Abstract Neeman Dualities

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Contents

Introduction **2**

1 Geometric $(\infty, 1)$-Categories **13**
1.1 Definition and Basic Properties . . . 14
1.2 Geometric and Tensor t-Structures . . . 19
1.3 Pseudo-Coherent and Coherent Objects . . . 24
1.4 Coherent $(\infty, 1)$-Categories . . . 26
1.5 Pseudo-Compact Objects . . . 30

2 Duality for Stable $(\infty, 1)$-Categories **33**
2.1 Morita Theory . . . 33
2.2 Duality Theory . . . 35

3 Neeman Dualities **37**
3.1 Quasi-Perfect and Quasi-Proper Functors . . . 38
3.2 Functors out of $\mathcal{C}_c^{op}$ . . . 41
3.3 Morphism of Universal Descent . . . 44
3.4 Functors out of $\mathrm{Coh}(\mathcal{C})$ . . . 47

4 Examples **49**
4.1 Modules . . . 50
4.2 Schemes . . . 53
4.3 Spectral Deligne-Mumford Stacks . . . 56

References **58**

Introduction

Finite objects are generally simpler to manipulate than complex ones and a large part of mathematics focuses on the development of methods that allow us to analyze infinite objects in terms of finite ones. Although this approach is natural, it is not independent of choices. In fact, a working mathematician must first select the appropriate notion of finiteness for the problem at hand and then determine the suitable approximation techniques to use. Among the many areas of mathematics, this paradigm is particularly relevant to derived and triangulated categories, which are the focus of this discussion.

Derived and triangulated categories were introduced by Verdier in his PhD thesis in the mid-1960s. In the 1970s, Illusie proposed in [BJG+71] and [Ill71] that an object in a triangulated category should be considered *finite*—or more precisely, *compact*—if any map out of it commutes with arbitrary coproducts. Many commonly used triangulated categories, such as *compactly generated triangulated categories*, contain a wealth of such objects. This background is well-established in the literature. However, what is more recent and less explored is the choice of approximation techniques, which necessitates the following.

Definition ([Nee18b, Definition 0.21])**.** Let $\mathcal{T}$ be a triangulated category with coproducts. We will say that $\mathcal{T}$ is *weakly approximable* if there exists a compact generator G, a t–structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ and an integer $N > 0$ such that:

(1) $G \in \mathcal{T}_{\geq -N}$ and $\mathrm{Hom}_{\mathcal{T}}(G, \mathcal{T}_{\geq N}) = 0$.

(2) Every object $x \in \mathcal{T}_{\geq 0}$ fits into a triangle $c \to x \to d$ with $c \in \langle\langle G \rangle\rangle^{[-N,N]}$ and $d \in \mathcal{T}_{\geq 1}$.

We will furthermore say $\mathcal{T}$ is *approximable* if the integer N can be chosen to further satisfy:

(3) In the triangle $c \to x \to d$ of (2) we may strengthen the condition on c, we may assume $c \in \langle G \rangle^{[-N,N]}$.

Here $\langle G \rangle^{[-N,N]}$ is the full thick subcategory built from G by taking arbitrary extensions of finite coproducts of the shifts $\Sigma^i G$ for $i \in [-N, N]$. The subcategory $\langle\langle G \rangle\rangle^{[-N,N]}$ is defined similarly; the only difference is that we instead allow infinite coproducts.

Nonetheless, the approximating technique is in assumption (2) for weakly approximable triangulated categories (and (3) for the approximable ones). It says that a 0-connective object $x \in \mathcal{T}_{\geq 0}$ can be approximated by some construction of a finite object G up to a 1-connective error. It also says that the construction can be iterated, thus allowing to approximate x up to smaller and smaller errors. Assumption (1) requires instead a compatibility of the t-structure with the compact generator.

One interesting feature of approximability is that it offers a new perspective on the second notion of finiteness available in triangulated categories. This second notion emerges when a t-structure is considered. Specifically, if the categorical properties of a given triangulated category allow us to define compact objects, then the choice of a t-structure enables us to define two subcategories, $\mathcal{T}^b_c \subseteq \mathcal{T}^-_c$, which consist of "finite objects in the geometry". Objects in $\mathcal{T}^-_c$ are those $x \in \mathcal{T}$ such that, for every integer $n > 0$, there exists a triangle $c \to x \to d$, where $c \in \mathcal{T}_c$ is compact and $d \in \mathcal{T}_{\geq n}$ is n-connective. On the other hand, the objects of $\mathcal{T}^b_c$ are the bounded objects in $\mathcal{T}^-_c$. These subcategories were introduced by Neeman in [Nee18b, Definition 0.16], under the names *bounded pseudo-compact* and *pseudo-compact objects*. The author traced this terminology to [CHNS24, Remark 4.1.5].

The new perspective provided by approximability is further illuminated by the following result, which can be found in [Nee18b, Theorem 0.3]).

Theorem (Neeman, Functors out of $\mathcal{T}^{op}_c$)**.** Let R be a commutative noetherian ring, and let $\mathcal{T}$ be an R-linear triangulated category with coproducts, and suppose it has a compact generator $G \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(-, G)$ is a G-locally finite cohomological functor. Assume further that $\mathcal{T}$ is approximable. Let $\mathcal{T}^-_c$ be the one corresponding to the preferred equivalence class of t–structures. Consider the following functors

$$\mathcal{T}^b_c \xrightarrow{i} \mathcal{T}^-_c \xrightarrow{よ} \mathrm{Hom}_R((\mathcal{T}_c)^{op}, R\text{-Mod}).$$

Then:

(1) The restricted Yoneda $よ$ is full, and its essential image consists of locally finite cohomological functors.

(2) The composition $よ \circ i$ is fully-faithful, and its essential image consists of finite cohomological functors.

In the statement, the *locally finite* cohomological functors are those R-linear cohomological functors $f : \mathcal{T}^{op} \to$ R-Mod such that, for every $x \in \mathcal{T}$, the R–module $f(\Sigma^i x)$ is finite for all $i \in \mathbb{Z}$ and vanishes for $i << 0$. The *finite* ones are those locally finite f such that, in addition to the above, $f(\Sigma^i x)$ vanishes for $i >> 0$.

Still in the word of approximable triangulated categories, we have a second and significantly more involved result, which classifies *locally finite* (and *finite*) homological functors. The locally finite homological functors are those homological functors $f : \mathcal{T} \to$ R-Mod such that, for every $x \in \mathcal{T}$, the R–module $f(\Sigma^i x)$ is finite for all $i \in \mathbb{Z}$ and vanishes for $i << 0$, whereas the finite ones satisfy, in addition to the above, $f(\Sigma^i x)$ vanishes for $i >> 0$.

Theorem (Neeman, Functors out of $\mathcal{T}^b_c$)**.** Let R be a commutative noetherian ring, and let $\mathcal{T}$ be an R-linear category, approximable triangulated category, and assume there is a compact generator $H \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(H, H[n])$ is a finite R–module for all $n \in \mathbb{Z}$. Let $\mathcal{T}^-_c$ and $\mathcal{T}^b_c$ be the ones corresponding to the preferred equivalence class of t–structures, and assume there is an object $G \in \mathcal{T}^b_c$ and an integer $N > 0$ with[1] $\mathcal{T} = \langle\langle G \rangle\rangle_N$. Consider the following functors

$$(\mathcal{T}^c)^{op} \xrightarrow{\tilde{\iota}} (\mathcal{T}^-_c)^{op} \xrightarrow{\tilde{\mathcal{Y}}} \mathrm{Hom}_R(\mathcal{T}^b_c, \text{R-Mod}).$$

Then:

(1) The restricted Yoneda $\tilde{\mathcal{Y}}$ is full, and its essential image consists of locally finite homological functors.

