

$$m|\xi|^2 \leq \langle A(y)\xi, \xi\rangle \leq M|\xi|^2 \qquad \text{for } y \in B_1(0),$$

so that the solution of the PDE is unique.

The Dirichlet to Neumann map of σ_A is the same as that of a system with conductivity

$$((F^{-1})_*\sigma_A)(x) = \begin{cases} ((F^{-1})_*A)(x) & \text{for } x \in B_\rho(0), \\ 1 & \text{for } x \in B_2(0) \setminus B_\rho(0). \end{cases}$$

The following Theorem holds.

Theorem 2.2.3. *Suppose that the shell $B_2(0) \setminus B_1(0)$ has conductivity F_*1. If ρ is small enough, then $B_1(0)$ is nearly cloaked, i.e. there exists some constant $C > 0$ such that*

$$\|\Lambda_{\sigma_A} - \Lambda_1\| \leq C\rho^2.$$

Proof. We use the monotonicity property given in Proposition 2.2.1 and the convergence relation of Proposition 2.2.2. We have

$$\lim_{\alpha\to 0} \Lambda_{\sigma_{\alpha,\rho}} = \Lambda_0^\rho \leq \Lambda_{\sigma_A} = \Lambda_{(F^{-1})_*\sigma_A} \leq \Lambda_\infty^\rho = \lim_{\alpha\to\infty} \Lambda_{\sigma_{\alpha,\rho}}.$$

Hence

$$\Lambda_0^\rho - \Lambda_1 \leq \Lambda_{\sigma_A} - \Lambda_1 \leq \Lambda_\infty^\rho - \Lambda_1.$$

The previous estimate, coupled with Theorem 2.2.2, gives the thesis. □

2.3 Results on transformation cloaking in elliptic geometry

We now present the results in elliptical geometry. We start by reformulating problems (2.5) and (2.6) in elliptic geometry. Let $\Omega = \mathcal{E}_R(0)$ and let $D = \mathcal{E}_\rho(0)$ with $\rho \in (0, R)$. We consider conductivity

$$\sigma_{\alpha,\rho}(x) = \begin{cases} \alpha & \text{for } x \in \mathcal{E}_\rho(0), \\ 1 & \text{for } x \in \mathcal{E}_R(0) \setminus \mathcal{E}_\rho(0). \end{cases}$$

Since Proposition 2.2.1 has been proved for general domains, we study equation (2.1) with $\sigma = \sigma_{\alpha,\rho}$ and Dirichlet data $u = f$ on $\partial\mathcal{E}_R(0)$ with $f \in H^1(\mathcal{E}_R(0))$ in the limits $\alpha \to 0$ and $\alpha \to +\infty$ in the elliptic case. Therefore we have the problems

$$\begin{cases} \Delta u_0^\rho = 0 & \text{in } \mathcal{E}_R(0) \setminus \overline{\mathcal{E}_\rho(0)}, \\ u_0^\rho = f & \text{on } \partial\mathcal{E}_R(0), \\ \frac{\partial u_0^\rho}{\partial \bar{n}} = 0 & \text{on } \partial\mathcal{E}_\rho(0), \end{cases} \tag{2.8}$$

and

$$\begin{cases} \Delta u_\infty^\rho = 0 & \text{in } \mathcal{E}_R(0) \setminus \overline{\mathcal{E}_\rho(0)}, \\ u_\infty^\rho = f & \text{on } \partial\mathcal{E}_R(0), \\ u_\infty^\rho = c_\infty & \text{on } \partial\mathcal{E}_\rho(0) \end{cases} \tag{2.9}$$

where the constant $c_\infty \in \mathbb{R}$ is uniquely determined from the condition

$$\int_{\partial\mathcal{E}_\rho(0)} \frac{\partial u_\infty^\rho}{\partial \bar{n}}\, d\sigma(x) = 0.$$

Since Proposition 2.2.2 is still valid, we only need to prove Theorem 2.2.2 in the elliptic case. Unfortunately there is no analogue of Theorem 2.2.2 in elliptic geometry, although elliptic and circular geometry are very similar. Indeed, the following Theorem holds.

Theorem 2.3.1. *In the above setting, let Λ_1 be the Dirichlet-to-Neumann map when $\sigma = 1$, and let $\Lambda_0^\rho, \Lambda_\infty^\rho$ be the Dirichlet-to-Neumann maps associated with the problems (2.8) and (2.9) respectively. Assume that $f = \sum_{k\in\mathbb{Z}} f_k e^{ik\nu}$ with $f_k = 0$ for all $|k| < k_0$, and $f_{k_0} \neq 0$. Then there exists $C > 0$ depending only on R such that*

$$\|\Lambda_1 - \Lambda_0^\rho\| \geq C k_0 |f_{k_0}| e^{2k_0\rho} \qquad \textit{and} \qquad \|\Lambda_1 - \Lambda_\infty^\rho\| \geq C k_0 |f_{k_0}| e^{2k_0\rho}. \tag{2.10}$$

Proof. We present our argument in three step. In the first one we construct the solution u_1 to the problem $\Delta u_1 = 0$ in $\mathcal{E}_R(0)$ with Dirichlet data $u = f$ on $\partial\mathcal{E}_R(0)$; and we find the Dirichlet-to-Neumann map. In the second one we construct the solution u_0^ρ of (2.8) and we prove the first estimate in (2.10). Finally, the third step is analogous to the second one: we construct the solution u_∞^ρ of (2.9) and we prove the second estimate in (2.10).

Step 1. Since we are looking for solution $u \in H^1(\mathcal{E}_R(0))$ we use the method of separation of variables we have seen in Section 1.1. The most general solution is

$$u_1(\mu,\nu) = \sum_{k\in\mathbb{Z}} (a_k e^{|k|\mu} + b_k e^{-|k|\mu}) e^{ik\nu}$$

with $\{a_k, b_k\}_{k\in\mathbb{Z}} \subset l^2(\mathbb{C})$ to be determined. First observe that $b_k = 0$ for all $k \in \mathbb{Z}$, because the solution u must be regular in $\mathcal{E}_R(0)$ and the change between elliptic coordinates to Cartesian coordinates does not produce a map that is regular in a neighborhood of the foci $(\pm a, 0)$. We now determine $\{a_k\}_{k\in\mathbb{Z}}$. Since $f \in H^1(\Omega)$, we can write for all $\nu \in [0, 2\pi)$

$$f(\nu) = \sum_{k\in\mathbb{Z}} f_k e^{ik\nu}$$

where

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\nu)e^{-ik\nu}\,d\nu.$$

is the k-th term of the Fourier series of the source f. The boundary condition $u_0(R, \nu) = f(\nu)$ for all $\nu \in [0, 2\pi)$ implies $a_k = f_k e^{|k|R}$ for all $k \in \mathbb{Z}$. The solution is then

$$u_0(\mu, \nu) = \sum_{k\in\mathbb{Z}} f_k e^{|k|(\mu-R)}e^{ik\nu}.$$

The Dirichlet-to-Neumann map is then

$$\Lambda_1(f) = \frac{\partial u_1}{\partial \bar{n}}(R, \nu) = \frac{\partial u_1}{\partial \mu}(R, \nu) = \sum_{k\in\mathbb{Z}} |k| f_k e^{ik\nu}.$$

Step 2. We now determine the unique solution u_0^ρ of (2.8). Similarly to the above, we write the problem in elliptical coordinates and determine the solution by separating the variables. A direct calculation shows that the solution is

$$u_0^\rho(\mu, \nu) = \sum_{k\in\mathbb{Z}} \frac{f_k}{e^{|k|R} + e^{2|k|\rho}e^{-|k|R}}(e^{|k|\mu} + e^{2|k|\rho}e^{-|k|\mu})e^{ik\nu}. \tag{2.11}$$

The current produced on the boundary $\partial\mathcal{E}_R(0)$ of the problem (2.8) is therefore

$$\Lambda_0^\rho(f) = \frac{\partial u_0^\rho}{\partial \bar{n}}(R, \nu) = \sum_{k\in\mathbb{Z}} |k| f_k \frac{1 - e^{2|k|(\rho-R)}}{1 + e^{2|k|(\rho-R)}} e^{ik\nu}.$$

We can now estimate the difference between the boundary Neumann data for the problem of an ellipse with uniform conductivity $\sigma = 1$ and problem (2.5). We have

$$\begin{aligned}\left\|\frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_0^\rho}{\partial \bar{n}}\right\|^2_{L^2(\partial\mathcal{E}_R(0))} &= \int_0^{2\pi}\left|\frac{\partial u_1}{\partial \bar{n}}(R,\nu) - \frac{\partial u_0^\rho}{\partial \bar{n}}(R,\nu)\right|^2 d\nu \\ &= 2\pi\sum_{k\in\mathbb{Z}} 4k^2|f_k|^2\frac{e^{4|k|(\rho-R)}}{(1+e^{2|k|(\rho-R)})^2}\end{aligned} \tag{2.12}$$

where the last line follows from Parseval's identity. Recall now that $f_{k_0} \neq 0$ with $k_0 > 0$. Since f is real we have $f_{-k_0} = \overline{f_{k_0}}$, hence

$$\left\|\frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_0^\rho}{\partial \bar{n}}\right\|^2_{L^2(\partial\mathcal{E}_R(0))} \geq 2\pi\sum_{k=\pm k_0} 4k^2|f_k|^2\frac{e^{4|k|(\rho-R)}}{(1+e^{2|k|(\rho-R)})^2} = 16\pi k_0^2|f_{k_0}|^2\frac{e^{4k_0(\rho-R)}}{(1+e^{2k_0(\rho-R)})^2}.$$

Taking the square root of each expression above and using the fact that $\rho < R$ lead to the bound

$$\left\| \frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_0^\rho}{\partial \bar{n}} \right\|_{L^2(\partial \mathcal{E}_R(0))} \geq 4\sqrt{\pi} k_0 |f_{k_0}| \frac{e^{2k_0(\rho - R)}}{1 + e^{2k_0(\rho - R)}} \geq 4\sqrt{\pi} k_0 |f_{k_0}| \frac{e^{2k_0\rho}}{2e^{k_0 2R}} \geq C k_0 |f_{k_0}| e^{2\rho}.$$

Step 3. A direct calculation shows that the solution to problem (2.9) is exactly (2.11). Therefore the proof is analogous as above. □

The previous Theorem states that cloaking by transformation optic does not happen in systems with elliptical geometry. Indeed if $\rho \to 0$, the difference between Dirichlet-to-Neumann maps is non-zero, and therefore an external observer is able to distinguish the two systems. Considering the topic from a qualitative stand point, we can understand the reason for this conclusion. Indeed, in the limit $\rho \to 0$, the ellipse $\mathcal{E}_\rho$ does not collapse to a point, i.e. the origin, but in the segment $[-a, a]$, where $a > 0$ is the focus of the ellipses, and as a matter of fact, we are removing a much heavier set in terms of the problems (2.8) and (2.9) than in the circular case in which we are removing a point.

