Poincaré duality and integration with kernels

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Abstract

These are the notes for my talk in the *HIOB* seminar of winter semester 2025-2026. We begin by quickly reviewing Grothendieck six operations on locally compact Hausdorff spaces. After having stated the main compatibility results between the six functors, we will focus on the study of Poincaré duality. We will introduce cohomologically smooth morphisms and show that any such morphism gives a Poincaré duality statement. We will then discuss how Poincaré duality can be rewritten in categorical terms by means of a 2-category of Kernels. Finally, we will discuss a large class of examples of cohomologically smooth morphisms as well as discussing relative purity.

1 Some words on what we did last time

Let R be a (commutative, with identity) ring. In the last talk we have showed that the usual triangulated derived category of R admits an enhancement, that is there exists a stable $(\infty, 1)$-category $\mathcal{D}(R)$ whose homotopy category gives back the derived category constructed by Verdier.

We have used this $(\infty, 1)$-category to do "homological algebra" in topological spaces. More precisely, to every locally compact Hausdorff space X we have associate the $(\infty, 1)$-category of sheaves $\mathrm{Shv}(X, R)$ on X with values in stable derived $(\infty, 1)$-category $\mathcal{D}(R)$ of R. Objects in this stable $(\infty, 1)$-category are sheaves F on X with values in $\mathcal{D}(R)$, that is a contravariant functors F from open subsets of X towards $\mathcal{D}(R)$ such that:

(1) $F(\emptyset) = *$.

(2) If $U = U_1 \cup U_2$, then $F(U) \to F(U_1) \times_{F(U_1 \cap U_2)} F(U_2)$ is an equivalence.

(3) If U is a filtered union of subsets $U_i \subseteq U$, then $F(U) \to \lim_i F(U_i)$ is an equivalence.

We have also showed that this construction supports Grothendieck *six operations*. Let us recall them.

(1) Since $\mathcal{D}(R)$ admits the structure of a symmetric monoidal $(\infty, 1)$-category under the derived tensor product $(\mathcal{D}(R), \otimes, R)$, we can extend this operation to sheaves on X. Given two sheaves, their tensor product is given by tensoring them pointwise, followed by a sheafification. The tensor product of sheaves is also closed, in the sense that there exists a functor

$$\underline{\mathrm{Hom}}(-, -) : \mathrm{Shv}(X, R)^{\mathrm{op}} \times \mathrm{Shv}(X, R) \to \mathrm{Shv}(X, R),$$

called the internal hom, such that

$$\mathrm{Hom}_{\mathrm{Shv}(X,R)}(F \otimes G, H) \simeq \mathrm{Hom}_{\mathrm{Shv}(X,R)}(F, \underline{\mathrm{Hom}}(G, H))$$

for every sheaf $F, G, H \in \mathrm{Shv}(X, R)$.

(2) Every map of locally compact Hausdorff spaces $f : X \to Y$ we can associate a pullback functor $f^* : \mathrm{Shv}(Y, R) \to \mathrm{Shv}(X, R)$, obtained by restricting to opens of X. This functor as a right adjoint $f_* : \mathrm{Shv}(X, R) \to \mathrm{Shv}(Y, R)$, which computes "cohomology". More precisely, if $f : X \to *$ is the unique map to the point, then the cohomology of X with coefficients in $A \in \mathrm{Shv}(X, R)$ is simply given by

$$\Gamma(X, A) = f_*(A) \in \mathcal{D}(R).$$

By picking the constant sheaf $\mathbb{Z}_X = f^*\mathbb{Z}$, one gets the usual notion of sheaf cohomology (and hence of singular cohomology).

(3) Finally, every map of locally compact Hausdorff spaces $f : X \to Y$ allows us to do "cohomology with compact support". Indeed, we can construct a proper pushforward functor $f_! : \mathrm{Shv}(X, R) \to \mathrm{Shv}(Y, R)$ by deriving the functor of sections with proper support. In particular, there is a natural transformation $f_! \to f_*$ that is an isomorphism when f is proper. Again, there is a right adjoint $f^! : \mathrm{Shv}(Y, R) \to \mathrm{Shv}(X, R)$, the exceptional inverse image functor. Cohomology with compact support is then obtained by considering the unique map to the point $f : X \to *$, in the sense that for $A \in \mathrm{Shv}(X, R)$ we think of

$$\Gamma_c(X, A) = f_!(A) \in \mathcal{D}(R).$$

as computing the sections with compact support of A.

These six operations satisfy a number of compatibility relations.