(2) The composition $\tilde{\mathcal{Y}} \circ \tilde{\iota}$ is fully-faithful, and its essential image consists of finite homological functors.

The reader can find it in [Nee18a, Theorem 0.4].

Overview

The objective of these pages is to prove a generalization of Neeman, Functors out of $\mathcal{T}^{op}_c$ and Neeman, Functors out of $\mathcal{T}^b_c$. The main goals are the followings.

(1) First of all, we would like to enhance Neeman's theorems. There are three natural candidates for enhancing triangulated categories: differential graded categories, A_∞-categories and stable $(\infty, 1)$-categories. Since all the three approaches have been shown to be equivalent in [Orn16] and [Coh16], we will work in the more natural and developed framework of *stable* $(\infty, 1)$*-categories*. Furthermore, because Neeman's results are formulated for linear triangulated categories, the appropriate generalizations should occur within the context of *enriched* $(\infty, 1)$*-categories*. A review of $(\infty, 1)$-categories and enriched $(\infty, 1)$-categories can be found in the following sections.

(2) Secondly, we aim to allow enrichments over more complicated bases. To clarify this goal, note that Neeman's statements can be interpreted in terms of schemes over an affine base. More precisely, if X is a proper scheme over a noetherian ring R, then a non-trivial result [Nee18b, Example 3.6] shows that the derived category of quasi-coherent sheaves $\mathcal{D}_{qc}(X)$ on X is approximable and has a single compact generator satisfying the assumptions of Neeman, Functors out of $\mathcal{T}^{op}_c$. The machinery of Neeman, Functors out of $\mathcal{T}^{op}_c$ then implies that every (locally) finite cohomological functor $\mathcal{D}_{qc}(X)^{op}_c \to$ R-Mod is represented by an object of $\mathcal{D}_{qc}(X)^b_c$ (or $\mathcal{D}_{qc}(X)^-_c$ in the local case). These subcategories are not mysterious: under the noetherianess assumption on X, they are actually $\mathcal{D}^b_{coh}(X)$ and $\mathcal{D}^-_{coh}(X)$. Now, if X is quasi-excellent and finite dimensional (or every closed subvariety admits a regular alteration in the sense of de Jong [DJ96]), then the assumptions of Neeman, Functors out of $\mathcal{T}^b_c$ are satisfied. Thus, every (locally) finite homological functor $\mathcal{D}^b_{coh}(X) \to$ R-Mod is represented by an object of $\mathcal{D}_{qc}(X)_c$ (or $\mathcal{D}_{qc}(X)^-_c$ in the local case).

A more general result should allow R to be a non-affine scheme, leading to a relative result

$$X \xrightarrow{\text{proper}} \mathrm{Spec}(R) \qquad \rightsquigarrow \qquad X \xrightarrow{\text{proper}} Y.$$

The noetherian assumption on R (and hence X) should be then regarded only as a computational tool.

[1] The subcategory $\langle\langle G \rangle\rangle_N$ is defined to be the full thick subcategory of $\mathcal{T}$ which is closed under coproducts and at most N extensions spanned by the shifts $\Sigma^i G$, for $i \in \mathbb{Z}$.

(3) Finally, it seems possible to remove the *approximability* assumption from Neeman, Functors out of $\mathcal{T}_c^{op}$. Let $\mathcal{T}$ and G be as in the theorem. Neeman's proof shows that every locally finite cohomological functor admits a *strong* $\langle G\rangle_n$*-approximating system*, as described in [Nee18b, Definition 7.3]. More specifically, if $f : \mathcal{T}_c^{op} \to R\text{-Mod}$ is a locally finite cohomological functor, then Proposition 7.10 in loc. shows that there exists a filtered object $f_1 \to f_2 \to \dots$ and an isomorphism $\text{よ}(\mathrm{colim}_{i\geq 1} f_i) \to f$. Since Remark 7.9 in loc. clarifies that these f_i are in $\mathcal{T}_c^-$, the theorem is nearly proved. However, approximability is still required only to show that $\mathrm{colim}_{i\geq 1} f_i$ lands in $\mathcal{T}_c^-$.

The third point seems the more unnatural, since we expect approximability to be, well, the approximation technique of triangulated categories. We will achieve this result by exploiting the relative point of view proposed in the second point and the higher categorical language of the first one. Indeed, our generalizations of Neeman, Functors out of $\mathcal{T}_c^{op}$ and Neeman, Functors out of $\mathcal{T}_c^{b}$ will be placed in the realm of *rigidly-compactly generated* $(\infty, 1)$*-categories* and rigid functors. The Ind-completion is what implements the choice of approximations[2].

Remark (On Finiteness)**.** Working with rigidly-compactly generated $(\infty, 1)$-categories (which, from now on, we will refer to as *geometric* $(\infty, 1)$*-categories*) allows us to establish an equality between two different notions of finiteness right from the start. Indeed, by definition, a geometric $(\infty, 1)$-category is a symmetric monoidal $(\infty, 1)$-category compactly generated by its dualizable objects, so that the *categorical* and *monoidal* structures coincide. This equality between *compact* and *dualizable objects* enables us to employ methods from both worlds. The third notion of finiteness will appear in our generalization of Neeman's statements since, by introducing the input of a t-structure, the finite objects in the t-structure are not immediately related to the other two finite objects. The challenge of understanding the relationships between these three (or, in reality, two) notions of finiteness is what makes the functional analysis of stable $(\infty, 1)$-categories so interesting[3]. In any case, the finite objects in the t-structure, the *coherent objects*, are the ones that capture the *geometry*. Indeed, one might think of the choice of a t-structure as a choice of a geometry on a geometric $(\infty, 1)$-category $\mathcal{C}$. This idea is strongly supported when $\mathcal{C}$ arises as the derived $(\infty, 1)$-category of quasi-coherent sheaves on a noetherian scheme X, since by [GS23, Theorem 0.1] the set of aisles of compactly generated tensor t-structures on $\mathrm{QCoh}(X)$ is in bijective correspondence with the set of Thomason filtrations of X. To conclude, our generalization of Neeman's statements, which is placed in the realm of geometric $(\infty, 1)$-categories, has to be thought as a new piece in the field of functional analysis of category theory.

Let us explain how the material is organized. Our work towards a generalization of Neeman's results begins in Chapter 1, where we introduce geometric $(\infty, 1)$-categories and geometric functors. Specifically, in Section 1.1, we define a geometric $(\infty, 1)$-category as a compactly generated stable $(\infty, 1)$-category $\mathcal{C}$ equipped with a symmetric monoidal structure that is compatible with colimits, and such that the compact objects coincide with the dualizable objects. These $(\infty, 1)$-categories are specializations of *stable homotopy theories*, since they are required to be compactly generated by the dualizable objects. A *geometric functor* is then a colimit-preserving, symmetric monoidal functor $f^* : \mathcal{B} \to \mathcal{C}$. This definition has many consequences. For instance, every geometric functor produces a double adjunction $f^* \dashv f_* \dashv f^{(1)}$ and satisfies a projection formula (see Proposition 1.1.9). Moreover, under certain finiteness assumptions, geometric functors satisfy a categorical version of *Grothendieck-Neeman duality* (see Theorem 1.1.11).