This fact can be formulated in mathematical terms using the 1-dimensional Hausdorff measure. Indeed, in the limit $\rho \to 0$ we have that $\mathcal{H}^1(\mathcal{E}_\rho) \neq 0$, although $\mathcal{H}^1(B_\rho) = 0$. Therefore, in the study of problems (2.8)-(2.9) and (2.5)-(2.6) we are significantly changing the domain, to such an extent that cloaking for transformation optics does not happen.

On the other hand, the proof of the previous Theorem can be adapted to prove the following result.

Theorem 2.3.2. *In the above setting, let u_0^ρ, u_∞^ρ be the solution to problems (2.8) and (2.9) respectively. Suppose the source $f : \mathcal{E}_R(0) \to \mathbb{R}$, $f \in C^l(\Omega)$, is high-frequency monochromatic, that is, there is a large $k_0 \in \mathbb{N}$ such that $f = f_{k_0} e^{ik_0\nu}$. Then there exists some constant $C > 0$ such that*

$$\left\| \frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_0^\rho}{\partial \bar{n}} \right\|_{L^2(\partial \mathcal{E}_R(0))} \leq \frac{C}{k_0} \qquad \textit{and} \qquad \left\| \frac{\partial u_1}{\partial \bar{n}} - \frac{\partial u_\infty^\rho}{\partial \bar{n}} \right\|_{L^2(\partial \mathcal{E}_R(0))} \leq \frac{C}{k_0}.$$

Proof. Starting from equation (2.12), we estimate

$$\frac{e^{4|k|(\rho - R)}}{(1 + e^{2|k|(\rho - R)})^2} \leq 1 \qquad \text{and} \qquad |f_{k_0}| \leq \frac{C}{k_0^l} \leq \frac{C}{k_0}.$$

□

Theorem 2.3.2 states that cloaking for transformation optics is still possible in elliptic geometries. Indeed, if we study a domain $\Omega = \mathcal{E}_R(0)$ through measurements on the boundary $\partial \mathcal{E}_R(0)$ using only high frequency monochromatic electrodes, with frequency k_0, then we are not able to distinguish the inside of Ω from an system Ω with an inclusion $D = \mathcal{E}_\rho(0)$, with $\rho < R$. Then, using a blow up map F as in (2.3), we can expand the ellipse $\mathcal{E}_\rho(0)$ into an ellipse $\mathcal{E}_r(0)$, with $r > \rho$, and hide objects inside it.

Chapter 3

Cloaking by anomalous localized resonance

In this Chapter we analyze the cloaking due to anomalous localized resonance (ALR). Many aspects of cloaking by ALR are known in circular geometry: in Section 3.1 we formulate the problem in this geometry. Then in Section 3.2 we present the ALR phenomenon in detail and discuss how it is related to cloaking. Next we make an overview of two different techniques used to prove that ALR occurs: spectral theory and variational principles. Focusing on the second, we show how the problem can be reformulated in terms of two dual variational principles. This is the content of Section 3.3. Finally, in Section 3.4 we build some suitable test functions to be used in these two principles and we present our main results in elliptical geometry.

3.1 The problem

Let us start with the formulation of the problem. We consider an infinite cylindrical structure with axis directed along the z-axis. Suppose that the section of this cylinder is given by the disk $B_R(0) \subset \mathbb{R}^2$ for some radius $R > 1$. Inside this cylinder we consider a cylinder with the same axis and section $\Sigma \subset B_1(0)$. We then assume the presence of an electric source that surrounds the structure.

Suppose the electrical permittivity in the *core* Σ and in the *shell* $B_R(0) \setminus \Sigma$ are $+1$ and -1, respectively. Since the region outside of the cylinder, called the *matrix* $\mathbb{R}^2 \setminus B_R(0)$, is supposed to be homogeneous, we assume that the permeability is $+1$. We assume that the cylinder and the environment are lossy, with losses controlled by the loss parameter $\eta > 0$.

Our goal is to determine the electrostatic potential for this system. Since the system has translational symmetry along the z-axis, we restrict the problem to the plane $\mathbb{R}^2$. The problem that determines the electrostatic potential $u_\eta : \mathbb{R}^2 \to \mathbb{C}$ is the elliptic problem

$$\begin{cases} \nabla \cdot (a_\eta \nabla u_\eta) = f & \text{in } \mathbb{R}^2 \\ u_\eta \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{3.1}$$

We notice that, by writing $u_\eta = v_\eta + i/\eta w_\eta$, the above equation can be seen as a system of two elliptic PDEs

$$\begin{cases} \nabla \cdot (A\nabla v_\eta) - \Delta w_\eta = f & \text{in } \mathbb{R}^2, \\ \nabla \cdot (A\nabla w_\eta) + \eta^2 \Delta v_\eta = 0 & \text{in } \mathbb{R}^2, \\ v_\eta \to 0 & \text{as } |x| \to +\infty, \\ w_\eta \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

The boundary condition in (3.1) is necessary: every real electric field must vanish at distances far from the structure. In accordance with our assumptions, the electric permittivity a_η is a piecewise constant and complex-valued function

$$a_\eta(x) = A(x) + i\eta, \tag{3.2}$$

where the real part $A(x)$ has a core-shell-matrix character

$$A(x) = \begin{cases} +1 & \text{in the core } \Sigma \\ -1 & \text{in the shell } B_R(0) \setminus \Sigma \\ +1 & \text{outside } B_R(0). \end{cases} \tag{3.3}$$

and $\eta > 0$ is the loss parameter.

The negative sign of the dielectric permeability in the shell is typical of *plasmonic* materials. Plasmonic metamaterials were first theorized by Veselago [26]. Without going in to details, they are characterized by a surprising property: energy is transported in a direction opposite to that of propagating wavefronts, rather than paralleling them, as is the case in positive index materials [25, 4] .

The source f represent the source of the electric field. To give physical and mathematical meaning to the equation in problem (3.1) we have to make assumptions on the nature of f. Physically, we assume that the source represents a charged wire with non-uniform charge density placed outside the plasmon structure, see Figure 3.1.

Mathematically, these physical demands lead to a class of sources f with precise assumptions. First of all, we assume that f is real-valued, since all charge densities are real-valued. Secondly, we assume that it is supported at a distance $q > R$ from the origin. Since f represent a one dimensional source and the equation of problem (3.1) must be understood in an appropriate Sobolev space, it is necessary to interpret f not as a function but as a distribution. Moreover, since f is supported on $\partial B_q(0)$, that is, on a set which is of Lebesgue measure zero, we cannot define f by integration with respect to the 2-dimensional Lebesgue measure dx. To solve this issue we interpret f as a distribution defined by integration with respect to the Hausdorff measure $\mathcal{H}^1$ supported at $\partial B_q(0)$. In this way we can write the source as $f = F\mathcal{H}^1 \lfloor \partial B_q(0)$, where $F : \partial B_q(0) \to \mathbb{R}$ is a $L^2(\partial B_q(0))$ function.

We require also that the source f has zero mean

$$\int_{\partial B_q(0)} F \, d\mathcal{H}^1 = 0. \tag{3.4}$$

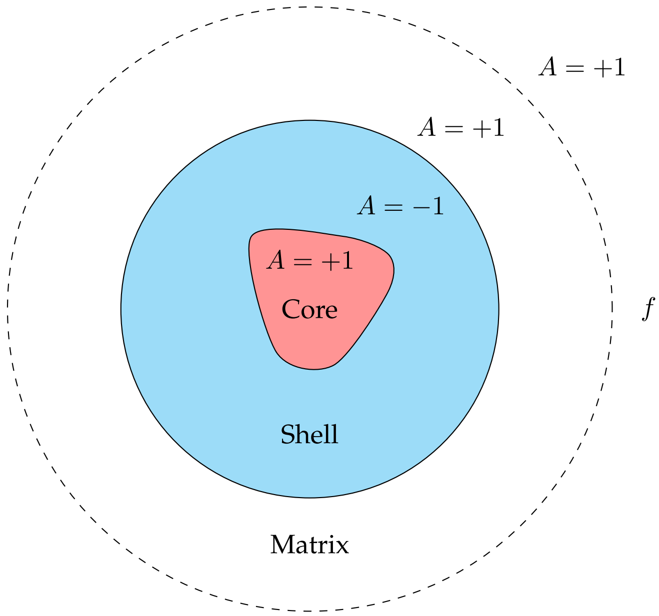

Figure 3.1: A section of the 2d coaxial cylinder structure. The source f is supported in some circumference $\partial B_q(0)$.

This assumption is a compatibility condition and is due to the structure of the differential equation of problem (3.1). Indeed, by integrating the equation on $\mathbb{R}^2$ and thanks to the behavior of the solution at infinity, from the divergence theorem we find the equation (3.4).

