Abstract Neeman Dualities

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Overview

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1. Given a quasi-compact quasi-separated scheme X, it is possible to construct a $(\infty, 1)$-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. This is a stable, compactly-generated and closed symmetric monoidal $(\infty, 1)$-category under the (derived) tensor product $- \otimes - : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$.

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1. Given a quasi-compact quasi-separated scheme X, it is possible to construct a $(\infty, 1)$-category QCoh(X) of quasi-coherent sheaves on X. This is a stable, compactly-generated and closed symmetric monoidal $(\infty, 1)$-category under the (derived) tensor product $- \otimes - : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$.
2. Furthermore, given a morphism $f : X \to Y$ of quasi-compact quasi-separated schemes, the pullback induces a symmetric monoidal and colimit-preserving functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$, the (derived) pullback.

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In particular, the functor

$$\mathrm{QCoh}(Y) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X), \qquad (y, x) \mapsto f^*(y) \otimes x$$

induces on $\mathrm{QCoh}(X)$ the structure of a $\mathrm{QCoh}(Y)$-module (in presentable $(\infty, 1)$-categories).

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The adjoint functor theorem provides then a $\mathrm{QCoh}(Y)$-enrichment of $\mathrm{QCoh}(X)$: that is, there exists a functor

$$\underline{\mathrm{QCoh}(X)} : \mathrm{QCoh}(X)^{\mathrm{op}} \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y), \qquad (x, x') \mapsto f_* \underline{\mathrm{Hom}}_{\mathrm{QCoh}(X)}(x, x').$$

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$\rightsquigarrow$ We are interested in restricting the source.

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and

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2. *If X is separated and of finite type scheme over an excellent scheme of dimension* ≤ 2*, then the restricted dual Yoneda embedding* $\tilde{\text{よ}}$ *induces an equivalence of* $(\infty, 1)$*-categories*

$$\mathrm{Perf}(X)^{op} \to \mathrm{Fun}^{ex}_{\mathrm{Perf}(Y)}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)).$$

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Here $\mathrm{Perf}(-)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(-) \subseteq \mathrm{D}^{-}_{\mathrm{coh}}(-)$ denote the stable $(\infty, 1)$-categories of perfect complexes, bounded and bounded below complexes with coherent (co)homology.

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3. Geometric $\rightsquigarrow$ coherent objects.

Geometric $(\infty, 1)$-categories and functors

We impose from the start an identification between the first two notion of finiteness.

Definition

A stable $(\infty, 1)$-category $\mathcal{C}$ is called geometric if:

1. It has a symmetric monoidal structure $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ compatible with colimits in both variables.
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1. They fit into adjunctions $f^* \dashv f_* \dashv f^{(1)}$. These functors satisfy a projection formula

$$f_*(x) \otimes_{\mathcal{B}} y \xrightarrow{\simeq} f_*(x \otimes_{\mathcal{C}} f^*(y)) \qquad \text{for every } x \in \mathcal{C}, y \in \mathcal{B},$$

and some internal realizations.

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   and some internal realizations.
2. By [BDS16], there exists a sensible Grothendieck-Neeman duality theory.

The enriched Yoneda embedding

Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor. Then the functor

$$\mathcal{B} \times \mathcal{C} \to \mathcal{C}, \qquad (y, x) \mapsto f^*(y) \otimes_{\mathcal{C}} x$$

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We can therefore construct a $\mathcal{B}$-enrichment of $\mathcal{C}$ via

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}, \qquad (x, y) \mapsto \mathcal{C}(x, y) = f_* \underline{\mathrm{Hom}}_{\mathcal{C}}(x, y)$$

by means of the pushfoward f_* and the internal hom $\underline{\mathrm{Hom}}_{\mathcal{C}}$. We think of $\mathcal{C}(x, y)$ as the $\mathcal{B}$-graph of morphisms $x \to y$ in $\mathcal{C}$.