(1) For example, the functors f^* and $f_!$ satisfy the Projection formula. That is, given $A \in \mathrm{Shv}(X, R)$ and $B \in \mathrm{Shv}(Y, R)$ there is a natural equivalence $f_!(A \otimes f^*B) \simeq f_!A \otimes B$. In categorical terms, this equivalence says that $f_! : \mathrm{Shv}(X, R) \to \mathrm{Shv}(Y, R)$ is $\mathrm{Shv}(Y, R)$-linear.

(2) The functors f^* and $g_!$ satisfy base change. That is, given a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

there are canonical equivalence $f^*g_! \simeq g'^!f'^*$. In particular, by transposition, we also have an equivalence $g^!f_* \simeq f'_*g'^!$.

(3) The functors f^* and $g_!$ satisfy proper base change, that is, we have an equivalence $g^*f_* \simeq f'_*g'^*$.

Together, these functors and compatibilities constitute the Grothendieck six operations for sheaves on locally compact Hausdorff spaces.

2 Poincaré duality

We now add to the above list of compatibility conditions another result, which is an important structural feature of cohomology: Duality.

Theorem 2.1 (Poincaré duality)**.** Let X be oriented manifold of dimension d. Then there exists an isomorphism

$$H^n_c(X) \cong H_{d-n}(X)$$

between the n-cohomology with compact support of X and the the $(d-n)$-homology group of X. Alternatively, there exists an isomorphism

$$H^n(X) \cong H^{BM}_{d-n}(X)$$

between the n-cohomology of X and the the $(d-n)$-Borel-Moore homology group of X.

I am sure that we all know this statement. What we do now is to see higher categorical versions of it: we want to convince ourselves that Poincaré duality is a relation between the $*$-functors and the !-functors associated to the unique map $f : X \to *$ to the point. The intuition is easy. Indeed, if we recall that cohomology with compact support of a sheaf $A \in \mathcal{D}(R)$ is defined as

$$H^n_c(X, A) = \pi_0 \mathrm{Hom}(\mathbb{1}_*, f_! f^*(A)[n])$$

and homology is defined as

$$H_{d-n}(X, A) = \pi_0 \mathrm{Hom}(\mathbb{1}_*, f_! f^!(A)[n-d])$$

then a simple comparison makes us believe that Poincaré duality should follow from an equivalence

$$f^*(A)[d] \simeq f^!(A).$$

This seems very believable. In fact, by defining $\omega_X = f^!(\mathbb{1}_*)$, we are lead to believe that the co-projection morphism $\omega_f \otimes f^*(-) \to f^!(-)$, constructed by taking the adjoint to the morphism

$$f_!(f^!(-) \otimes f^*(-)) \simeq f_!(f^!(-)) \otimes (-) \to - \otimes -,$$

could be the one giving the required equivalence. What is missing in this picture is an identification between the *dualizing object* ω_X and the twist $[d]$ by d, which follow since X is oriented. This is true and a proof can be found in [KS90]. In any case, this leads us to the following.

Definition 2.2. Let $f : X \to Y$ be a morphism and let us denote $\omega_f = f^!(\mathbb{1}_Y)$ the dualizing object of f. We will say that f is *weakly cohomologically smooth* if:

(1) The the co-projection morphism $\omega_f \otimes f^*(-) \to f^!(-)$ is an equivalence.

(2) The dualizing object ω_f is an invertible object of $\mathrm{Shv}(X, R)$. Moreover, it is stable under base change, that is, for any Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

the natural morphism $(g')^*(\omega_f) \longrightarrow \omega_{f'}$ is an equivalence.

We will furthermore say that f is *cohomologically smooth* if, for any morphism $g : Y' \to Y$ in $\mathcal{C}$, the base change $f' : X' \to Y'$ is weakly cohomologically smooth.

Remark 2.3. Of course the previous definition can be stated for general 6-functor formalisms. In particular, one can show that whenever the categorical Künneth formula holds, then any weakly cohomologically smooth morphism is cohomologically smooth. This happens, in particular, in our setting.

Lemma 2.4. Let $f : X \to Y$ be weakly cohomologically smooth. Then we have an adjunction

$$\begin{tikzcd} \mathrm{Shv}(X, R) \arrow[r, bend left, "f_!(-\otimes\omega_X)"] \arrow[r, phantom, "\perp"] & \mathrm{Shv}(Y, R). \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Proof. This is a computation:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Shv}(Y,R)}(f_!(- \otimes \omega_f), -) &\simeq \mathrm{Hom}_{\mathrm{Shv}(X,R)}(- \otimes \omega_f, f^!(-)) \\ &\simeq \mathrm{Hom}_{\mathrm{Shv}(X,R)}(-, f^!(-) \otimes \omega_f^{-1}) \\ &\simeq \mathrm{Hom}_{\mathrm{Shv}(X,R)}(-, f^*). \end{aligned}$$