Next, in Section 1.2, we introduce the third notion of finiteness by incorporating a *geometric* t*-structure*. A t-structure on a presentable stable $(\infty, 1)$-category is said to be geometric if it is accessible, compatible with filtered colimits, and right complete. Our main result is Lemma 1.2.12, which shows that if a presentable stable $(\infty, 1)$-category $\mathcal{C}$ is equipped with a collection of compact generators, then there exists a geometric t-structure whose connective part $\mathcal{C}_{\geq 0}$ is compactly generated by the given collection. We can view this result as providing an interaction between the categorical and geometric structures. To ensure an interaction with the monoidal structure, we restrict to t-structures whose connective part also inherits a monoidal structure. We call these *tensor* t*-structures*. The monoidal unit $\mathbb{1}_{\mathcal{C}}$ always determines a tensor t-structure. If the $(\infty, 1)$-category is also equipped with a connective compact generator, this standard t-structure defines a *geometric tensor* t*-structure*.

[2] The language of $(\infty, 1)$-categories is really needed here. Indeed, it is well known that the Ind-completion of a small triangulated category need not to be triangulated.

[3] Who knows, in the future, it may become interesting to formulate results where the categorical and monoidal structures do not coincide.

We will use geometric t-structures to define *pseudo-coherent objects* $\mathrm{PCoh}(\mathcal{C})$, essentially by adopting Lurie's definition of almost perfect complexes. In Section 1.3, we show that the subcategories spanned by coherent and pseudo-coherent objects, $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$, are stable thick subcategories, and that the 0-connective pseudo-coherent objects are closed under geometric realizations. However, the main result of this section is Remark 1.3.9. This result demonstrates that, under certain mild generation assumptions, the standard t-structure is in the preferred equivalence class, and is geometric. Furthermore, it shows that coherent and pseudo-coherent objects are closed under tensoring with compacts.

To explicitly compute the coherent and pseudo-coherent objects, in Section 1.4 we introduce *coherent t-structures*. Roughly speaking, a t-structure is called coherent if it is generated by a single compact generator G, and the heart defines a locally coherent abelian 1-category. We will also require that the higher homotopy groups $\pi_n G$ of the compact generator be compact in the heart. With this notion in hand, we will prove the following.

Theorem. Let $\mathcal{C}$ be a presentable stable $(\infty,1)$-category equipped with a coherent t-structure. Then:

(1) $\mathrm{Coh}(\mathcal{C})^\heartsuit = \mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^\heartsuit$ consists precisely of the compact objects of $\mathcal{C}^\heartsuit$.

(2) $x \in \mathrm{PCoh}(\mathcal{C})$ if and only if $\pi_n x \in \mathrm{Coh}(\mathcal{C})^\heartsuit$ and $\pi_n x = 0$ for $n << 0$.

(3) $x \in \mathrm{Coh}(\mathcal{C})$ if and only if $\pi_n x \in \mathrm{Coh}(\mathcal{C})^\heartsuit$ and $\pi_n x = 0$ for all but finitely many n.

In particular, $\mathrm{PCoh}(\mathcal{C})$ is the left t-completion of $\mathrm{Coh}(\mathcal{C})$.

In the last section, Section 1.5, we prove that, under certain mild assumptions, our pseudo-coherent objects coincide with Neeman's pseudo-compact objects. The result is Proposition 1.5.7. The proof will require a compact generator that satisfies assumption (1) of the definition of approximable triangulated categories. This will conclude Chapter 1.

In Chapter 2, we study the categorical properties of all the different $(\infty,1)$-categories we have introduced so far. We begin in Section 2.1 by analyzing the $(\infty,1)$-category $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega}$, which describes *compactly generated stable homotopy theories*. In particular, we will show that taking compact objects exhibits $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega}$ as equivalent to $\mathrm{Cat}_{(\infty,1)}^{\mathrm{perf}}$, the $(\infty,1)$-category of stable idempotent-complete $(\infty,1)$-categories, with the Ind-completion serving as the quasi-inverse. We will then show that both of these $(\infty,1)$-categories can be equipped with symmetric monoidal structures, and that Ind-completing refines to a symmetric monoidal equivalence. Finally, we will identify the $(\infty,1)$-category of geometric $(\infty,1)$-categories with a full subcategory of $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega})$ and discuss the construction of limits in the former. We will then prove in Section 2.2 the main result of the chapter.

Theorem (Geometric Categories define Frobenius Algebra Objects)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor, and let $\Gamma : \mathcal{C} \to \mathcal{B}$ denote the functor given by $\Gamma(x) = \mathcal{C}(\mathbb{1}_{\mathcal{C}}, x)$. Then $(\mathcal{C}, \Gamma)$ is a Frobenius algebra object of $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega})$. In other words, the composite map

$$u : \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{C} \xrightarrow{\Gamma} \mathcal{B}$$

is a duality datum in the symmetric monoidal $(\infty,1)$-category $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega})$. In particular, $\mathcal{C}$ is not only dualizable in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\omega})$, but also self-dual.

Our generalization of Neeman's results will occupy the all of Chapter 3. We begin with Section 3.1 where we will introduce *quasi-perfect* and *quasi-proper* functors. These are t-geometric functors $f^* : \mathcal{B} \to \mathcal{C}$, meaning geometric functors that are also right t-exact, and whose right adjoint f_* is right t-exact up to a finite shift and preserves compact and pseudo-coherent objects, respectively. In particular, quasi-perfect functors are the one satisfying the abstract Grothendieck-Neeman duality, while quasi-proper functors will be the main object of Neeman dualities (which we still need to formulate). Corollary 3.1.7 shows that every quasi-perfect functor is quasi-proper[4], establishing a relationship between these two dualities.

Next, in Section 3.2, we prove our generalization of Neeman, Functors out of $\mathcal{T}_c^{\mathrm{op}}$. The proof crucially relies on the fact that for a geometric functor $f^* : \mathcal{B} \to \mathcal{C}$, the enriched Yoneda embedding induces an equivalence of $(\infty,1)$-categories $\mathcal{C} \to \mathrm{Fun}_{\mathcal{B}_c}^{\mathrm{ex}}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{B})$. We then have two possible strategies. Either we restrict

[4]We also expect that the converse should hold under the assumption of finite tor-dimension.

the Yoneda embedding at the source and prove that it still gives an equivalence with some subcategory of functors, or either we restrict the target of the Yoneda embedding and identify its kernel. The second approach appears more tractable and leads to the following result, which we will refer to as the *first abstract Neeman duality*.

Theorem (Functors out of $\mathcal{C}_c^{op}$)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor. Assume that $\mathcal{B}$ is coherent. Assume furthermore that the compact generator G of $\mathcal{C}$ is such that $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ detects connective and coconnective objects and that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some integer $N > 0$. Then there are equivalences of $(\infty, 1)$-categories

$$\operatorname{PCoh}(\mathcal{C}) \to \operatorname{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \operatorname{PCoh}(\mathcal{B})), \qquad \operatorname{Coh}(\mathcal{C}) \to \operatorname{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \operatorname{Coh}(\mathcal{B}))$$

induced by the restricted Yoneda embedding.

We will then turn our attention to our generalization of Neeman, Functors out of $\mathcal{T}_c^b$. We will begin in Section 3.3 by introducing *morphisms of universal descent*, following [Mat16] and [BS17]. Roughly speaking, these are geometric functors for which the source can be written as a totalization of augmented cosimplicial object built from the target. The main results of the section are Proposition 3.3.8 and Lemma 3.3.12. With these results in our hands, we finally prove in Section 3.4 our generalization of Neeman, Functors out of $\mathcal{T}_c^b$. We will first define *regular* $(\infty, 1)$*-categories* as those geometric $(\infty, 1)$-categories compactly generated by their coherent objects, so that a generalization of Neeman, Functors out of $\mathcal{T}_c^b$ is immediate. We will then exploit morphisms of universal descent with target a regular $(\infty, 1)$-categories to deduce the *second abstract Neeman duality* from the first one.