To sum up, we assume that

$$f = F\mathcal{H}^1 \lfloor \partial B_q(0), \qquad F : \partial B_q(0) \to \mathbb{R}, \qquad F \in L^2(\partial B_q(0)), \qquad \text{and} \int_{\partial B_q(0)} F \, d\mathcal{H}^1 = 0. \tag{3.5}$$

Definition 3.1.1. The plasmonic structure is defined by the coefficient a_η and (3.3). The configuration of the system is the set of hypotheses on the plasmonic structure (3.3) and the source (3.5).

Considering that the equation of problem (3.1) is in divergence form, it is natural to investigate the properties of the configuration by using energy methods. The energy associated with the problem (3.1) is the functional $\mathcal{E}_\eta : X \to \mathbb{R}$ given by

$$\mathcal{E}_\eta = \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla u_\eta|^2 \, dx, \tag{3.6}$$

where X is an appropriate Sobolev space. Physically, the energy functional is proportional to the energy produced by the source f and dissipated into heat by Joule's effect. Indeed, if we call E_η the electric field generated by the source f and $D_\eta = a_\eta \vec{D}_\eta$ the electric displacement field, we have

$$\mathcal{E}_\eta \sim \operatorname{Im} \int_{\mathbb{R}^2} D_\eta \cdot E_\eta \, dx = \operatorname{Im} \int_{\mathbb{R}^2} (A(x) + i\eta) \nabla u_\eta \cdot \nabla u_\eta \, dx = \eta \int_{\mathbb{R}^2} |\nabla u_\eta|^2 \, dx.$$

We will use the energy functional $\mathcal{E}_\eta$ to establish the connection to cloaking.

Definition 3.1.2. Let a configuration be given by coefficients A and source f as in (3.3)–(3.5). We call the configuration resonant if

$$\limsup_{\eta\to 0} \mathcal{E}_\eta = \infty.$$

Otherwise we call the configuration non-resonant.

3.2 Cloaking by anomalous localized resonance

In this section we establish the connection between cloaking and the behavior of the energy functional $\mathcal{E}_\eta$ as $\eta \to 0$.

When the loss parameter η goes to zero, the system of PDE (3.1) loses its ellipticity. In this limit, the plasmonic structure exhibits a striking feature: the magnitude of the electric field diverges in a specific area, called the anomalous resonance region, but converges to a smooth field outside that region. Surprisingly, the anomalous resonance region has sharp boundary which is not defined by any discontinuity of the electrical conductivity a_η. This behavior is called anomalous localized resonance (ALR). The ALR was first presented numerically in [6].

The connection with cloaking is then clear. Suppose that the source f, which by the hypotheses (3.5) can be assumed to be a dipole, is placed within a critical distance of the plasmonic structure such that ALR occurs. Since $\mathcal{E}_\eta$ is the rate at which the energy produced by the source f is dissipated into heat, the total power absorbed by the plasmonic structure becomes infinite when the loss parameter goes to zero.

Physically, the source interacts with the resonant field that is generated by the source itself. The resonant field creates a sort of electromagnetic molasses against which the source has to do a huge amount of work to maintain its amplitude; which in the limit $\eta \to 0$ becomes infinite. This situation in unphysical.

The energy (3.6) is related with the electric field generated by the dipole source, which is proportional to the dipole moment of f. Therefore, any realistic dipole source f within the anomalous resonance region must have dipole moment which vanishes as $\eta \to 0$. Accordingly, it makes sense to consider a normalized dipole source $\alpha_\eta f$, where $\alpha_\eta \to 0$ as $\eta \to 0$. Then, the physical electric field $\alpha_\eta u_\eta$ tends to zero outside the region where ALR occurs: the source f becomes cloaked. In this case we say that cloaking by anomalous localized resonance (CALR) occurs.

Definition 3.2.1. We say that anomalous localized resonance (ALR) occurs if

1. the energy diverges

$$\lim_{\eta\to 0} \mathcal{E}_\eta = +\infty. \tag{3.7}$$

2. the solution u_η remains bounded outside some ball with radius a

$$|u_\eta(x)| < C, \qquad \text{when } |x| > a. \tag{3.8}$$

for some constant $C > 0$.

We say that weak ALR takes place if

$$\limsup_{\eta\to 0} \mathcal{E}_\eta = +\infty. \tag{3.9}$$

in addition to (3.8).

Conditions (3.7) and (3.8) are sufficient to guarantee CALR. If instead only weak ALR occurs, then the source f will be invisible for an infinite sequence of parameters η which tend to zero, but it will be clearly visible for all the other parameters for which weak ALR does not occur.

3.2.1 CALR by spectral theory

There are different techniques to provide a necessary and sufficient condition on the source term under which CALR and weak CALR takes place.

In [2] Ammari et al. studied a problem similar to (3.1)-(3.2)-(3.3)-(3.5). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let Σ be a domain whose closure is contained in Ω. Suppose also that $\partial\Omega, \partial\Sigma$ are smooth. Consider the electric conductivity a_η given by

$$a_\eta = \begin{cases} +1 & \text{in } \Sigma \\ -1 + i\eta & \text{in } \Omega \setminus \overline{\Sigma} \\ +1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases}$$

Let f be a complactly supported source with zero average

$$\int_{\mathbb{R}^2} f\, dx = 0$$

and consider the PDE problem

$$\begin{cases} \nabla \cdot (a_\eta \nabla u_\eta) = f & \text{in } \mathbb{R}^2 \\ |\nabla u_\eta| \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

with energy

$$\tilde{\mathcal{E}}_\eta = \int_{\Omega\setminus\overline{\Sigma}} |\nabla u_\eta|^2\, dx.$$

By using layer potentials and symmetrization techniques, Ammari et al. were able to give a necessary and sufficient condition on the fixed source term f for electromagnetic power dissipation $\tilde{\mathcal{E}}_\eta$ to blow up as the loss parameter of the plasmonic material goes to zero. Indeed, they gave this condition in terms of the Newtonian potential of the source term, i.e. the convolution between f and the fundamental solution Γ to Laplace's equation. See [23] for references on potential theory.

Studying the case $\Sigma = B_r(0)$, $\Omega = B_R(0)$, they were also able to provide an explicit condition to the location of the source for cloaking to occur. In particular, for any source supported outside the ball with critical radius

$$R^* = r\left(\frac{R}{r}\right)^{3/2} \tag{3.10}$$

CALR does not take place. Conversely, for sources located inside the critical radius which satisfy certain conditions, CALR does take place as the loss parameter goes to zero.

The main problem of this the approach is that one needs detailed information on the spectrum of certain boundary integral operators, which depend on the geometry of the problem. It is not easy in general to determine this information, and consequently their approach is not easily usable to obtain explicit results in geometries different from the circular one.

In the case of elliptic geometries, some results can be found in [9]. In particular, Chung et al. have shown that the configuration is resonant only if the source is supported inside the ellipse with critical radius

$$R^* = \begin{cases} (3R-r)/2 & \text{for } R \leq 3r, \\ 2(R-r) & \text{for } R > 3r. \end{cases}$$

where r and R are the elliptic radii of the core and the shell respectively. To obtain quantitative results in elliptic geometry we will use a different approach based on variational principles. This approach was first introduced in [20] by Khon et al and it is explained in the next section.

3.3 Cloaking via variational methods

The variational approach introduced by Kohn et al. is based on some ideas presented in [8] for a complex conductivity problem. In [8] Cherkaev et al. wrote a system of two complex PDEs of the first order in terms of four potentials, deriving some variational principles of minimax and minima for the latter. Minimax problems are written in terms of the potentials that are most useful for controlling the energy of the system. Unfortunately, however, the minimax nature of these problems does not allow to control the energies involved, since they do not provide any upper or lower bounds.

However, the authors noted that, through the Legendre transform, it is possible to pass from the minimax problems to the principles of minima. This is a great advantage: we can control the bounds of the energies. However, the principles of minimum are written in terms of potentials that are not easily controlled.

In any case, following the ideas of Khon et al. we rewrite the original problem (3.1) to obtain a pair of dual variational principles.

3.3.1 The primal variational principle

First of all we rewrite the problem (3.1) in terms of two real potentials. Hence let $v_\eta, w_\eta : \mathbb{R}^2 \to \mathbb{R}$ and set

$$u_\eta = v_\eta + \frac{i}{\eta} w_\eta.$$

A direct calculation shows that the complex equation (3.1) is equivalent to a system of two coupled real PDEs. Indeed we have

$$\begin{aligned} \nabla \cdot (A \nabla v_\eta) - \Delta w_\eta &= f, \\ \nabla \cdot (A \nabla w_\eta) + \eta^2 \Delta v_\eta &= 0, \end{aligned} \tag{3.11}$$

on $\mathbb{R}^2$. Moreover, the energy of the system (3.6) is written as

$$\mathcal{E}_\eta(u_\eta) = \frac{\eta}{2}\int_{\mathbb{R}^2} |\nabla v_\eta|^2\,dx + \frac{1}{2\eta}\int_{\mathbb{R}^2} |\nabla w_\eta|^2\,dx. \tag{3.12}$$

Using the system (3.11) and the energy expression (3.12) we can state two dual variational principles. First of all we build an appropriate functional analytic framework. Let

$$\dot{H}^1(\mathbb{R}^2) = \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : \nabla u \in L^2(\mathbb{R}^2;\mathbb{R}^2) \right\}$$

equipped with the norm

$$\begin{aligned}\|u\|^2_{\dot{H}^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{B_1(0)} |u|^2\,dx \\ &= \|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|u\|^2_{L^2(B_1(0))}.\end{aligned}$$

Fix a source $f \in H^{-1}(\mathbb{R}^2)$ and consider the energy functional $\mathcal{I}_\eta : \dot{H}^1(\mathbb{R}^2) \times \dot{H}^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\mathcal{I}_\eta(v,w) = \frac{\eta}{2}\int_{\mathbb{R}^2} |\nabla v|^2\,dx + \frac{1}{2\eta}\int_{\mathbb{R}^2} |\nabla w|^2\,dx.$$

The primal variational problem is the following.