Here in the first equivalence we have used the adjunction $f_! \dashv f^!$, in the second one the fact that ω_f is $\otimes$-invertible, and in the third one the equivalence $f^*(-) \otimes \omega_f \to f^!(-)$. ☐

3 Categorifying Poincaré duality

Checking that $f : X \to Y$ is cohomologically smooth seems highly nontrivial. Indeed, for any base change of f, we need to prove that some map is an isomorphism for all $B \in \mathrm{Shv}(Y, R)$; and the map involves $f^!(B)$ which is abstractly defined as an adjoint, so we have to compute the morphisms from any $A \in \mathrm{Shv}(X)$ towards $f^!(B)$. The goal of this section is to show that in fact, it is enough to construct a surprisingly small amount of data (and check the commutativity of two diagrams), and this data involves only some very simple sheaves on X, Y and $X \times_Y X$. For that, we need the machinery of 2-categories.

Remark 3.1. Since not everyone is familiar with the theory of 2-categories, let us take the time to explain the main properties they have (and fix the notation). There are several models of 2-categories. On the top of my head I can mention the following.

(1) One possible model is the one of *strict 2-categories*, that is categories enriched over the cartesian monoidal category Cat_1. Such a thing $\mathcal{C}$ has a collection of objects, and for each pair $a, b \in \mathcal{C}$ of objects a category $\mathcal{C}(a, b)$. The objects of these hom-categories are the morphisms, and the morphisms of these hom-categories are the 2-morphisms. We also require the existence of functors $1_a : 1 \to \mathcal{C}(a, a)$ and $\mathrm{comp}_{a,b,c} : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ for each object $a, b, c \in \mathcal{C}$, satisfying associativity and unitality axioms on the nose.

(2) Another possible model is the one of *bicategories*, that is of categories weakly enriched in over the cartesian monoidal category Cat_1. Roughly speaking a bicategory $\mathcal{C}$ has a collection of objects, and for each pair $a, b \in \mathcal{C}$ of objects a category $\mathcal{C}(a, b)$, and there are again units and compositions functor. The only difference with strict 2-categories is that the associativity and unitality axioms hold only up to an invertible 2-cell (which may not be the identity).

Fortunately, all the models of 2-categories one can imagine are equivalent, so that we don't have really to distinguish them.

Example 3.2. The 2-category of small 1-categories 2-Cat_1 is an example of a strict 2-category. Its objects are the small categories, its 1-morphism are the functor between those and the 2-morphisms are the natural transformation between functors.

We now construct the 2-category of kernels.

Definition 3.3. The *2-category of kernels* Ker is given by the following data.

(1) Objects of Ker are the locally compact Hausdorff spaces.

(2) Given two locally compact Hausdorff spaces X and Y, we let

$$\mathrm{Ker}(X, Y) = \mathrm{hShv}(X \times Y, R)$$

be the homotopy category of $\mathcal{D}(R)$-valued sheaves on the product $X \times Y$. We will refer to objects in $\mathrm{Ker}(X, Y)$ as *Fourier-Mukai kernels*, or simply as *kernels*.

(3) Given three locally compact Hausdorff spaces X, Y and Z, the composition functor

$$\mathrm{comp}_{X,Y,Z} : \mathrm{Ker}(X, Y) \times \mathrm{Ker}(Y, Z) \to \mathrm{Ker}(X, Z), \qquad (F, G) \mapsto p_{XZ,!}(p_{XY}^*(F) \otimes p_{YZ}^*(G))$$

is given by composition of Fourier-Mukai kernels. Here p_{AB} is the projection from the product $X \times Y \times Z$ to the factor $A \times B$, for $A, B = X, Y, Z$.

(4) The identity functor id_X of X is given by $\Delta_!(\mathbb{1}_X)$, where $\Delta : X \to X \times X$ is the diagonal.

The above definition is not complete, since to define a strict 2-category we have also to provide proof of the commutativity of the relevant diagrams (that is, the pentagon for associativity of composition and the two triangle for the unit). We don't have the time (and neither the space) to do so, and we ask the reader to believe us on faith.