Theorem (Functors out of $\operatorname{Coh}(\mathcal{C})$)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor satisfying the assumption of the first abstract Neeman duality. Assume furthermore that $\mathcal{C}$ admits a morphism of $\mathcal{B}$-universal descent to a regular $(\infty, 1)$-category. Then there exists an equivalence of $(\infty, 1)$-categories

$$\mathcal{C}_c^{op} \to \operatorname{Fun}^{ex}_{\mathcal{B}_c}(\operatorname{Coh}(\mathcal{C}), \operatorname{Coh}(\mathcal{B}))$$

induced by the restricted dual Yoneda embedding.

The "Assume furthermore" part of the theorem is the main obstruction. Indeed, the existence of *morphism of* $\mathcal{B}$*-universal descent* to a regular $(\infty, 1)$-category is not known in general, due to the technical definition of $\mathcal{B}$-morphism of universal descent (Remark 3.3.11).

Remark. We would like also to point out that some generalization of Neeman's results is already available. The master's thesis of Paul Bärnreuther [BÏ8] generalizes Neeman, Functors out of $\mathcal{T}_c^b$. His result, which is a *dg-enhancement* of Neeman's theorem, works under the assumption $\mathcal{T}_c \subseteq \mathcal{T}_c^b$.

Finally, we Chapter 4 will turn our attention to applications. We will first describe the case of *modules* $(\infty, 1)$*-categories* in Section 4.1. We will work in great generality, but the example to keep in mind is the $(\infty, 1)$-category of spectra Sp. The main result is the following.

Theorem (Functors out of $\mathcal{C}_c^{op}$ for module categories)**.** Let $\mathcal{C}$ be a t-geometric $(\infty, 1)$-category and assume that monoidal unit is a compact generator. Assume also that the t-structure is induced by Lemma 1.2.12. Let $f : x \to y$ be a finitely presented map in $\operatorname{CAlg}(\mathcal{C})$ between coherent objects with y connective in $\operatorname{Mod}_x(\mathcal{C})$. Then the restricted Yoneda embedding induces equivalences

$$\operatorname{PCoh}(y) \to \operatorname{Fun}^{ex}_{\operatorname{Perf}(x)}(\operatorname{Perf}(y)^{op}, \operatorname{PCoh}(x)), \qquad \operatorname{Coh}(y) \to \operatorname{Fun}^{ex}_{\operatorname{Perf}(x)}(\operatorname{Perf}(y)^{op}, \operatorname{Coh}(x))$$

of $(\infty, 1)$-categories. Here $\operatorname{Perf}(-)$ and $\operatorname{Coh}(-) \subseteq \operatorname{PCoh}(-)$ stand for the full subcategory of perfect, that is compact, coherent and pseudo-coherent objects.

The proof will not rely on the first abstract Neeman duality, but rather on the more general statement Theorem 3.2.2.

More concrete examples will be presented in Section 4.2, where we will deal with schemes, and in Section 4.3 where we will discuss spectral Deligne-Mumford stacks. Our result for schemes is the following. It can be regarded as a generalization of [Nee18b, Corollary 0.5] and [Nee18a, Theorem 0.2].

Corollary. Let $f : X \to Y$ be a proper map and assume that Y is noetherian.

(1) Then we have equivalences of $(\infty,1)$-categories

$$\mathrm{D}^{-}_{\mathrm{coh}}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{D}^{-}_{\mathrm{coh}}(Y)), \qquad \mathrm{D}^{b}_{\mathrm{coh}}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{D}^{b}_{\mathrm{coh}}(Y)).$$

induced by the the restricted Yoneda embedding.

(2) Assume that X is separated and of finite type scheme over an excellent scheme of dimension ≤ 2. Then we have an equivalence of $(\infty,1)$-categories

$$\mathrm{Perf}(X)^{\mathrm{op}} \to \mathrm{Fun}_{\mathrm{Perf}(Y)}(\mathrm{D}^{b}_{\mathrm{coh}}(X), \mathrm{D}^{b}_{\mathrm{coh}}(Y))$$

induced by the the restricted dual Yoneda embedding.

In the statement, $\mathrm{Perf}(-)$ and $\mathrm{D}^{b}_{\mathrm{coh}}(-) \subseteq \mathrm{D}^{-}_{\mathrm{coh}}(-)$ denote the stable $(\infty,1)$-categories of perfect complexes, bounded and bounded below complexes with coherent (co)homology.

The first two equivalences are a consequence of the first abstract Neeman duality, whereas the second a consequence of the second abstract Neeman duality (and some results by de Jong on regular alterations). In the case of spectral Deligne-Mumford stacks we have the following.

Corollary. Let $f : X \to Y$ be a morphism of finite cohomological dimension of quasi-compact quasi-separated spectral algebraic spaces which is proper and locally almost of finite presentation. Assume that $\mathrm{QCoh}(X)$ comes equipped with a compact generator G such that $\pi_0 \mathrm{Hom}_{\mathrm{QCoh}(X)}(G, \mathrm{QCoh}(X)_{\geq N}) = 0$ for some integer $N > 0$. Assume also that Y is noetherian. Then we have equivalences of $(\infty,1)$-categories

$$\mathrm{PCoh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{PCoh}(Y)), \qquad \mathrm{Coh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{Coh}(Y)).$$

induced by the restricted Yoneda embedding. As before, $\mathrm{Perf}(-)$ and $\mathrm{Coh}(-) \subseteq \mathrm{PCoh}(-)$ denote the stable $(\infty,1)$-categories of perfect complexes, coherent and pseudo-coherent sheaves.

Unfortunately we do not know any application of the second abstract Neeman duality in $\mathbb{E}_\infty$-geometry.

Some Words on $(\infty, 1)$-Categories

In these pages, we will freely speak the language of $(\infty,1)$-categories. In a nutshell, an $(\infty,1)$-category should consist of a set of objects and a set of 1-morphisms between these objects. But, as opposed to 1-categories, an $(\infty,1)$-category also has 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, and so on. But now that we've started counting, and once mathematicians start counting, they cannot stop: this is the ∞ appearing in $(\infty,1)$-categories. The 1 stands for reminding us that all the k-morphisms for $k > 1$ are "invertible", at least up to higher invertible morphisms.

Despite the very simple intuition, $(\infty,1)$-category theory is generally considered an impenetrable subject by non-experts in the field. This is not the case for 1-category theory, and the reason is quite simple. The higher coherence conditions needed to define what an $(\infty,1)$-category *is* make it impossible to write an explicit definition of $(\infty,1)$-categories. As simple as it is, if it is not possible to define them, it is not possible to use them.

Nonetheless, many natural objects in algebraic topology, homological algebra, and algebraic geometry naturally exhibit the structure of an $(\infty,1)$-category. Experience tells us that this higher structure cannot simply be discarded, as it leads to many unpleasant features. Great work has been done by Quillen [Qui06] and many others to systematically organize these higher morphisms and coherence conditions. The common belief is that behind any *homotopical* information, there should be an $(\infty,1)$-category capturing it, and *model categories* are exactly designed for this purpose. In particular, it is now accepted that $(\infty,1)$-categories appear as objects in some model category of "$(\infty,1)$-categories". Unfortunately, this perspective has an enormous drawback: model categories of $(\infty,1)$-categories can be presented in a variety of formulations, connected via a zig-zag of Quillen equivalences.