Definition 3.3.1 (Primal variational problem)**.** Minimize the energy $\mathcal{I}_\eta(v,w)$ over all pairs (v,w) which satisfy the PDE constraint $\nabla\cdot(A\nabla v) - \Delta w = f$ in $\mathbb{R}^2$.

The primal variational principle (3.3.1) is well posed in the sense that it is assumed at a pair (v_η, w_η), and this pair is unique. Moreover, the function $u_\eta = v_\eta + i\eta^{-1}w_\eta$ is the unique solution of the original problem (3.1) and energies coincide. These results are stated in the following Lemma (see [20]).

Lemma 3.3.1. *Fix a source $f \in H^{-1}(\mathbb{R}^2)$ with compact support and vanishing average. Then*

1. *the infimum*

$$\inf\left\{ \mathcal{I}_\eta(v,w) : (v,w) \in \dot{H}^1(\mathbb{R}^2)\times\dot{H}^1(\mathbb{R}^2), \nabla\cdot(A\nabla v) - \Delta w = f \right\} \tag{3.13}$$

   is attained at a pair $(v_\eta, w_\eta) \in \dot{H}^1(\mathbb{R}^2)\times\dot{H}^1(\mathbb{R}^2)$.

2. *The minimizing pair (v_η, w_η) is unique (up to an additive constant).*
3. *The function $u_\eta = v_\eta + i\eta^{-1}w_\eta$ is the unique solution of the original problem (3.1).*
4. *$\mathcal{E}_\eta(u_\eta) = \mathcal{I}_\eta(v_\eta, w_\eta)$.*

Proof. Let $s > 0$ be such that the source f is contained in the open ball $B_s(0)$, i.e. $\text{supp}(f) \subset B_s(0)$.

1. If we are able to define the functional $\mathcal{I}_\eta$ on a non-empty subspace $X \subset \dot{H}^1(\mathbb{R}^2)$ in such a way that it is convex, then a known result of the calculus of variation guarantees the existence of a pair $(v_\eta, w_\eta) \in X \times X$ such that

$$\mathcal{I}_\eta(v_\eta, w_\eta) \leq \mathcal{I}_\eta(\tilde{v}, \tilde{w})$$

for all $(\tilde{v}, \tilde{w}) \in X \times X$ with $\nabla \cdot (A \nabla \tilde{v}) - \Delta \tilde{w} = f$. Therefore the infimum (3.13) is attained on $X \times X$. It is easy to verify that the space

$$X = \left\{ u \in \dot{H}^1(\mathbb{R}^2) \; : \; \int_{B_s(0)} u \, dx = 0 \right\}$$

satisfies the previous requirements.

2. If we show that the original problem (3.1) is equivalent to the primal variational problem and that the original problem has a unique solution, then the minimizing pair is unique. We prove the latter in 3); we now prove that the two problems are equivalent.

Since the pair (v_η, w_η) is a minimizer of the functional $\mathcal{I}_\eta$, then for any $(\tilde{v}, \tilde{w}) \in X \times X$ the real function $j : \mathbb{R} \to \mathbb{R}$, defined by

$$j(t) = \mathcal{I}_\eta(v_\eta + t\tilde{v}, w_\eta + t\tilde{w}),$$

is $C^1(\mathbb{R})$ and has a stationary point at $t = 0$. Therefore

$$\frac{d}{dt} j(t) \bigg|_{t=0} = 0$$

for every $(\tilde{v}, \tilde{w}) \in X \times X$ that satisfies $\nabla \cdot (A \nabla \tilde{v}) - \Delta \tilde{w} = 0$. From the previous equation we obtain

$$0 = \eta \int_{\mathbb{R}^2} \nabla v_\eta \cdot \nabla \tilde{v} \, dx + \frac{1}{\eta} \int_{\mathbb{R}^2} \nabla w_\eta \cdot \nabla \tilde{w} \, dx = \eta \int_{\mathbb{R}^2} \nabla v_\eta \cdot \nabla \tilde{v} \, dx + \frac{1}{\eta} \int_{\mathbb{R}^2} \nabla w_\eta \cdot A \nabla \tilde{v} \, dx$$

where we used the constraint for the second equality. The preceding equation is the weak formulation of the second equation in (3.11). Since (v_η, w_η) is a solution of (3.11), then

$$u_\eta = v_\eta + \frac{i}{\eta} w_\eta$$

is a solution of the original problem.

3. We now prove that the original problem (3.1) as a unique solution. To see that, define the sesquilinear form

$$\begin{aligned} b : X \times X &\to \mathbb{C} \\ (u_1, u_2) &\mapsto b(u_1, u_2) = -i \int_{\mathbb{R}^2} a_\eta \nabla u_1 \nabla \overline{u}_2 \, dx. \end{aligned}$$

The form b is bounded on X: for all $u_1, u_2 \in X$ we have

$$\begin{aligned} |b(u_1, u_2)| &\leq \int_{\mathbb{R}^2} |a_\eta \nabla u_1 \nabla \overline{u}_2| \, dx \\ &\leq (1 + \eta^2)^{1/2} \|\nabla u_1\|_{L^2(\mathbb{R}^2)} \|\nabla u_2\|_{L^2(\mathbb{R}^2)} \end{aligned}$$

by Cauchy-Schwartz inequality. Now, $\|\nabla u\|^2_{L^2(\mathbb{R}^2)} = \|u\|^2_X - \|u\|^2_{L^2(B_1(0))} \le \|u\|^2_X$, so

$$|b(u_1, u_2)| \le \|u_1\|_X \|u_2\|_X.$$

The form b is also coercive on X. Indeed for all $u \in X$ we have

$$|b(u,u)| \ge \operatorname{Re} b(u,u) = \eta \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = \eta \|\nabla u\|^2_{L^2(\mathbb{R}^2)}$$

but

$$\begin{aligned} \|\nabla u\|^2_{L^2(\mathbb{R}^2)} &= \|u\|_X - \|u\|_{L^2(B_1(0))} \\ &\ge \|u\|_X - c\|\nabla u\|^2_{L^2(\mathbb{R}^2)}, \end{aligned}$$

where the last estimate follows from the Poincaré inequality. Hence $\|\nabla u\|^2_{L^2(\mathbb{R}^2)} \ge c\|u\|^2_X$ and therefore $|b(u,u)| \ge c\|u\|_X$ for any $u \in X$.

Lax-Milgram Lemma yields that there exists a unique weak solution $u_\eta \in \dot{H}^1(\mathbb{R}^2)$ of the original problem (3.1).

4. A direct calculation verifies 4).

□

The connection between the primal variational principle and the configuration of the system is as follows. Let u_η be the unique solution of the original problem (3.1). Then, since the Lemma 3.3.1 implies there exists a unique pair (v_η, w_η) such that $\mathcal{I}_\eta(v_\eta, w_\eta)$ as the smallest value, and such that

$$\mathcal{E}_\eta(u_\eta) = \mathcal{I}_\eta(v_\eta, w_\eta). \tag{3.14}$$

Then it follows that

$$\mathcal{E}_\eta(u_\eta) \le \mathcal{I}_\eta(v, w) \tag{3.15}$$

for all $(v, w) \in \dot{H}^1(\mathbb{R}^2) \times \dot{H}^1(\mathbb{R}^2)$ satisfying $\nabla \cdot (A\nabla v) - \Delta w = f$. The inequality (3.15) can be used to prove non-resonance results as $\eta = \eta_i \to 0$. Indeed, in order to show that a given configuration is non-resonant, it suffices to construct a sequence of trial functions (ϕ_η, ψ_η) that satisfy the constraint of the primal variational problem

$$\nabla \cdot (A\nabla \phi_\eta) - \Delta \psi_\eta = f \qquad \text{in } \mathbb{R}^2 \tag{3.16}$$

and have bounded energies $\mathcal{I}_\eta(\phi_\eta, \psi_\eta)$.

3.3.2 The dual variational principle

We now introduce the dual variational principle, which is a maximum principle and characterizes the energy $\mathcal{E}_\eta$ as a constrained maximum. For a fixed source $f \in H^{-1}(\mathbb{R}^2)$ we introduce the dual energy $\mathcal{J}_\eta : \dot{H}^1(\mathbb{R}^2) \times \dot{H}^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\mathcal{J}_\eta(v, \psi) = \int_{\mathbb{R}^2} f\psi \, dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx$$

Definition 3.3.2 (The dual variational principle)**.** Maximize the energy $\mathcal{J}_\eta(v,w)$ over all pairs (v,ψ) which satisfy the PDE constraint $\nabla\cdot(A\nabla\psi)+\eta\Delta v=0$ in $\mathbb{R}^2$.

As the primal variational principle, problem (3.3.2) is well posed: the following Lemma holds.

Lemma 3.3.2. *Fix a source $f\in H^{-1}(\mathbb{R}^2)$ with compact support and vanishing average. Then*

1. *the supremum*

$$\sup\left\{\mathcal{J}_\eta(v,\psi):(v,\psi)\in\dot{H}^1(\mathbb{R}^2)\times\dot{H}^1(\mathbb{R}^2),\nabla\cdot(A\nabla\psi)+\eta\Delta v=0\right\}$$

   is attained at a pair $(v_\eta,\psi_\eta)\in\dot{H}^1(\mathbb{R}^2)\times\dot{H}^1(\mathbb{R}^2)$.

2. *The maximizing pair (v_η,ψ_η) is unique (up to an additive constant).*
3. *The function $u_\eta=v_\eta+i\psi_\eta$ is the unique solution of the original problem (3.1).*
4. *$\mathcal{E}_\eta(u_\eta)=\mathcal{J}_\eta(v_\eta,\psi_\eta)$.*

Proof. Let us prove one point at a time. Since the proof of this Lemma is similar to the previous one, we only discuss the different points.