Remark 3.4. We point out that there exists a beautiful argument that Ker is indeed a 2-category. The argument works for every 3-functor formalism $\mathcal{D} : \mathrm{Span}(\mathcal{C}) \to \mathrm{Cat}_{(\infty,1)}$ and is as follows. First of all, by using diagonal spans, one shows that every object in the symmetric monoidal $(\infty, 1)$-category $\mathrm{Span}(\mathcal{C})$ is not only dualizable, but actually self dual. But a symmetric monoidal $(\infty, 1)$-category in which every object is dualizable is also closed, hence enriched over itself. In particular, $\mathrm{Span}(\mathcal{C})$ is $\mathrm{Span}(C)$-enriched. Since enrichment can be transferred along lax symmetric monoidal functors, and we have such a thingy $\mathcal{D} : \mathrm{Span}(\mathcal{C}) \to \mathrm{Cat}_{(\infty,1)}$, we deduce that $\mathrm{Span}(\mathcal{C})$ is a $\mathrm{Cat}_{(\infty,1)}$-enriched $(\infty, 1)$-category, and hence an $(\infty, 2)$-category. Take now the homotopy 2-category to construct the 2-category of kernels $\mathrm{Ker}_{\mathcal{D}}$. Our case follows then by noting that Grothendieck operations introduced in section 1 assemble into a 3-functor formalism.

To better express the idea behind 1-cells of Ker as Fourier-Mukai kernels for Fourier-Mukai transforms, it is useful to study the corepresentable 2-functor

$$h^{\mathbb{1}} = \mathrm{Ker}(*, -) : \mathrm{Ker} \to 2\text{-}\mathrm{Cat}_1$$

that is a 2-functor from the 2-category of kernels to the 2-category of 1-categories. Indeed, we can easily see that

(1) On objects $h^{\mathbb{1}}$ sends an object X of $\mathcal{C}$ to the homotopy 1-category $h\mathrm{Shv}(X, R)$. This follows since

$$h^{\mathbb{1}}(X) = \mathrm{Ker}(*, X) = h\mathrm{Shv}(* \times X, R) = h\mathrm{Shv}(X, R).$$

(2) On 1-cells instead, given two objects X and Y the 2-functor $h^{\mathbb{1}}$ provides the functor

$$h\mathrm{Shv}(X \times Y, R) \to \mathrm{Fun}(h\mathrm{Shv}(X, R), h\mathrm{Shv}(Y, R))$$

given on objects by $F \mapsto h^{\mathbb{1}}(F)(-) = p_{Y!}(F \otimes p_X^*(-))$.

(3) Composition is given by convolution of kernels, as in the theory of Fourier-Mukai transforms. The proof of this claim requires the projection formula and proper base-change.

The last ingredient needed for formalizing Poincaré duality is the notion of adjunctions in 2-categories. The following definition is such that in $2\text{-}\mathrm{Cat}_1$, adjunctions in the 2-categorical sense are the usual adjunctions of functors.

Definition 3.5. Let $\mathcal{C}$ be a 2-category and let $f : X \to Y$ be a 1-morphism in $\mathcal{C}$. A *right adjoint* of f is a triple (g, α, β) consisting of a 1-morphism $g : Y \to X$ and 2-morphisms $\alpha : \mathrm{id}_X \Rightarrow gf$ and $\beta : fg \Rightarrow \mathrm{id}_Y$ such that the following composites

$$\begin{tikzcd} f \arrow[r, Rightarrow, "f\alpha"] \arrow[dr, equal, "\mathrm{id}_f"'] & fgf \arrow[d, Rightarrow, "\beta f"] \\ & f \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, Rightarrow, "\alpha g"] \arrow[dr, equal, "\mathrm{id}_g"'] & gfg \arrow[d, Rightarrow, "g\beta"] \\ & g \end{tikzcd}$$

are the identity 2-morphisms. In this case one usually writes $f \dashv g$.

The next result shows some properties of adjunctions in 2-categories.

Lemma 3.6 (Properties of adjunctions)**.**

(1) Adjunctions are unique up to unique isomorphism.

(2) Every functor of 2-categories preserves adjunctions.

(3) The triangle identities detect units.

(4) Triangle identities can be strictified.

Proof. We don't prove point (1) since its proof is essentially the same that we have when working with $2\text{-}\mathrm{Cat}_1$. For (2) it is easy to check that if $F : \mathcal{C} \to \mathcal{D}$ is a 2-functor and $f : X \to Y$ and $g : X \to Y$ are 1-cells in $\mathcal{C}$ equipped with an adjunction $f \dashv g$, with unit $\eta : \mathrm{id}_X \Rightarrow gf$ and counit $\epsilon : fg \Rightarrow \mathrm{id}_Y$, then $F(f) \dashv F(g)$ in $\mathcal{D}$, with unit $F(\eta)$ and counit $F(\epsilon)$.