Among all such models, it is worth mentioning topological categories, simplicial enriched categories, Segal categories, and last but not least, *quasi-categories*. We refer the reader to the survey [Ber09] for a comparison of all these models. For the purpose of this introduction, we would like to closely analyze the most successful definition of $(\infty, 1)$-categories. Joyal's theory of quasi-categories, introduced in [Joy02] and fully developed in Lurie's book [Lur09], stands out among all models for its simplicity and categorical behavior. Indeed, within Joyal and Lurie's framework, an $(\infty, 1)$-category is just a simplicial set that satisfies a weakened form of the Kan condition, which ensures the fillability of certain horns. Roughly speaking, the fillable horns provide all the higher coherence conditions needed for our simplistic idea of $(\infty, 1)$-categories. On the other hand, the categorical behavior of Joyal and Lurie's framework becomes apparent when we look for standard category-theoretic concepts. For example, their theory supports the notion of limits and colimits [Lur09, Chapter 4], a Yoneda embedding [Lur09, Section 5.1], adjunctions [Lur09, Section 5.2], ind-categories and compact objects [Lur09, Section 5.3], and presentable categories [Lur09, Section 5.5]. The list can go on, especially if we look at Lurie's second book [Lur17]. There, all the tools needed for commutative algebra are developed. In particular, quasi-categories support the notion of operads [Lur17, Chapter 2], which allows the construction of (symmetric) monoidal categories [Lur17, Chapter 2], associative and commutative algebra objects [Lur17, Section 4.1], as well as an entire hierarchy of operads interpolating increasing commutativity conditions [Lur17, Section 5.1], left and right modules [Lur17, Section 4.2], and bimodules [Lur17, Section 4.3]. The list can go even further.

Despite all of these successes, it is unnatural to treat quasi-categories as the true model of $(\infty, 1)$-category theory. In fact, none of the above mentioned "presentations of $(\infty, 1)$-categories" should prevail over the others. Many arguments can be found to support this idea, ranging from technical to philosophical ones. Our argument is of the latter kind. Theoretically speaking, it is possible to argue that the theory of quasi-categories is so successful because all the problems we have encountered in the framework of $(\infty, 1)$-category theory can be (more or less easily) solved within the framework of quasi-categories. There is no reason for this to be the case in the future: mathematics may produce a problem that cannot be solved within (or, hopefully, the solution is not within reach) the framework of quasi-categories.

Such arguments lead higher category theorists to wonder if it is possible to work in a *model-independent* way. Even if a fully model-independent foundation for $(\infty, 1)$-category theory has not yet been established, it is possible to work model-independently, thanks to the work of Riehl and Verity [RV16]. Their theory of ∞-*cosmoi* allows us to treat all models of $(\infty, 1)$-categories on the same footing. The drawback is that ∞-cosmoi require quasi-categories to be defined. Nonetheless, ∞-cosmoi delineate a new perspective on $(\infty, 1)$-categories: rather than asking what an $(\infty, 1)$-category *is*, mathematicians should only know how to manipulate them. This *synthetic* approach is taken to the extreme in the work of Cisinski, Cnossen, Nguyen, and Walde [CCNW24], which seems to construct $(\infty, 1)$-category theory axiomatically[5].

The takeaway from this section is that many articles in the literature are now written without even mentioning simplicial sets or any particular model of $(\infty, 1)$-categories and these pages will not be any less.

Enriching $(\infty, 1)$-Categories

Neeman's theorems involve R-linear triangulated categories, placing their statement within the framework of enriched 1-category theory. The corresponding $(\infty, 1)$-categorical counterpart requires *enriched* $(\infty, 1)$-*categories*, and since this theory is *difficult* and still relatively new, we provide a brief overview in this section, along with detailed references for the reader's convenience. We warn the reader: this section is rather technical compared to the previous one. If the reader feels confident enough to believe that there is a sensible Yoneda embedding for enriched $(\infty, 1)$-categories, then it is possible to jump directly to Chapter 1.

As for $(\infty, 1)$-categories, the idea behind enriched $(\infty, 1)$-categories is quite simple. Suppose that $\mathcal{B}$ is a monoidal $(\infty, 1)$-category, that is an $(\infty, 1)$-category equipped with a tensor product $- \otimes_{\mathcal{B}} - : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ satisfying the usual associativity and unital conditions up to higher invertible morphisms. Roughly speaking, a $\mathcal{B}$-enriched $(\infty, 1)$-category $\mathcal{C}$ should consist of a space of objects $\mathcal{C}^{\simeq}$, and, for every pair of objects $x, y \in \mathcal{C}^{\simeq}$, a $\mathcal{B}$-object $\mathcal{C}(x, y)$ called the *graph of* $\mathcal{C}$. These data should come with composition morphisms $\mathcal{C}(x, y) \otimes_{\mathcal{B}} \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ and identity morphisms $\mathbb{1}_{\mathcal{B}} \to \mathcal{C}(x, x)$ equipped with the usual and coherence conditions. Under suitable assumptions on $\mathcal{B}$, the collection of (small) $\mathcal{B}$-enriched $(\infty, 1)$-categories should assemble into an $(\infty, 1)$-category $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$. In particular, $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ should enjoy all the feature of enriched

[5]We point out that this work is still under construction.

1-category theory: it should be functorial in $\mathcal{B}$ as well as inheriting a tensor product from $\mathcal{B}$, it should have a sensible theory of weighted limits and colimits, a Yoneda embedding, and so on and so forth. There are essentially two ways of achieving this construction, each one with its one strengths and weaknesses.

The first one is due to Gepner-Haugseng [GH15] and its based on *categorical algebras*. Gepner-Haugseng associate to every space $\mathcal{C}^{\simeq}$ a (generalized) non symmetric $(\infty,1)$-operad $\Delta^{\mathrm{op}}_{\mathcal{C}^{\simeq}}$. This operad is constructed by right Kan extending the inclusion $\{0\} \hookrightarrow \Delta^{\mathrm{op}}$ to a functor $\mathrm{Cat}_{(\infty,1)} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_{(\infty,1)})$, restricting it to spaces $\mathrm{Spc} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_{(\infty,1)})$ and then applying the $(\infty,1)$-Grothendieck construction to produce a cocartesian fibration $\Delta^{\mathrm{op}}_{\mathcal{C}^{\simeq}} \to \Delta^{\mathrm{op}}$. Given a monoidal $(\infty,1)$-category $\mathcal{B}$, categorical algebras in $\mathcal{B}$ with space of objects $\mathcal{C}^{\simeq}$ are then simply $\Delta^{\mathrm{op}}_{\mathcal{C}^{\simeq}}$-algebras in $\mathcal{B}$. These objects encode the combinatorics of composition by satisfying a Rezk-Segal condition, see [GH15, Section 2.4].

By letting the space vary, categorical algebras assemble into an $(\infty,1)$-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{B})$, which, unfortunately, does not exhibit $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ straightaway. The reason is simple. Consider the case where $\mathcal{B}$ is the $(\infty,1)$-category of spaces Spc endowed with the cartesian monoidal structure. By [GH15, Section 4.4], categorical algebras in Spc correspond to Segal spaces. Since they do not form a model for $(\infty,1)$-categories, the $(\infty,1)$-category $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Spc})$ cannot define[6] $\mathrm{Cat}_{(\infty,1)}$. *Completeness* is missing. Fortunately, it is possible to localize $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{B})$ at the complete $\mathcal{B}$-enriched precategories to obtain a Bousfield localization

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{B}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$$

which constructs the $(\infty,1)$-category of $\mathcal{B}$-enriched $(\infty,1)$-categories $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$. This $(\infty,1)$-category has many of the expected properties.