1. Using arguments analogous to those of the previous Lemma, we find the existence of a maximizing pair (v_η,ψ_η).
2. Since the pair (v_η,ψ_η) is a maximizer of the functional $\mathcal{J}_\eta$, then for any $\tilde{v},\tilde{\psi}\in\dot{H}^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ the real function $i:\mathbb{R}\to\mathbb{R}$, defined by $i(t)=\mathcal{J}(v_\eta+t\tilde{v},\psi_\eta+t\tilde{\psi})$, has a minimum in $t=0$. Hence, by imposing that the first derivative vanishes in $t=0$, we have

$$0=\int_{\mathbb{R}^2}f\tilde{\psi}\,dx-\eta\int_{\mathbb{R}^2}\nabla v_\eta\cdot\nabla\tilde{v}\,dx-\eta\int_{\mathbb{R}^2}\nabla\psi_\eta\nabla\tilde{\psi}\,dx.$$

   The constraint $\nabla\cdot(A\nabla\tilde{\psi})+\eta\Delta\tilde{v}$ allows you to replace $\tilde{v}$ in the previous equation, hence we have

$$0=\int_{\mathbb{R}^2}f\tilde{\psi}\,dx+\int_{\mathbb{R}^2}\nabla v_\eta\cdot A\nabla\tilde{\psi}\,dx-\int_{\mathbb{R}^2}\nabla\psi_\eta\cdot\nabla\tilde{\psi}\,dx.$$

   The previous equation is the weak formulation of the first equation of (3.11), for which we conclude that the function $u_\eta=v_\eta+i\psi_\eta$ is a solution of $\nabla\cdot(A\nabla u_\eta)=f$ on $\mathbb{R}^2$.

3. Using techniques similar to the previous Lemma we have the uniqueness of (3.1).
4. For the energy equality, we have

$$\begin{aligned}\mathcal{E}_\eta(u_\eta)-\mathcal{J}_\eta(v_\eta,\psi_\eta)&=\frac{\eta}{2}\int_{\mathbb{R}^2}|\nabla u_\eta|^2\,dx-\mathcal{J}_\eta(v_\eta,\psi_\eta)\\&=\eta\int_{\mathbb{R}^2}|\nabla v_\eta|^2\,dx+\eta\int_{\mathbb{R}^2}|\nabla\psi_\eta|^2\,dx-\int_{\mathbb{R}^2}f\psi\,dx\\&=-\int_{\mathbb{R}^2}\nabla v_\eta\cdot A\nabla\psi_\eta-\int_{\mathbb{R}^2}(-\psi_\eta f+\psi_\eta\nabla\cdot(A\nabla v_\eta))\,dx-\int_{\mathbb{R}^2}f\psi\,dx=0.\end{aligned}$$

□

The connection between the dual variational principle and the configuration of the system is as follows. Let u_η be the unique solution of the original problem (3.1). Then, since Lemma 3.3.2 implies that there exists a unique pair (v_η, ψ_η) such that $\mathcal{J}_\eta(v_\eta, \psi_\eta)$ has the maximum value, and such that

$$\mathcal{E}_\eta(u_\eta) = \mathcal{I}_\eta(v_\eta, \psi_\eta).$$

Hence, for any $(v, \psi) \in \dot{H}^1(\mathbb{R}^2) \times \dot{H}^1(\mathbb{R}^2)$ satisfying $\nabla \cdot (A\nabla\psi) + \eta\Delta v = 0$ in $\mathbb{R}^2$ we have

$$\mathcal{E}_\eta(u_\eta) \geq \mathcal{I}_\eta(v, \psi). \tag{3.17}$$

The inequality (3.17) can be used to prove resonance results as $\eta = \eta_i \to 0$. Indeed, in order to show that a given configuration is resonant, it suffices to construct a sequence of trial functions (ϕ_η, ψ_η) that satisfy the constraint of the dual variational problem

$$\nabla \cdot (A\nabla\psi) + \eta\Delta\phi_\eta = 0 \tag{3.18}$$

and have unbounded energies $\mathcal{J}_\eta(\phi_\eta, \psi_\eta)$ at the limit $\eta \to 0$.

3.4 Results in elliptic geometry

In this section we discuss the results obtained in elliptical geometry. First we derive the analytic expression of the trial functions that we will use in the variational principles 3.3.1 and 3.3.2. Finally we prove our results.

3.4.1 Perfect plasmon waves in elliptic geometry

Since our proofs of resonance and non-resonance of a configuration are based on the primal and dual variational principle, it is suffices to construct a sequence of trial functions (ϕ_η, ψ_η), defined on all $\mathbb{R}^2$, that satisfy (3.16), or (3.18), and that have bounded, or unbounded energy. This trial functions are built from the *perfect plasmon waves*.

A perfect plasmon wave of bounded metallic particle $\Omega \subset \mathbb{R}^n$ is an harmonic bounded function on $\mathbb{R}^n \setminus \partial\Omega$, which is continuous at the interface $\partial\Omega$ between the metallic object and the surroundings, and whose exterior and interior normal derivatives at interface have a constant ratio. The study of perfect plasmon waves has been extensively addressed through explicit calculations in simple geometries in the physical literature. A more theoretical study can be found in [14].

We now construct the perfect plasmon waves in elliptical geometry. Fix an elliptic radius $R > 0$ and consider the following problem

$$\begin{cases} \nabla \cdot (A\nabla\psi) = 0 & \text{in } \mathbb{R}^2 \\ \psi = O(|x|^{-1}), & \text{as } |x| \to +\infty \end{cases} \tag{3.19}$$

where

$$A(x) = \begin{cases} -1 & \text{in } \mathcal{E}_R(0) \\ +1 & \text{in } \mathbb{R}^2 \setminus \mathcal{E}_R(0) \end{cases} \tag{3.20}$$

for the perfect plasmon wave $\psi : \mathbb{R}^2 \to \mathbb{R}$. The problem (3.19)-(3.20) is equivalent to the problem

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbb{R}^2 \setminus \partial\mathcal{E}_R(0) \\ \psi\big|_+ = \psi\big|_- & \text{on } \partial\mathcal{E}_R(0) \\ \frac{\partial\psi}{\partial\bar{n}}\big|_+ = \frac{\partial\psi}{\partial\bar{n}}\big|_- & \text{on } \partial\mathcal{E}_R(0) \\ \psi(x) = O(|x|^{-1}) & \text{as } |x| \to +\infty, \end{cases}$$

By using the method of separation of variables, we can derive the functions $\psi^\pm$, and, consequently, the function

$$\psi_k(\mu,\nu) = \begin{cases} e^{k\mu}\cos(k\nu) & \text{for } (\mu,\nu) \in [0,R) \times [0,2\pi), \\ e^{2kR}e^{-k\mu}\cos(k\nu) & \text{for } (\mu,\nu) \in [R,+\infty) \times [0,2\pi), \end{cases} \tag{3.21}$$

is a solution of (3.19) for any $k \in \mathbb{N}$.

3.4.2 Resonance results

First of all we prove that if the configuration, (see Definition 3.1.1), does not have a core, then the configuration is always resonant, and therefore the source f is always cloaked.

Theorem 3.4.1 (No core implies resonance for sources at any distance)**.** *Assume that the configuration has no core (i.e. $\Sigma = \emptyset$). Let $f = F\mathcal{H}^1\lfloor\partial\mathcal{E}_R(0)$ with $0 \neq F : \partial\mathcal{E}_R(0) \to \mathbb{R}$ be a source at a distance $q > R$. Then the configuration is resonant, i.e.*

$$E_\eta(u_\eta) \to +\infty \qquad \text{as } \eta \to 0.$$

Proof. Fix the radii $R, q > 0$ and let η be an arbitrary sequence such that $\eta_i \to 0$ as $i \to +\infty$. Since f is a source with zero average and $F \in L^2(\partial\mathcal{E}_q(0))$, we expand f in Fourier series

$$f = \sum_{k=1}^{+\infty}(a_k\cos(k\nu) + b_k\sin(k\nu))\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$$

where $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}} \in l^2(\mathbb{R})$. Without loss of generality, assume that the source is composed only by even harmonics, i.e. $b_k = 0$ for all $k \in \mathbb{N}$. Hence we write

$$f = \sum_{k=1}^{+\infty} a_k f_k,$$

where $f_k = \cos(k\nu)\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$. Since $F \neq 0$ there exists at least one $\bar{k} \geq 1$ such that $a_{\bar{k}} \neq 0$.

Choose

$$v_\eta(\mu,\nu) = 0$$
$$\tilde{w}_\eta(\mu,\nu) = \lambda_k\psi_k(\mu,\nu)$$

where $\psi_k(\mu,\nu)$ is the perfect plasmon wave found in (3.21) for the radius q, and $\lambda_k \in \mathbb{R}$ has to be defined below. Clearly, the pair $(v_\eta, \tilde{w}_\eta)$ satisfies the PDE constraint of the dual variational problem, Definition 3.3.2. Moreover we have

$$\begin{aligned}
\mathcal{E}_\eta(u_\eta) &\geq \mathcal{J}_\eta(v_\eta, \tilde{w}_\eta) = \mathcal{J}_\eta(0, \tilde{w}_\eta) = \int_{\mathbb{R}^2} f\tilde{w}_\eta\, dx - \frac{\eta}{2}\int_{\mathbb{R}^2} |\nabla \tilde{w}_\eta|^2\, dx \\
&= \int_{\partial\mathcal{E}_q(0)} \alpha_k \cos(k\nu)\lambda_\eta \frac{e^{2kR}}{e^{kq}}\cos(k\nu)\, d\nu - \frac{\eta}{2}\pi k(2e^{2kR}-1)\lambda_\eta^2 \\
&= \pi\alpha_k \frac{e^{2kR}}{e^{kq}}\lambda_\eta - \frac{\eta}{2}\pi k(2e^{2kR}-1)\lambda_\eta^2 \\
&\geq \pi\alpha_k \frac{e^{2kR}}{e^{kq}}\lambda_\eta - \eta\pi k\lambda_\eta^2 e^{2kR}.
\end{aligned}$$

Choosing $\lambda_\eta \to +\infty$ with $\eta\lambda_\eta^2 \to 0$ we obtain $\mathcal{E}_\eta(u_\eta) \to +\infty$ for $\eta \to +\infty$. □

Let us now consider the configuration with core. In circular geometry, according to [2, 20], the configuration is resonant only if the source f is supported inside the disk of radius R^*. In particular, although in [2] and in [20] the energy can only be dissipated in the shell $B_R(0)$ and in all $\mathbb{R}^2$ respectively, the critical radii (3.10) found coincide.