For the next two claims we fix a 2-category $\mathcal{C}$, and $f : X \to Y$ and $g : Y \to X$ be 1-morphisms in $\mathcal{C}$. Consider now (3). We claim that whenever we have three 2-morphisms $\epsilon : fg \to \mathrm{id}_Y$ and $\eta : \mathrm{id}_X \to fg, \eta' : \mathrm{id}_X \to gf$ such that the composites

$$f \xrightarrow{f\eta'} fgf \xrightarrow{\epsilon f} f, \qquad g \xrightarrow{\eta g} gfg \xrightarrow{g\epsilon} g$$

are the identity, then $\eta = \eta'$. This is straightforward:

$$\begin{aligned} \mathrm{id}_X \xrightarrow{\eta} gf &= \mathrm{id}_X \xrightarrow{\eta} gf \xrightarrow{gf\eta'} gfgf \xrightarrow{g\epsilon f} gf \\ &= \mathrm{id}_X \xrightarrow{\eta'} gf \xrightarrow{\eta gf} gfgf \xrightarrow{g\epsilon f} gf \\ &= \mathrm{id}_X \xrightarrow{\eta'} gf. \end{aligned}$$

Here in the first equality we have applied g to the first triangle identity involving η', in the second one... and in the third one we have just recognised the identity (given by the second triangle precomposed with f).

Consider (4). Suppose that we have 2-morphisms $\eta : \mathrm{id}_X \to gf$ and $\epsilon : fg \to \mathrm{id}_Y$ such that the composite

$$f \xrightarrow{f\eta} fgf \xrightarrow{\epsilon f} f, \qquad g \xrightarrow{\eta g} gfg \xrightarrow{g\epsilon} g$$

are isomorphisms. We claim we can find some $\eta' : \mathrm{id}_X \to gf$ such that (g, η', ϵ) is a right adjoint of f. Changing η by an isomorphism $g \cong g$, we can arrange that $g \xrightarrow{\eta g} gfg \xrightarrow{g\epsilon} g$ is the identity. We can also find some $\eta' : \mathrm{id}_X \to gf$ such that $f \xrightarrow{f\eta'} fgf \xrightarrow{\epsilon f} f$ is the identity. By point (3), we necessarily have $\eta = \eta'$, and we are done. $\square$

Recall that our goal is to give equivalent conditions for $f : X \to Y$ to be cohomologically smooth. In order to simplify the discussion, from now one we will work in the slice category over Y, so that we can assume that Y to be the terminal object.

Definition 3.7. Let $f : X \to *$ be the unique map to the point and let $A \in \mathrm{Shv}(X, R)$ be a sheaf. We say that A is f-*smooth* if $A \in \mathrm{Ker}(X, *)$ is a left adjoint in Ker.

Explicitly, this means we can find $B \in \mathrm{Ker}(*, X)$, that is a sheaf $B \in \mathrm{Shv}(X, R)$, and 2-cells $\eta : \mathrm{id}_X \Rightarrow BA$ and $\epsilon : AB \Rightarrow \mathrm{id}_*$ such that the compositions

$$A \xrightarrow{A\eta} ABA \xrightarrow{\epsilon B} A, \qquad B \xrightarrow{\eta B} BAB \xrightarrow{B\epsilon} B$$

are the identity. More down to earth, we can compute the compositions BA and AB.

(1) For BA let us note that the rule of composition in Ker tells us that we have to compute the span

$$\begin{tikzcd} & & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\ X & & * & & X \end{tikzcd}$$

and the take the pullbacks, so that $BA = p_1^*(A) \otimes p_2^*(B)$.

(2) Similarly, for AB we need to compute the span

$$\begin{tikzcd} & & X \arrow[dl, "="'] \arrow[dr, "="] & & \\ & X \arrow[dl] \arrow[dr, "="'] & & X \arrow[dl, "="] \arrow[dr] & \\ * & & X & & * \end{tikzcd}$$

so that $AB = f_!(A \otimes B)$.

Since $\Delta_!(\mathbb{1}_*) = \mathbb{1}_*$ for the diagonal $\Delta : * \to * \times *$, we deduce that unit and counit are maps

$$\eta : \Delta_!(\mathbb{1}_X) \to p_1^*(A) \otimes p_2^*(B), \qquad \epsilon : f_!(A \otimes B) \to \mathbb{1}_*.$$

Our first result deals with some easy properties of f-smooth objects.

Proposition 3.8. Let $A \in \mathrm{Shv}(X, R)$ be f-smooth, with right adjoint $B \in \mathrm{Shv}(X, R)$. Then:

(1) The object $B \in \mathrm{Shv}(X, R)$ is f-smooth, with right adjoint A.