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A deep theorem by Heine (coupled with some easy computations with compactly-generated categories) shows that the there exists a fully-faithful enriched Yoneda embedding $\text{よ} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{B})$ defined by $x \mapsto \mathcal{C}(-, x)$.

Geometric *t*-structures

The third notion of finiteness appears when certain *t*-structures are considered.

Definition

Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category. A geometric *t*-structure is a *t*-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that:

1. The *t*-structure is accessible. That is, $\mathcal{C}_{\geq 0}$ is presentable.
2. The *t*-structure is compatible with filtered colimits. That is, $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$.
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Geometric t-structures

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We will furthermore say that the geometric t-structure is tensor if:

(4) The connective objects $\mathcal{C}_{\geq 0}$ inherits the symmetric monoidal structure of $\mathcal{C}$.

Point (4) ensures a compatibility between the "geometric" objects and the compact-dualizable ones.

Black Box

Let $\mathcal{C}$ *be a geometric* $(\infty, 1)$*-category and* $\mathcal{G} \subseteq \mathcal{C}$ *a collection of compact generators. Then there exists a geometric t-structure* $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ *such that:*

1. *The coconnective objects are given by* $\mathcal{C}_{\leq 0} = \{x \in \mathcal{C} \,|\, \pi_n \mathrm{Hom}_{\mathcal{C}}(g, x) = 0 \text{ for all } g \in \mathcal{G}, n > 0\}$.
2. *Let* $\mathcal{E}$ *be the smallest full subcategory which contains* $\mathcal{G}$ *and is closed under finite colimits and extensions. Then the inclusion* $\mathcal{E} \hookrightarrow \mathcal{C}$ *extends to an equivalence of* $(\infty, 1)$*-categories* $Ind(\mathcal{E}) \to \mathcal{C}_{\geq 0}$.

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Warning

In these slides, we will always assume that every geometric tensor t-structure is in the preferred equivalence class.

Coherent and pseudo-coherent objects

With the datum of a geometric (tensor) t-structure we can define the finite objects in the geometry.

Definition

Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category with a geometric t-structure. We say that an object $x \in \mathcal{C}$ is:

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Remark

If $\mathcal{C}$ admits a connective compact generator $G \in \mathcal{C}_{\geq N}$, then $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are also closed under tensor product with compact objects.

How do we compute $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$?

Theorem

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Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category equipped with a t-structure. Then:

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2. The heart $\mathcal{C}^\heartsuit$ is a locally coherent abelian 1-category.

Quasi-perfect and quasi-proper functors

We now need geometric functors preserving the "geometry". Let $f^* : \mathcal{B} \longrightarrow \mathcal{C}$ be a geometric functor and assume that both $\mathcal{B}$ and $\mathcal{C}$ are equipped with geometric tensor t-structures in the preferred equivalence classes.

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Notice that quasi-perfect satisfy the abstract Grothendieck-Neeman duality. Quasi-proper functors, on the other hand, will satisfy the abstract Neeman dualities.

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Definition

A right t-exact geometric functor $f^* : \mathcal{B} \longrightarrow \mathcal{C}$ is of finite tor-dimension if it is left t-exact up to a shift.

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induced by the restricted Yoneda embedding.

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The proof uses a more general statement about the Yoneda embedding between geometric $(\infty, 1)$-categories and an explicit computation of its kernel.

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Our second (and significantly more involved) duality result is the:

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Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor satisfying the assumption of the first abstract Neeman duality.

Then there exist an equivalence of $(\infty, 1)$-categories

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Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor satisfying the assumption of the first abstract Neeman duality. Assume furthermore that $\mathcal{C}$ admits a morphism of $\mathcal{B}$-universal descent to a regular $(\infty, 1)$-category. Then there exist an equivalence of $(\infty, 1)$-categories

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A geometric $(\infty, 1)$-category is regular if compact and coherent objects coincide.

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3. Can we formulate these results for more general $(\infty, 1)$-categories?
   We are interested in prestable and dualizable $(\infty, 1)$-categories.

Thank you!

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