Using the equation (1.22) we expect to find similar results, and in particular we expect the configuration to be resonant only if the source is located inside the ellipse with critical radius

$$R^* = \frac{3R - r}{2}, \tag{3.22}$$

where r is the elliptical radius of the core, R is that of the shell. This result is expressed in the following Theorem. However, this attempt is only partially confirmed. Indeed Chung et al. [9] showed that if the energy is dissipated only in the shell, then the critical radius is given by

$$R^* = \begin{cases} (3R-r)/2 & \text{for } R \leq 3r, \\ 2(R-r) & \text{for } R > 3r. \end{cases} \tag{3.23}$$

The reason for this discrepancy cannot be fully attributed to the region where the energy is dissipated. Indeed, the variational principles depend on the choice of test functions, which bound the energy of the solution. With the choice of test functions made in the following theorem, we were able to prove that the configuration is resonant for R^* as in (3.22) and not as in (3.23). Hence, for $R > 3r$, we obtain a weaker estimate for R^*, i.e. the one that we obtain is smaller than the one obtained in [9].

However, on one point our result agrees with that expressed in [9]: the critical radius R^* does not depend on the foci $(\pm a, 0)$ of the ellipses.

Theorem 3.4.2 (Non-resonance beyond R^*)**.** *Let $\Sigma = \mathcal{E}_r(0) \subset \mathcal{E}_R(0)$ and let*

$$A(x) = \begin{cases} +1 & \text{in } \Sigma, \\ -1 & \text{in } \mathcal{E}_R(0) \setminus \Sigma, \\ +1 & \text{in } \mathbb{R}^2 \setminus \mathcal{E}_R(0). \end{cases}$$

Let $f = F\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$, $0 \neq F : \partial\mathcal{E}_q(0) \to \mathbb{R}$ R, be a source at a distance $q > R$ with zero average and $F \in L^2(\partial\mathcal{E}_q(0))$. Then the configuration is non-resonant if $q > R^*$ where

$$R^* = (3R - r)/2$$

Proof. Since the source $F \in L^2(\partial\mathcal{E}_q(0)$, let us expand it into Fourier series

$$F(\nu) = \sum_{k\geq 1}(\alpha_k \cos(k\nu) + \beta_k \sin(k\nu)) = F_{\text{even}} + F_{\text{odd}}.$$

Clearly, it suffices to show that the configuration is non-resonant for $f_{\text{even}} = F_{\text{even}}\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$ and $f_{\text{odd}} = F_{\text{odd}}\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$ separately Let us now prove this for f_{even}.

Accordingly set

$$f = \sum_{k\geq 1}\alpha_k f_k$$

where $f_k = \cos(k\nu)\mathcal{H}^1\lfloor\partial\mathcal{E}_q(0)$ and $\{\alpha_k\}_{k\geq 1} \subset l^2(\mathbb{R})$.

The scheme of the proof is the following. In Step 1 we build the appropriate test functions to be used in the primal variational principle; in Step 2 we calculate the energies associated with these test functions and show that the energy of the solution is limited by the previous energies.

Step 1: Construction of test functions. Fix a sequence of loss parameters $\eta = \{\eta_i\}$ tending to zero. In order to use the primal variational principle, we need to construct test functions (v_η, w_η) such that

$$\nabla\cdot(A\nabla v_\eta) - \Delta w_\eta = f \tag{3.24}$$

and such that they have finite energy $\mathcal{I}_\eta(v_\eta, w_\eta) < +\infty$ uniformly with respect to η. Our strategy is to decompose the source f into a low frequency part and a high frequency part as

$$f = f^{\text{low}} + f^{\text{high}}, \qquad f^{\text{low}} = \sum_{k\leq k^*}\alpha_k f_k, \qquad f^{\text{high}} = \sum_{k>k^*}\alpha_k f_k$$

where k^* has to be chosen and will depend on η. We also decompose the solution into a high and low frequency part by setting $v_\eta = v_\eta^{\text{low}} + v_\eta^{\text{high}}$. With these choices, we rewrite the constraint equation (3.24) in the form

$$\begin{aligned} v_\eta^{\text{low}} \text{ satisfies } \nabla\cdot(A\nabla v_\eta^{\text{low}}) &= f^{\text{low}} \qquad \text{on } \mathbb{R}^2 \\ v_\eta^{\text{high}} \text{ satisfies } \nabla\cdot(A\nabla v_\eta^{\text{high}}) &= f^{\text{high}} \qquad \text{on } \partial\mathcal{E}_q(0) \\ w_\eta \text{ satisfies } -\Delta w_\eta &= -\nabla\cdot(A\nabla v_\eta^{\text{high}}) + f^{\text{high}} \qquad \text{on } \mathbb{R}^2 \end{aligned}$$

We now derive the individual test functions.

Step 1a: construction of v_η^{low}. The function v_η^{low} is pieced together using variants of the perfect plasmon waves (3.21). We set

$$\hat{v}_k(\mu,\nu) = \begin{cases} e^{k\mu}\cos(k\nu) & \text{for } 0 \leq \mu < r, \\ e^{2kr}e^{-k\mu}\cos(k\nu) & \text{for } r \leq \mu < R, \\ e^{2k(r-R)}e^{k\mu}\cos(k\nu) & \text{for } R \leq \mu < q, \\ e^{2k(q+r-R)}e^{-k\mu}\cos(k\nu) & \text{for } \mu > q, \end{cases}$$

and $\nu \in [0, 2\pi)$. The functions $\hat{v}_k(\mu,\nu)$ have the following properties. First, they are continuous on $\mathbb{R}^2$. Then for every $x \in \mathbb{R}^2 \setminus \partial\mathcal{E}_q(0)$ they satisfy $\nabla\cdot(A\nabla\hat{v}_k) = 0$, since along $\partial\mathcal{E}_q(0)$ they have a jump into the normal flux

$$(\bar{n}\cdot\nabla\hat{v}_k)\big|_{\partial\mathcal{E}_q(0)} = -2ke^{k(q+2r-2R)}\cos(k\nu).$$

Therefore, $\lambda_k \hat{v}_k$ satisfies

$$\nabla \cdot (A \nabla \lambda_k \hat{v}_k) = \alpha_k f_k \qquad \text{on } \mathbb{R}^2$$

if we set

$$\lambda_k = -\frac{\alpha_k}{2k e^{k(q+2r-2R)}}. \tag{3.25}$$

Therefore by setting $v_\eta^{\text{low}} = \sum_{k \le k^*} \lambda_k \hat{v}_k$ we have that

$$\begin{aligned}
\nabla \cdot (A \nabla v_\eta^{\text{low}}) &= (\bar{n} \cdot \nabla v_\eta^{\text{low}})\big|_{\partial \mathcal{E}_q(0)} \mathcal{H}^1 \lfloor \partial \mathcal{E}_q(0) \\
&= \sum_{k \le k^*} \lambda_k \big(-2k e^{k(q+2r-2R)} \cos(k\nu)\big) \mathcal{H}^1 \lfloor \partial \mathcal{E}_q(0) \\
&= \sum_{k \le k^*} \alpha_k \cos(k\nu) \mathcal{H}^1 \lfloor \partial \mathcal{E}_q(0) \\
&= f^{\text{low}}
\end{aligned}$$

Step 2a: construction of v_η^{high} and w_η. The function v_η^{high} is constructed from the elementary plasmon waves $\hat{V}_k$ for the elliptic radius q. This functions are not tuned to solve $\nabla \cdot (A \nabla \hat{V}_k) = 0$ on $\partial \mathcal{E}_r(0)$ or $\partial \mathcal{E}_R(0)$. We set

$$\hat{V}_k(\mu, \nu) = \begin{cases} e^{k\mu} \cos(k\nu) & \text{for } 0 \le \mu < q \\ e^{2kq} e^{-k\mu} \cos(k\nu) & \text{for } \mu \ge q \end{cases}$$

for $\nu \in [0, 2\pi)$. Since $A(x) = 1$ near $\partial \mathcal{E}_q(0)$ we have

$$(\bar{n} \cdot \nabla \hat{V}_k)\big|_{\partial \mathcal{E}_q(0)} = -2k e^{kq} \cos(k\nu).$$

Therefore by setting $v_\eta^{\text{high}} = \sum_{k > k^*} \Lambda_k \hat{V}_k$ with

$$\Lambda_k = -\frac{\alpha_k}{2k e^{kq}} \tag{3.26}$$

we have on $\partial \mathcal{E}_q(0)$ that

$$\nabla \cdot (A \nabla v_\eta^{\text{high}}) = f^{\text{high}}.$$

We emphasize that v_η^{high} is not a solution on all of $\mathbb{R}^2$ due to normal flux jumps at $\partial \mathcal{E}_r(0)$ and $\partial \mathcal{E}_R(0)$. To correct this, we choose w_η such that

$$\begin{aligned}
-\Delta w_\eta &= -\nabla \cdot (A \nabla v_\eta^{\text{high}}) + f^{\text{high}} \\
&= -\sum_{k > k^*} \Lambda_k (\bar{n} \cdot \nabla \hat{V}_k)\big|_{\partial \mathcal{E}_r(0)} \mathcal{H}^1 \lfloor \partial \mathcal{E}_r(0) \\
&\quad - \sum_{k > k^*} \Lambda_k (\bar{n} \cdot \nabla \hat{V}_k)\big|_{\partial \mathcal{E}_R(0)} \mathcal{H}^1 \lfloor \partial \mathcal{E}_R(0).
\end{aligned}$$

We use the previous equation to define w_η.