(2) There is a natural isomorphism of functors $B \otimes f^*(-) \simeq \underline{\mathrm{Hom}}(A, f^!(-)) : \mathrm{Shv}(Y, R) \to \mathrm{Shv}(X, R)$. In particular, $B \simeq \underline{\mathrm{Hom}}(A, f^!(\mathbb{1})) = D_f(A)$ is the (relative) Verdier dual of A.

(3) The Verdier biduality map $A \to D_f(D_f(A))$ is an isomorphism.

(4) The formation of the Verdier dual $D_f(A)$ commutes with any base change.

Proof. Consider first point (1). Its proof follows by noting that $\mathrm{Ker}^{\mathrm{op}} \simeq \mathrm{Ker}$, essentially because the product of two objects is insensitive to the order of the objects (we also need some base change argument to deal with composition). This duality transform left adjoint into right adjoint, exchanging the role of A and B.

For point (2) we note that by applying the functor $h^{\mathbb{1}}$, we get that $B \otimes f_*(-)$ is the right adjoint of $f^!(A \otimes -)$, but that right adjoint is $\underline{\mathrm{Hom}}(A, f^!(-))$. Point (3) is then point (2) applied twice, and point (4) follows since any functor of 2-categories preserves right adjoints by Lemma 3.6. □

We can now discuss the main result.

Theorem 3.9. Let $f : X \to *$ be the unique map to the point. Then f is cohomologically smooth if and only if $\mathbb{1}_X$ is f-smooth and $\omega_f = f^!(\mathbb{1}_*)$ is $\otimes$-invertible.

Proof. Let us start with ($\Rightarrow$). Since f is cohomologically smooth, the dualizing object ω_f is $\otimes$-invertible. We need to show that $\mathbb{1}_X \in \mathrm{Ker}(X, *)$ is f-smooth, and to do that we allow ourselves to make some choices. In particular, we pick $\omega_f \in \mathrm{Ker}(*, X)$ as possible adjoint, so that we have two construct two morphisms

$$\eta : \Delta_!(\mathbb{1}_X) \to p_1^*(\mathbb{1}_X) \otimes p_2^*(\omega_f) = p_2^*(\omega_f), \qquad \epsilon : f_!(\mathbb{1}_X \otimes \omega_f) = f_!(\omega_f) \to \mathbb{1}_*$$

satisfying the triangle identities. The counit ϵ is easy to construct: we just pick the counit of the adjunction $f_! \dashv f^!$ and we evaluate it at 1_*. For the unit, let us note that the $\Delta_! \dashv \Delta^!$ provides us an equivalence

$$\mathrm{Hom}_{\mathrm{Shv}(X \times X, R)}(\Delta_!(\mathbb{1}_X), p_2^*(\omega_f)) \simeq \mathrm{Hom}_{\mathrm{Shv}(X,R)}(\mathbb{1}_X, \Delta^! p_2^*(\omega_f))$$

so that to construct η it suffices to produce an element of the second space. However,

$$\Delta^! p_2^*(\omega_f) \simeq \Delta^! p_1^!(\mathbb{1}_X) \simeq \mathbb{1}_X.$$

Here the first equivalence follows since the dualizing objects are stable under base change (and here we are computing the base change of f along itself), whereas the second equivalence follows by functoriality of upper-!, together with the fact that $p_1 \Delta = \mathrm{id}_X$. In particular, we can pick η to be the adjoint of the identity map on $\mathbb{1}_X$.

We are now left to prove the triangle identities. The first one is straightforward (or at least, if you stare at it long enough you can figure out how to prove it), but for the second one... we need to use the 2-category of kernels. Consider again $h^{\mathbb{1}} : \mathrm{Ker} \to 2\text{-}\mathrm{Cat}_1$. By its explicit description we see that:

(1) $h^{\mathbb{1}}(\mathbb{1}_X) = f_!$,

(2) $h^{\mathbb{1}}(\omega_f) = \omega_f \otimes f^*$.

Since f is cohomologically smooth, we have $f_! \dashv \omega_f \otimes f^*$. But this implies that there are natural transformations

$$\eta_0 : \mathrm{id}_{\mathrm{Shv}(X,R)} \to (\omega_f \otimes f^*) \circ f_! \qquad \epsilon_0 : f_! \circ (\omega_f \otimes f^*) \to \mathrm{id}_{\mathrm{Shv}(*,R)}$$

satisfying the triangle identities in $2\text{-}\mathrm{Cat}_1$.