(1) It is functorial in $\mathcal{B}$ with respect to lax monoidal functors of monoidal $(\infty,1)$-categories by [GH15, Corollary 5.7.6]. To be precise, if $f : \mathcal{B} \to \mathcal{B}'$ is lax monoidal, then a $\mathcal{B}$-enriched $(\infty,1)$-category $\mathcal{C}$ is sent to the $\mathcal{B}'$-enriched $(\infty,1)$-category $f_*\mathcal{C}$ whose space of objects is the same of $\mathcal{C}$, but whose graph of morphism is given by applying f to the graph of $\mathcal{C}$.

(2) Moreover, if $\mathcal{B}$ is a presentably monoidal $(\infty,1)$-category, then $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ is presentable, and $\mathrm{Cat}^{(-)}_{(\infty,1)}$ is lax monoidal with respect the tensor product of presentable $(\infty,1)$-categories thanks to [GH15, Proposition 5.7.8 and Proposition 5.7.16].

(3) If $\mathcal{B}$ is an $\mathbb{E}_n$-monoidal $(\infty,1)$-category then $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ is $\mathbb{E}_{n-1}$-monoidal. In particular, if $\mathcal{B}$ is symmetric monoidal then so is $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ by [GH15, Corollary 5.7.12]. Moreover, consider two presentably monoidal $(\infty,1)$-categories $\mathcal{B}$ and $\mathcal{B}'$. Consider also a colimit preserving monoidal functor $f^* : \mathcal{B} \to \mathcal{B}'$ and its right adjoint $f_* : \mathcal{B}' \to \mathcal{B}$. The right adjoint f_* is lax monoidal and, by [GH15, Proposition 5.7.17], the adjoints $f^* \dashv f_*$ induce an adjunction $f^* \dashv f_* : \mathrm{Cat}^{\mathcal{B}}_{(\infty,1)} \to \mathrm{Cat}^{\mathcal{B}'}_{(\infty,1)}$.

Gepner-Haugseng model achieved tons of results. For example, they proved the homotopy hypothesis [GH15, Corollary 6.1.10], the Baez-Dolan stabilization hypothesis [GH15, Corollary 6.2.9], showed that $(\infty,1)$-categories enriched in spaces are modeled by (complete) Segal spaces [GH15, Theorem 4.4.7] and that stable $(\infty,1)$-categories are enriched in the $(\infty,1)$-category of spectra, and that, for R an $\mathbb{E}_2$-ring spectrum, every R-linear $(\infty,1)$-category is enriched in the $(\infty,1)$-category of R-module spectra. Nonetheless, it is still complicated to work within the model. For example, it is non-trivial to show that this model exhibits a Yoneda embedding.

The second approach solve this issue, by incorporating a sensible theory of presheaves directly in its foundations. Hinich's [Hin20] construction of enriched $(\infty,1)$-categories differs from Gepner-Haugseng from the start. Instead of defining $\mathcal{B}$-enriched (pre)categories with space of objects $\mathcal{C}^{\simeq}$ as $\Delta^{\mathrm{op}}_{\mathcal{C}^{\simeq}}$-algebras in the monoidal category $\mathcal{B}$, Hinich defines $\mathcal{B}$-enriched (pre)categories as associative algebra objects in a non-symmetric[7] $(\infty,1)$-operad $\mathrm{Quiv}_{\mathcal{C}^{\simeq}}(\mathcal{B})$. This non-symmetric $(\infty,1)$-operad is the operad of *$\mathcal{B}$-enriched quivers with space of objects* $\mathcal{C}^{\simeq}$, and algebra objects therein correspond to *$\mathcal{B}$-enriched precategories with space of objects* $\mathcal{C}^{\simeq}$. These precategories, like categorical algebras, encode a notion of composition satisfying the required coherence conditions. Nevertheless, Hinich's model suffers of the same completeness issue of

[6] Here we are tacitly assuming that $(\infty,1)$-categories should be the same of Spc-enriched $(\infty,1)$-categories.
[7] Called *planar* in loc.

Gepner-Haugseng. The problem is solved by localizing at complete $\mathcal{B}$-enriched precategories.

In Hinich's model, a Yoneda embedding satisfying the usual Yoneda lemma is available although its construction is quite convoluted and not internal to the model. The construction starts in [Hin20, Section 6.1]. Given a $\mathcal{B}$-enriched $(\infty, 1)$-category $\mathcal{C}$, and a left $\mathcal{B}$-module $\mathcal{D}$, the functor category $\mathrm{Fun}(\mathcal{C}^{\simeq}, \mathcal{D})$ comes equipped with a left action of $\mathrm{Quiv}_{\mathcal{C}^{\simeq}}(\mathcal{B})$. If $\mathcal{B}$ has enough colimits and the left action on $\mathcal{D}$ respects these colimits, then it is possible to define *$\mathcal{B}$-enriched functors* $\mathcal{C} \to \mathcal{D}$ as left $\mathcal{C}$-module in $\mathrm{Fun}(\mathcal{C}^{\simeq}, \mathcal{D})$. Informally, the left $\mathcal{C}$-module structure on a functor $f : \mathcal{C}^{\simeq} \to \mathcal{D}$ determines compatible maps

$$\mathcal{C}(x, y) \otimes f(x) \to f(y)$$

for every objects $x, y \in \mathcal{C}$ with the expected functoriality. The collection of all $\mathcal{B}$-enriched functors organize into an $(\infty, 1)$-category $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ with the expected functoriality in $\mathcal{C}$ and $\mathcal{D}$.

Defining enriched presheaves is then a matter of keeping track of the opposite. Indeed, if $\mathcal{C}$ is a $\mathcal{B}$-enriched $(\infty, 1)$-category, then the opposite $\mathcal{C}^{\mathrm{op}}$ will be a $\mathcal{B}^{\mathrm{rev}}$-enriched $(\infty, 1)$-category; here $\mathcal{B}^{\mathrm{rev}}$ is the *reverse monoidal category*, where the tensor product is given by reversing the tensor product of $\mathcal{B}$, that is $-_1 \otimes^{\mathrm{rev}} -_2 = -_2 \otimes -_1$. The space of object is now $(\mathcal{C}^{\simeq})^{\mathrm{op}}$, and the graph is given in "reverse". In particular, since $\mathcal{B}$ is a $\mathcal{B}$-bimodule, hence a left $\mathcal{B}$-module, it is possible to define the $(\infty, 1)$-category of $\mathcal{B}$-enriched presheaves on $\mathcal{C}$ as $\mathrm{Fun}_{\mathcal{B}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{B})$. More explicitly,

$$\mathrm{Fun}_{\mathcal{B}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}) = \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Fun}((\mathcal{C}^{\simeq})^{\mathrm{op}}, \mathcal{B}).$$

In this construction, the existence of the Yoneda embedding follows by a general construction of commutative algebra: every associative algebra A in a monoidal category $\mathcal{M}$ as the structure of a left $A \otimes A^{\mathrm{op}}$-module. If $\mathcal{M} = \mathrm{Quiv}_{\mathcal{C}^{\simeq}}(\mathcal{B})$ and $\mathcal{A} = \mathcal{C}$, then a left $\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$-action is induced on $\mathcal{C}$. A *folding construction*[8] then shows that $\mathrm{Fun}(\mathcal{C}^{\simeq} \times (\mathcal{C}^{\simeq})^{\mathrm{op}}, \mathcal{B})$ gets a left $\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$-action, where $\mathcal{B}$ is considered as a left $\mathcal{B} \otimes \mathcal{B}^{\mathrm{rev}}$-module. This defines a $\mathcal{B} \otimes \mathcal{B}^{\mathrm{rev}}$ functor $\tilde{よ} : \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}} \to \mathcal{B}$ called the *folded Yoneda*. Unfolding produces then a map

$$よ : \mathcal{C} \to \mathrm{Fun}_{\mathcal{B}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{B})$$

called the *unfolded Yoneda embedding*. The main result is then [Hin20, Corollary at the end of Section 6.2]. It shows that $よ : \mathcal{C} \to \mathrm{Fun}_{\mathcal{B}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{B})$ is fully-faithful in a suitable sense. Furthermore, it has also been showed in [BM24] that this Yoneda embedding is natural.