Step 2: Calculation of energies. Now let us calculate the energy for the choice $v_\eta = v_\eta^{\text{low}} + v_\eta^{\text{high}}$ and w_η. It is in this step that we choose the low-high frequency cutoff, $k^* = k^*(\eta)$ to ensure that $\mathcal{I}_\eta(v_\eta, w_\eta)$ remains uniformly bounded as $\eta \to 0$.

Step 2a: Energy of v_η^{low}. A long but straightforward calculation shows that

$$\begin{aligned}\eta \int_{\mathbb{R}^2} |\nabla v_\eta^{\text{low}}|\, dx &= \eta \sum_{k\le k^*} |\lambda_k|^2 \int_{\mathbb{R}^2} |\nabla v_k|^2\, dx \\ &= \eta\pi \sum_{k\le k^*} |\lambda_k|^2 k(2e^{2kr} + 2e^{2k(2r+q-2R)} - 2e^{2k(2r-R)} - 1) \\ &= \frac{\eta\pi}{4} \sum_{k\le k^*} \frac{|\alpha_k|^2}{ke^{2k(q+2r-2R)}}(2e^{2kr} + 2e^{2k(2r+q-2R)} - 2e^{2k(2r-R)} - 1) \\ &\le C\eta \sum_{k\le k^*} |\alpha_k|^2 e^{2k(2R-r-q)} \max\{1, e^{k(r+q-2R)}\}^2\end{aligned}$$

where we used the definition of λ_k equation (3.25). Then if $q \ge 2R - r$ we have

$$\eta \int_{\mathbb{R}^2} |\nabla v_\eta^{\text{high}}|\, dx \le C\eta \sum_{k\le k^*} |\alpha_k|^2 \le C\eta$$

which is obviously bounded. If $R^* < q < 2R - r$ we have

$$\eta \int_{\mathbb{R}^2} |\nabla v_\eta^{\text{low}}|\, dx \le C\eta \sum_{k\le k^*} |\alpha_k|^2 e^{2k^*(2R-r-q)}. \tag{3.27}$$

Step 2b: Energy of v_η^{high}. The energy of v_η^{high} is easier to control

$$\begin{aligned}\eta \int_{\mathbb{R}^2} |\nabla v_\eta^{\text{high}}|\, dx &= \eta \sum_{k>k^*} |\Lambda_k|^2 \int_{\mathbb{R}^2} |\nabla V_k|^2\, dx \\ &= \frac{\eta\pi}{4} \sum_{k>k^*} \frac{|\alpha_k|^2}{ke^{2kq}}(e^{2kq} - 1) \\ &\le \frac{\eta\pi}{2} \sum_{k>k^*} |\alpha_k|^2 < C\eta\end{aligned}$$

where we used the definition of Λ_k equation (3.26).

Step 2c: Energy of w_η. We have

$$\begin{aligned}\frac{1}{\eta} \int_{\mathbb{R}^2} |\nabla w_\eta|^2\, dx &\le \frac{C}{\eta} \left\| \nabla \cdot (A\nabla v_\eta^{\text{high}}) - f^{\text{high}} \right\|_{L^2(\mathbb{R}^2)} \\ &= \frac{C}{\eta} \sum_{k>k^*} \frac{|\alpha_k|^2}{4k^2 e^{2kq}} \left(2k^2 e^{2kr} + 2k^2 e^{2kR}\right) \\ &\le \frac{C}{\eta} \sum_{k>k^*} |\alpha_k|^2 \left(e^{2k^*(r-q)} + e^{2k^*(R-q)}\right) \\ &\le \frac{C}{\eta} \sum_{k>k^*} |\alpha_k|^2 e^{2k^*(R-q)}.\end{aligned} \tag{3.28}$$

We now balance the right hand sides of the bounds (3.27) and (3.28). We have

$$\eta e^{2k^*(2R-r-q)} \sim \frac{1}{\eta} e^{2k^*(R-q)}$$

that is

$$\eta \sim e^{k^*(r-R)}.$$

Hence we choose $k^* = k^*(\eta)$ to be the smallest integer such that

$$\eta \leq e^{(k^*+1)(r-R)} \qquad \text{and} \qquad \frac{1}{\eta} < e^{k^*(R-r)}.$$

With this choice of η we have

$$\frac{1}{\eta} \int_{\mathbb{R}^2} |\nabla w_\eta|^2 \, dx \leq C \sum_{k \geq 1} |\alpha_k|^2 e^{k^*(R-r)} e^{2k^*(R-q)}$$

and

$$\eta \int_{\mathbb{R}^2} |\nabla v_\eta^{\text{low}}| \, dx \leq C \sum_{k \leq 1} |\alpha_k|^2 e^{(k^*+1)(r-R)} e^{2k^*(2R-r-q)}.$$

Thus if $q > R^* = (3R-r)/2$, the energy $\mathcal{I}_\eta(v_\eta, w_\eta)$ is bounded as $\eta \to 0$. The proof is complete. $\square$

Conclusion

In this work we presented the basics of electromagnetic cloaking, focusing on two different schemes: cloaking by transformation optics and cloaking by anomalous localized resonance. It is known that the geometry of the problem has a strong influence on the cloaking properties of the system, and a full understanding is still missing. In this thesis we carefully analyzed the relatively simple case of elliptical geometry and we have seen that some results have remarkable differences compared to the ones for circular geometry. Indeed, cloaking is a relatively new phenomenon and still little is known, both on practical implementation and on theoretical study. Our work was theoretical and, although dedicated only to a particular geometry, problems and research challenges appeared several times.

First of all, the result expressed by Theorem 3.4.2 is a partial result that we have not been able to improve. Future work must improve this result, or at least find out why it is not possible to obtain it, in view of a better understanding of the phenomenon of cloaking by anomalous localized resonance. Furthermore, CALR must also be studied from a geometric point of view: it is in fact necessary to understand how it occurs in relation to the geometry of the system. In particular, in [24] it is stated that the radial configuration is the one that hides best among all possible geometries. We aim at proving a rigorous mathematical justification of this result for CALR.

Another technological problem is the following. In Chapter 3 we have considered a dielectric permittivity with a matrix-shell-core character. The method presented in [20] was stated with $A(x) = +1$ in the core. It would be interesting to extend the results obtained to the case in which $A(x) \neq 1$ in the core, and, in particular, to find permittivities $A(x)$ that maximize or minimize the cloaking phenomenon for the same sets. In fact, the resonance of the configuration depends on the loss parameter η and on $A(x)$, therefore it would be interesting to study the dependence of the energy on these two parameters, and to relate the speed with which it diverges to $A(x)$.

A final problem we pose is the following. In both Chapter 2 and Chapter 3 we have considered systems in $\mathbb{R}^2$. It is known [13] that cloaking for transformation optics can happen in dimensions $n \geq 2$, meaning that if σ is non-negative, then boundary measurements determine σ up to change of variables. Therefore, to obtain cloaking, we only need to find systems for which the analogue of Theorem 2.2.2 holds. It would be interesting to study the case of prolate and oblate ellipsoidal systems in $\mathbb{R}^3$. For cloaking by anomalous localized resonance the question is different. Indeed, in [3] it is proved that CALR does not occur for spherical systems in $\mathbb{R}^3$, except with anisotropic permittivity. It would be interesting to study the case of ellipsoidal systems in $\mathbb{R}^3$, both with the spectral methods and with the variational methods. Unfortunately, variational methods need a more careful study, since they are essentially based on perfect plasmon waves, which do not exist in $\mathbb{R}^3$ for circular systems.

Appendix A

Sturm-Liouville problem

Consider the equation

$$\Delta\theta = 0 \quad \text{in } \Omega, \tag{A.1}$$

where $\Omega = E \times F = [0,1] \times [0,1]$. Suppose we can write $\theta(x,y) = \phi(x)\psi(y)$ with $\phi(x)$ and $\psi(y)$ to be determined. Proceeding formally we have

$$\Delta\theta(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x)\psi(y) = \psi(y)\frac{\partial^2\phi}{\partial x^2}(x) + \phi(x)\frac{\partial^2\psi}{\partial y^2}(y) = 0$$

Since the two terms of the right hand sine of this equation are functions of x, y separately, they must be constants. In particular, we have

$$\begin{cases} \ddot{\phi}(x) = -\lambda\phi(x) & \text{for } x \in [0,1], \\ \ddot{\psi}(y) = +\lambda\psi(y) & \text{for } y \in [0,1], \end{cases} \tag{A.2}$$

with use of Newton's notation for derivatives. The number $\lambda \in \mathbb{R}$ is the eigenvalue of problem (A.2) and must be determined.

To completely determine the solutions of (A.2) it is necessary to assign initial conditions. In fact the original problem has some conditions at the boundary. WLOG we assume that

$$\begin{cases} \theta(x,0) = 0, & \theta(0,y) = 0, \\ \theta(x,1) = f, & \theta(1,y) = 0 \end{cases} \tag{A.3}$$

for $f : [0,1] \to \mathbb{R}$ given. The boundary conditions (A.3) on θ translate into conditions for ϕ, ψ

$$\begin{cases} \phi(0) = 0, & \psi(0) = 0, \\ \phi(1) = 0 & \phi(x)\psi(1) = f(x) \end{cases} \tag{A.4}$$

Focusing only on the conditions on ϕ, we se that they are not Cauchy conditions: they are assigned at different points in the interval $[0,1]$. Therefore the problem (A.2)-(A.4) is not a Cauchy problem, and we cannot solve it using the existence and uniqueness theorems. Indeed the problem (A.2), (A.4) is a global problem: the solution must exist in the whole interval $[0,1]$ for the boundary conditions to be applied. The equation (A.2) with the relative data at the boundary (A.4) constitutes the *Sturm-Liouville problem*.