Notice that ϵ_0 is given by $h^{\mathbb{1}}(\epsilon)$; this follows since we picked ϵ to be the counit of $f_! \dashv f^!$. A note: I was for long confused about this argument, so let me say some words. Until now we have constructed a possible unit and counit for the adjunction $\mathbb{1}_X \dashv \omega_f$, and we know that the first triangle of the triangle identities commutes. We also know that, after applying $h^{\mathbb{1}}$, the counit goes to the counit of the adjunction $f_! \dashv \omega_f \otimes f^*$. This seems enough to conclude: we write the second triangle identity for $f_! \dashv \omega_f \otimes f^*$ and we are done... This is not the case, since the diagram involves the unit η_0, which is not related to η! Fortunately, We are now in the situation of point (3) of Lemma 3.6 : we obtain that η_0 is induced by $h^{\mathbb{1}}(\eta)$, and we are done.

Consider now ($\Leftarrow$). We need to show that f is cohomologically smooth. By assumption, ω_f is $\otimes$-invertible, so we need to show that $f_!$ admits $\omega_f \otimes f^*$ as a right adjoint. We consider again X and the point $*$ as objects of Ker. Since $\mathbb{1}_X \in \mathrm{Ker}(X, *)$ is f-smooth, we can find an object $B \in \mathrm{Ker}(*, X)$, hence a sheaf $B \in \mathrm{Shv}(X, R)$, together with two morphisms

$$\eta : \Delta_!(\mathbb{1}_X) \to p_1^*(\mathbb{1}_X) \otimes p_2^*(B) = p_2^*(B), \qquad \epsilon : f_!(\mathbb{1}_X \otimes B) = f_!(B) \to \mathbb{1}_*$$

satisfying the triangle identities. Once again, we consider the 2-functor $h^{\mathbb{1}} : \mathrm{Ker} \to 2\text{-}\mathrm{Cat}_1$. Since $h^{\mathbb{1}}$ is a functor of 2-categories, it preserves adjunctions by Lemma 3.6. Hence $\mathbb{1}_X \dashv B$ implies $h^{\mathbb{1}}(\mathbb{1}_X) \dashv h^{\mathbb{1}}(B)$. The definition of $h^{\mathbb{1}}$ implies then that

$$f_! = h^{\mathbb{1}}(\mathbb{1}_X) \dashv h^{\mathbb{1}}(B) = B \otimes f^*$$

are adjoint functors $h\mathrm{Shv}(X, R) \to h\mathrm{Shv}(*, R)$. By lifting this adjunction through [Lur25, Tag 02FX], we deduce that the right adjoint $f^!$ is given by a kernel, and in particular it is $\mathrm{Shv}(*, R)$-linear, so $\omega_f \otimes f^* \to f^!$ is an equivalence. Moreover, we must have $B = \omega_f$.

To show stability under any base change of f along $g : Y' \to *$, it suffices to show that the base change along g defines a 2-functor of 2-category $\mathrm{Ker} \to \mathrm{Ker}_{/Y'}$. Here $\mathrm{Ker}_{/Y'}$ is the category of kernels where everything lives over Y'. In particular, this 2-functor preserves everything that we need. □

4 Some computations

The remaining goal of the seminar is to show that topological submersions are cohomologically smooth and deduce relative purity from cohomological smoothness. Let us start with the first task and let us work with $\mathbb{Z}$-coefficients.

Lemma 4.1. The map $f : \mathbb{R} \to *$ is cohomologically smooth.

Proof. We wish to apply Theorem 3.9. First of all, notice that the proof of the $(\Leftarrow)$ can be weakened: we don't have really to look at ω_f, since it suffices to ask for the right adjoint B of $\mathbb{1}_X$ to be $\otimes$-invertible. For this reason, we consider the $\otimes$-invertible object $\mathbb{Z}[1]$ and we show that $\mathbb{1}_{\mathbb{R}}$ is f-smooth by proving that $\mathbb{1}_{\mathbb{R}} \dashv \mathbb{Z}[1]$.

We need to construct a unit $\eta : \Delta_!(\mathbb{1}_{\mathbb{R}}) \to p_2^*(\mathbb{Z}[1])$ and a counit $\epsilon : f_!(\mathbb{Z}[1]) \to \mathbb{1}_*$ satisfying the triangle identities. Let us start with the counit. First of all $\Gamma_c(\mathbb{R}, \mathbb{Z}[1]) = f_!(\mathbb{Z}[1])$. If we now pick an orientation of $\mathbb{R}$, we can use the isomorphism

$$H_c^1(\mathbb{R}, \mathbb{Z}) \cong \mathbb{Z}$$

to construct ϵ as the evaluation on the class of $[\mathbb{R}]$. On the other side, to construct the unit we need a map $\Delta_!(\mathbb{Z}) \to \mathbb{Z}[1]$, where everything lives on $\mathbb{R}^2$. But by considering the distinguished triangle

$$0 \longrightarrow \Delta_! \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow j_! \mathbb{Z} \to 0$$