Despite providing a theory of presheaves, Hinich's approach to enriched $(\infty, 1)$-categories is rather technical and convoluted. Fortunately, Macpherson [Mac19] showed that the two models presented so far are actually equivalent, thus providing a Yoneda embedding in the more friendly Gepner-Haugseng' model of enriched $(\infty, 1)$-categories. Nonetheless, there is another subtlety worth mentioning that appears having applications in mind.

Indeed, many of the enriched $(\infty, 1)$-categories appearing in algebraic topology and algebraic geometry do not posses just a space of objects and a graph of morphism, but actually do form an $(\infty, 1)$-category. The enrichment is then given by some *closed left action* of a monoidal $(\infty, 1)$-category. More explicitly, an ∞-category $\mathcal{C}$ carries a closed left action of a monoidal $(\infty, 1)$-category $\mathcal{B}$ if for any object x in $\mathcal{C}$ the action functor $-\otimes x : \mathcal{B} \to \mathcal{C}$ admits a right adjoint $\mathcal{C}(x, -) : \mathcal{C} \to \mathcal{B}$. We think of $\mathcal{C}(x, y) \in \mathcal{B}$ as graph of morphisms $x \to y$ in $\mathcal{C}$ for any $x, y \in \mathcal{C}^{\simeq}$. This picture is particularly fruitful within the realm of (derived and spectral) algebraic geometry, since most of the $(\infty, 1)$-categories appearing are presentable. Indeed, presentability makes available the adjoint functor theorem [Lur09, Corollary 5.5.2.9], thus providing a right adjoint to the action $- \otimes - : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$.

On one side it is clear that this structure is much simpler compared to Gepner-Haugseng and Hinich homotopy-coherent enrichment, since all the required coherence conditions are inherited from the action $- \otimes - : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$. On the other side, one can expect that this simplistic point of view is a loss of generality. This is not exactly the case. Indeed, truncating the above mentioned coherence conditions lead to a naturally 1-enrichment of $h\mathcal{C}$ over $h\mathcal{B}$, and, surprisingly enough, both Gepner-Haugseng [GH15, Theorem 7.4.7] and Hinich [Hin20, Proposition 6.3.1] showed that this 1-enrichment canonically refines to

[8] If two monoidal categories $\mathcal{M}_a$ and $\mathcal{M}_b$ act from the left and from the right on a category $\mathcal{M}$, then these data can be also encoded by a left $\mathcal{M}_a \times \mathcal{M}_b^{\mathrm{rev}}$ action on $\mathcal{M}$. Folding and unfolding correspond to these different perspectives.

an homotopy-coherent enrichment of $\mathcal{C}$ in $\mathcal{B}$. In particular, a result of [Hei23, Theorem 1.1] (and Theorem 1.2 in loc. for the presentable case) shows that $(\infty,1)$-categories with a closed left action of a monoidal $(\infty,1)$-category $\mathcal{B}$ do indeed model the class of *tensored* $\mathcal{B}$-enriched $(\infty,1)$-categories[9]. In formulas, Theorem 1.1 in loc. shows that there is an equivalence of $(\infty,1)$-categories

$$\mathrm{LMod}_{\mathcal{B}}(\mathrm{Cat}_{(\infty,1)})^{\mathrm{cl}} \simeq \mathrm{Cat}^{\mathcal{B},\mathrm{ten}}_{(\infty,1)} \tag{1}$$

between the full subcategory $\mathrm{LMod}_{\mathcal{B}}(\mathrm{Cat}_{(\infty,1)})^{\mathrm{cl}} \subseteq \mathrm{LMod}_{\mathcal{B}}(\mathrm{Cat}_{(\infty,1)})$ of $(\infty,1)$-categories with closed left $\mathcal{B}$-action and a non-full subcategory $\mathrm{Cat}^{\mathcal{B},\mathrm{ten}}_{(\infty,1)} \subseteq \mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$ of tensored $\mathcal{B}$-enriched $(\infty,1)$-categories.

Equation 1 is a special case of the more general [Hei23, Theorem 1.5], which express different equivalences depending on the action of $\mathcal{B}$ on $\mathcal{C}$ that it is considered on the left hand side, and the type of enrichment on the right hand side.

Going from the most special to the most general notion, the left hand side of Equation 1 can be generalized to include Lurie's pseudo-enriched in $\mathcal{B}$ in [Lur17, Definition 4.2.1.25] and weakly enriched in $\mathcal{B}$ $(\infty,1)$-categories [Lur17, Definition 4.2.1.12]. These are generalization of Lurie's $(\infty,1)$-*categories enriched in* $\mathcal{B}$ defined in [Lur17, Definition 4.2.1.19]. On the right hand side instead one will then obtain $\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathrm{Spc})$-enriched $(\infty,1)$-categories, and $\mathrm{Fun}(\mathrm{Env}(\mathcal{B})^{\mathrm{op}},\mathrm{Spc})$-enriched $(\infty,1)$-categories. Here $\mathrm{Env}(\mathcal{B})$ is the enveloping monoidal $(\infty,1)$-category of $\mathcal{B}$, studied by Lurie [Lur17, Section 2.2.4].

Let us focus on those $(\infty,1)$-categories which are weakly left tensored over $\mathcal{B}$. Some of these categories are already $\mathcal{B}$-enriched (see [Hei23, Theorem 1.5]). Denote by $\mathrm{Cat}^{\mathcal{B},\mathrm{Lur}}_{(\infty,1)}$ the $(\infty,1)$-category they span. The first non-trivial fact is that for every pair of $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{D}$ weakly enriched in $\mathcal{B}$, the collection of functors $\mathcal{C} \to \mathcal{D}$ compatible with the weak left $\mathcal{B}$-action forms an $(\infty,1)$-category $\mathrm{LaxLinFun}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$. The objects of this $(\infty,1)$-category are called *lax $\mathcal{B}$-linear functors*. The second non-trivial fact is that this $(\infty,1)$-category can be identified with the right adjoint of a closed left $\mathrm{Cat}_{(\infty,1)}$-action on $\mathrm{Cat}^{\mathcal{B},\mathrm{Lur}}_{(\infty,1)}$. That is, there is an adjunction

$$\mathrm{Cat}_{(\infty,1)} \underset{\mathrm{LaxLinFun}_{\mathcal{B}}(\mathcal{C},-)}{\overset{-\otimes\mathcal{C}}{\rightleftarrows}} \mathrm{Cat}^{\mathcal{B},\mathrm{Lur}}_{(\infty,1)} \quad (\perp)$$

for every $(\infty,1)$-category $\mathcal{C}$ weakly enriched in $\mathcal{B}$. Now, as soon as the monoidal $(\infty,1)$-category $\mathcal{B}$ is compatible with small colimits, there is a canonical left action of the $(\infty,1)$-category of spaces Spc, endowed with the cartesian structure, on $\mathcal{B}$. See [Lur09, Remark 5.5.1.7]. This left action is compatible with the monoidal structures of $\mathcal{B}$ and it can be shown that it induces a left closed action of $\mathrm{Cat}_{(\infty,1)}$ on the $(\infty,1)$-category $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$. Given two $\mathcal{B}$-enriched $(\infty,1)$-categories $\mathcal{X}$ and $\mathcal{Y}$, let us denote by $\mathrm{Fun}_{\mathcal{B}}(\mathcal{X},\mathcal{Y})$ the $\mathcal{B}$-enriched $(\infty,1)$-category corresponding to the morphism object of $\mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$. The following result clarifies the role of these left actions.