We now give an abstract formulation of the Sturm-Liuville problem

$$-y'' = \lambda y \tag{A.5}$$

on $[0,1]$ with $\lambda \in \mathbb{C}$ and with boundary condition

$$\begin{cases} \alpha_0 y(0) + \beta_0 y'(0) = 0 & \text{with } \alpha_0^2 + \beta_0^2 \neq 0, \quad \text{(A.6)} \\ \alpha_1 y(1) + \beta_1 y'(1) = 0 & \text{with } \alpha_1^2 + \beta_1^2 \neq 0. \quad \text{(A.7)} \end{cases}$$

Our intent is to show that there exists a family of solutions which constitutes an orthonormal basis of L^2, as well as to determine the eigenvalues. In order to do that we introduce the spaces[1]

$$D_0 = \{g : [0,1] \to \mathbb{R}, g \in C^1, g' \in AC, g'' \in L^2, g \text{ satisfies (A.6)}\}$$

$$D_1 = \{g : [0,1] \to \mathbb{R}, g \in C^1, g' \in AC, g'' \in L^2, g \text{ satisfies (A.7)}\}.$$

We define also $D = D_0 \cap D_1$. We will look for the solutions to the Sturm-Liouville problem in the real vector space D. We define $L : D \to L^2([0,1])$ to be the operator with rule $g \mapsto Lg = -g''$. Clearly the operator L is linear. It is also injective on D.

Definition A.1. Let $h_0 \in D_0$, $Lh_0 = 0$, $h_0 \neq 0$. Let $h_1 \in D_1$, $Lh_1 = 0$, $h_1 \neq 0$.

The functions h_0, h_1 clearly exist. Indeed, by taking an initial condition that satisfies (A.6) or (A.7) respectively, we can use the global existence and uniqueness theorem and guarantee not only existence of the solutions, but also $h_0, h_1 \in C^2$.

Definition A.2. Let $W : [0,1] \to \mathbb{R}$ be the function given by

$$W(x) = \det \begin{pmatrix} h_0(x) & h_1(x) \\ h_0'(x) & h_1'(x) \end{pmatrix}$$

We claim that W is constant, and $W \neq 0$. Indeed, $W \in C^1$ with

$$W'(x) = \det \begin{pmatrix} h_0'(x) & h_1'(x) \\ h_0'(x) & h_1'(x) \end{pmatrix} + \det \begin{pmatrix} h_0(x) & h_1(x) \\ h_0''(x) & h_1''(x) \end{pmatrix} = 0 + \det \begin{pmatrix} h_0(x) & h_1(x) \\ qh_0(x) & qh_1(x) \end{pmatrix} = 0,$$

and this proves that W is constant. If it were $W(0) = 0$, the columns would be linearly dependent and would exist some constant $c \neq 0$ with $h_0(x) = ch_1(x)$ and $h_0'(x) = ch_1'(x)$ for all $x \in [0,1]$. But

[1] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is absolutely continuous $f \in AC$ if for every $\epsilon > 0$ there is a positive number $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (a_k, b_k) of I satisfies $\sum_k (b_k - a_k) < \delta$ then $\sum_k (f(b_k) - f(a_k)) < \epsilon$.

then $\alpha_1 h_0(1) + \beta_1 h_0'(1) = c(\alpha_1 h_1(1) + \beta_1 h_1'(1)) = 0$ and therefore also (A.7) would be satisfied. So $h_0 \in D_0 \cup D_1$, but the hypothesis is that the only function $h \in D$ with $Lh = 0$ is $h = 0$, which contradicts the fact that $h_0 \neq 0$.

Considering that W is constant and not zero, by rescaling h_0 we can assume that $W(x) = -1$ for all $x \in [0,1]$.

Definition A.3. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be the function given by

$$k(x,z) = \begin{cases} h_0(x)\, h_1(z) & 0 \leq x \leq z \leq 1 \\ h_0(z)\, h_1(x) & 0 \leq z \leq x \leq 1. \end{cases}$$

Note that k is continuous and symmetric. Since $k \in L^2([0,1] \times [0,1])$, we define the Hilberts-Schmidt operator $T : L^2([0,1]) \to L^2([0,1])$ by

$$(Tf)(x) = \int_{[0,1]} k(x,z) f(z)\, dz.$$

Since k is real and symmetric, T is self-adjoint. Moreover, since T is a Hilbert-Schmidt operator, T is also compact.

Lemma A.1. *The range of T is D, i.e. $Tf \in D$ for every $f \in L^2([0,1])$. Furthermore, for all $h \in D$ there exists $f \in L^2([0,1])$ such that $Tf = h$.*

The lemma implies that

$$L(Tf) = f \qquad \forall f \in L^2([0,1]), T(Lh) = h \qquad \forall h \in D.$$

So T is the inverse map of L.

Proof. First of all, $f \in L^2([0,1])$ implies $f \in L^1([0,1])$. Since $k \in C^1([0,1]^2)$, we have $k(x,\cdot)f(\cdot) \in L^1([0,1])$, therefore $(Tf)(x)$ is well defined for every $x \in [0,1]$. By the dominated convergence theorem we have $Tf \in C([0,1])$. Moreover, by the definition of k it follows that

$$(Tf)(x) = h_0(x) \int_x^1 h_1(z) f(z)\, dz + h_1(x) \int_0^x h_0(z) f(z)\, dz \tag{A.8}$$

and given that $h_0, h_1 \in C^1$, $h_0 f, h_1 f \in L^1$, it follows that $Tf \in AC$.

From the above formula it follows that

$$\begin{aligned}(Tf)'(x) &= h_0(x) \int_x^1 h_1(z) f(z)\, dz - h_0(x = h_1(x) f(x) + h_1'(x) \int_0^x h_0(z) f(z)\, dz + h_0(x) h_1(x) f(x) \\ &= h_0(x) \int_x^1 h_1(z) f(z)\, dz + h_1'(x) \int_0^x h_0(z) f(z)\, dz := S\end{aligned}$$

almost everywhere. Nevertheless, we note that S is well defined everywhere and is continuous. Since $Tf \in AC$ we have

$$(Tf)(x) = (Tf)(0) + \int_0^x (Tf)'(u)\, du = Tf)(0) + \int_0^x S(u)\, du \in C^1([0,1])$$

hence $Tf \in C^1$, with $(Tf)' = S$ everywhere. S is also in AC, so $(Tf)''$ exists in L^1, with

$$\begin{aligned}(Tf)''(x) &= h_0''(x)\int_x^1 h_1(z)f(z)\,dz - h_0(x)h_1(x)f(x) + h_1''(x)\int_0^x h_0(z)f(z)\,dz + h_1'(x)h_0(x)f(x)\\ &= -f(x) \qquad \text{a.e.,}\end{aligned} \tag{A.9}$$

where we used the fact that $h_0'' = h_1'' = 0$ and the definition of $W = -1$. Hence $(Tf)'' \in L^2([0,1])$. Moreover it follows from (A.8)

$$(Tf)(0) = h_0(0)\int_0^1 h_1(z)f(z)\,dz, \qquad (Tf)'(0) = h_0'(0)\int_0^1 h_1(z)f(z)\,dz,$$

$$(Tf)(1) = h_1(1)\int_0^1 h_0(z)f(z)\,dz, \qquad (Tf)'(1) = h_1'(1)\int_0^1 h_0(z)f(z)\,dz,$$

and Tf satisfies (A.6) and (A.7). This proves that $Tf \in D$, i.e. that the range of T is a subset of D.

From (A.9) it follows that $L(Tf) = f$ for all $f \in L^2$. Furthermore, for every $h \in D$ we have $Lh \in L^2$. So choosing $f = Lh$ in the previous relation we find $L(TLh - h) = 0$. But $TLh - h \in D$ and by hypothesis L is injective in D, so $TLh = h$. Since h is arbitrary, this gives the range of T is D. □

Since T is a Hilbert-Schmidt operator, its spectrum is given by $\{0\} \cup \{\mu_n\}_{n\in\mathbb{N}}$, where every μ_n is a real eigenvalue. Moreover $\mu_n \to 0$ when n diverges, and $\dim \mathrm{Ker}(T - \mu_n\mathbb{I}) < +\infty$. Actually

Lemma A.2. *$\dim \mathit{Ker}(T - \mu_n\mathbb{I}) = 1$ for every $n \in \mathbb{N}$.*

Proof. Since μ_n is an eigenvalue, we have $\dim \mathrm{Ker}(T - \mu_n\mathbb{I}) \geq 1$. Suppose that the dimension is greater than 2, and let ϕ_1, ϕ_2 be two independent functions in Ker$(T - \mu_n\mathbb{I})$. Since $\mu_n\phi_j = T\phi_j$ we have $\phi_j \in D$ for $j = 1, 2$. However, they are solutions of $y'' = 0$, and since thery are linearly independent, they are a basis of the solutions of $y'' = 0$. So every solution of $y'' = 0$ should satisfy (A.6) and (A.7) at the same time, which is impossible since the Cauchy problem

$$\begin{cases} y'' = 0 \\ y(0) = a \\ y'(0) = b \end{cases}$$

has a solution even when $\alpha_0 a + \beta_0 b \neq 0$. □

Note that $T\phi = \mu\phi$ is equivalent to $L\phi = 1/\mu LT\phi = 1/\mu\phi$ when $\mu \neq 0$. So if μ is an eigenvalue for T, then $1/\mu$ is an eigenvalue for L. We then deduce the following theorem.

Theorem A.1. *There exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and an orthonormal basis $\{e_n\}_{n\in\mathbb{N}} \subset L^2([0,1])$ such that*

1. *$0 < |\lambda_0| < |\lambda_1| < \ldots$, with $|\lambda_n| \to +\infty$ when $n \to +\infty$;*
2. *$e_n \in D$ and $Le_n = \lambda_n e_n$ for every $n \in \mathbb{N}$.*

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