of sheaves on $\mathbb{R}^2$ for $j : \mathbb{R}^2 \setminus \Delta \to \mathbb{R}^2$ the complementary open immersion, we compute

$$\mathrm{Hom}(\Delta_! \mathbb{Z}[-1], \mathbb{Z}) \simeq \mathbb{Z},$$

giving us η. Indeed, the triangle shows that $\mathrm{Hom}(\Delta_! \mathbb{Z}[-1], \mathbb{Z})$ is the cone of

$$\Gamma(\mathbb{R}^2, \mathbb{Z}) \longrightarrow \Gamma(\mathbb{R}^2 \setminus \Delta, \mathbb{Z}),$$

and this map is the diagonal embedding $\mathbb{Z} \to \mathbb{Z}^2$.

Notice that the choice of η and ϵ depends on the choice of a sign. If the signs are chosen compatibly, then we are in business; but a priori we might worry that this cannot be done. Fortunately, point (4) of Lemma 3.6 tells us that this choice is not necessary. □

Proposition 4.2. Let $f : X \to Y$ be a topological manifold bundle. Then f is cohomologically smooth.

Proof. Cohomological smoothness can be checked locally on the source, and is stable under pullback and passage to open subsets. This reduces the question to $p : \mathbb{R} \to *$. We conclude by applying the Lemma 4.1. □

We now deduce relative purity from cohomological smoothness.

Proposition 4.3 (Realtive purity)**.** Let $f : X \to Y$ be continuous, and assume that ω_Y is $\otimes$-invertible. Then we have an equivalence $f^*(\omega_Y) \otimes \omega_f \to \omega_X$.

Proof. Consider the co-projection map $\omega_f \otimes f^* \to f^!$ and evaluated it at ω_Y to get a map $\omega_f \otimes f^*(\omega_Y) \to f^!(\omega_Y)$. Now by functoriality we have that $f^!(\omega_Y) \simeq \omega_X$, so that we end up with a map $\omega_f \otimes f^*(\omega_Y) \to \omega_X$. Since f^* is symmetric monoidal and ω_Y is $\otimes$-invertible we deduce that $f^*(\omega_Y)$ is $\otimes$-invertible, with inverse $(f^*(\omega_Y))^{-1} \simeq f^*(\omega_Y^{-1})$. In particular, we get equivalences

$$\begin{aligned}
\omega_f \otimes f^*(\omega_Y) &\simeq \underline{\mathrm{Hom}}(f^*(\omega_Y^{-1}), \omega_f) \\
&\simeq f^! \underline{\mathrm{Hom}}(\omega_Y^{-1}, \mathbb{1}) \\
&\simeq f^!(\omega_Y) \\
&\simeq \omega_f.
\end{aligned}$$

Here the second equivalence follows by the projection formula, whereas the third one by definition of $\otimes$-invertible, and the fourth one by construction. Since this equivalence is exactly induced by the map $f^*(\omega_Y) \otimes \omega_f \to \omega_X$, we are done. □

Corollary 4.4. Let $i : X \subseteq Y$ be an inclusion of topological manifolds, and assume that both X and Y are orientable. Let codim(X, Y) be the codimension of X in Y. Then the cohomology of Y supported in X is given by

$$H^n_X(Y, \mathbb{Z}) \simeq H^{n+\mathrm{codim}(X,Y)}(X, \mathbb{Z}).$$

that is, by a shift of the cohomology of the subspace X.

Proof. First of all, notice that since Y is a topological manifold, the dualizing object ω_Y is $\otimes$-invertible; this follows by Lemma 4.1. We now apply relative purity to get an equivalence $i^*(\omega_Y) \otimes \omega_i \simeq \omega_X$, so that $\omega_i \simeq \omega_X \otimes (i^*(\omega_Y))^{-1}$. Since we have an orientation on X and Y, we can trivialize both ω_X and ω_Y, and the right hand side of the previous equivalence turns out to be

$$\omega_i \simeq \mathbb{Z}_X[-\mathrm{codim}(X, Y)].$$

This is not trivial, and requires the study of orientations. In particular we get

$$\begin{aligned} H^n_X(Y, \mathbb{Z}) &\simeq \pi_{-n}\Gamma(X, \omega_i) \\ &\simeq \pi_{-n}\Gamma(X, \mathbb{Z}_X[-\mathrm{codim}(X, Y)]) \\ &\simeq \pi_{-n-\mathrm{codim}(X,Y)}\Gamma(X, \mathbb{Z}_X) \\ &\simeq H^{n+\mathrm{codim}(X,Y)}(X, \mathbb{Z}). \end{aligned}$$

□

References

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel. 3

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