Theorem ([Hei23, Corollary 8.17)]). Let $\mathcal{B}$ be a monoidal $(\infty,1)$-category compatible with small colimits. Then there is a $\mathrm{Cat}_{(\infty,1)}$-enriched equivalence

$$\chi : \mathrm{Cat}^{\mathcal{B},\mathrm{Lur}}_{(\infty,1)} \to \mathrm{Cat}^{\mathcal{B}}_{(\infty,1)}$$

given by sending a small $(\infty,1)$-category weakly left tensored over $\mathcal{B}$ to its underlying $(\infty,1)$-category enriched in $\mathcal{B}$.

In particular, any $(\infty,1)$-category $\mathcal{C}$ weakly left tensored over $\mathcal{B}$ which is $\mathcal{B}$-enriched determines (and is determined by) a $\mathcal{B}$-enriched $(\infty,1)$-category in the sense of Gepner-Haugseng and Hinich. Moreover, given two $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{D}$ weakly left tensored over $\mathcal{B}$, the $(\infty,1)$-category of lax linear functors $\mathrm{LaxLinFun}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ corresponds to $\mathrm{Fun}_{\mathcal{B}}(\chi(\mathcal{C}),\chi(\mathcal{D}))$. This makes the Yondea embedding available in the former model, and we phrase it as follows.

[9]More explicitly, a $\mathcal{B}$-enriched $(\infty,1)$-category $\mathcal{C}$ is tensored if for any object $b \in \mathcal{B}$ and $x \in \mathcal{C}$ there is an object $b \otimes x$ in $\mathcal{C}$ equipped with a morphism $b \to \mathcal{C}(x, b \otimes x)$ in $\mathcal{B}$ such that for any object $y \in \mathcal{C}$ the canonical composition $\mathcal{C}(b \otimes x, y) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{C}(x, b \otimes x), \mathcal{C}(x,y)) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(b, \mathcal{C}(x,y))$ is an equivalence. The object $b \otimes x$ is called the tensor of b and x.

Theorem ([Hei23, Theorem 10.11])**.** Let $\mathcal{B}$ be a monoidal $(\infty, 1)$-category and let $\mathcal{C}$ be weakly enriched in $\mathcal{B}$. Then the canonical map

$$\text{よ} : \mathcal{C} \to \mathrm{LaxLinFun}_{\mathcal{B}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}), \qquad x \mapsto \mathcal{C}(-, x)$$

is fully-faithful.

In these pages we will apply this result in the case where $\mathcal{C}$ has a closed (left) action of a symmetric monoidal $(\infty, 1)$-category $\mathcal{B}$. Futhermore, to simplify the notation, we will denote by $\mathrm{Fun}_{\mathcal{B}}$ what Heine denotes by $\mathrm{LaxLinFun}_{\mathcal{B}}$.

Remark. We conclude by recalling one more result in the literature: Berman [Ber20] constructed its own theory of enriched $(\infty, 1)$-categories. His theory actually supports a Yoneda embedding having all the expected features. His approach seems also equivalent to Gepner-Haugseng construction. See Remark 1.1 *in loc*. Unfortunately, Berman work is still incomplete.

Conventions

Stable $(\infty, 1)$-Categories. Recall that an $(\infty, 1)$-category $\mathcal{C}$ is called stable if it is pointed and finite limits and colimits coincide. Equivalently, $\mathcal{C}$ is pointed and every morphism admits a fibre and a cofibre, which canonically agree.

We will denote by $\mathrm{Cat}^{\mathrm{ex}}_{(\infty,1)}$ the pointed $(\infty, 1)$-category of small stable $(\infty, 1)$-categories and exact functors. Recall that a functor is exact if it preserves finite limits and colimits. Given two small stable $(\infty, 1)$-categories $\mathcal{C}$ and $\mathcal{D}$, we will denote by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ the $(\infty, 1)$-category of exact functors between; this is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the exact functors.

t-structures. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category. A t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ will be graded homologically. That is, we imagine $\mathcal{C}$ as linearized in the following way

$$\dots \longrightarrow \bullet_{n+1} \longrightarrow \bullet_n \longrightarrow \bullet_{n-1} \longrightarrow \dots$$

We will think of objects in $\mathcal{C}_{\geq n}$ as existing at the left on n, whereas objects in $\mathcal{C}_{\leq n}$ will exists on the right. In particular, we will call these objects n-*connective* and n-*coconnective*. The inclusions of the connective and coconnective objects admit a left and right adjoint respectively

$$\mathcal{C} \underset{i_{\leq n}}{\overset{\tau_{\leq n}}{\rightleftarrows}} \mathcal{C}_{\leq n} \quad (\tau_{\leq n} \dashv i_{\leq n}) \qquad\qquad \mathcal{C}_{\geq n} \underset{\tau_{\geq n}}{\overset{i_{\geq n}}{\rightleftarrows}} \mathcal{C}. \quad (i_{\geq n} \dashv \tau_{\geq n})$$

Let $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ denote the *heart of the t-structure*. We will denote by $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ the functor $\tau_{\leq 0}\tau_{\geq 0}[-n]$ for any $n \in \mathbb{Z}$, and refer to π_n as the *n-th homotopy group of the t-structure*. We will also add a superscript (for example, $\pi_n^{\mathcal{C}}$) if more categories with t-structure are considered. Finally, we will denote by

$$\mathcal{C}^- = \bigcup_{n>0} \mathcal{C}_{\geq -n}, \qquad \mathcal{C}^+ = \bigcup_{n>0} \mathcal{C}_{\leq n}, \qquad \mathcal{C}^b = \mathcal{C}^- \cap \mathcal{C}^+$$

the full subcategories of $\mathcal{C}$ spanned by the connective, coconnective and bounded objects.

Presentable $(\infty, 1)$-Categories. Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 1)$-categories. If $\mathcal{C}$ and $\mathcal{D}$ have finite limits, we will denote by $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functor which are left exact: that is, those functors which preserve finite limits.

We will denote by Pr^{L} the $(\infty, 1)$-category of presentable $(\infty, 1)$-categories. Its objects are presentable $(\infty, 1)$-categories, that is $(\infty, 1)$-categories which are closed under all colimits (as well as small limits by

[Lur09, Proposition 5.5.2.4]), and moreover generated in a weak sense by a small category. The morphisms of Pr^{L} are continuous functors, that is, functors that preserve all colimits. If $\mathcal{C}$ and $\mathcal{D}$ are presentable, we let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ denote the $(\infty, 1)$-category of colimit preserving functors from $\mathcal{C}$ to $\mathcal{D}$.

We will regard Pr^{L} with the symmetric closed monoidal structure presented in [Lur17, Section 4.8]. The tensor product of two presentable $(\infty, 1)$-categories $\mathcal{C}$ and $\mathcal{D}$ is the universal target of a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves colimits separately in each variable. The monoidal unit of this symmetric $(\infty, 1)$-category is given by the $(\infty, 1)$-category of spaces. We will denote it by Spc.

We will denote by $\mathrm{Pr}^{\mathrm{L},\omega}$ the $(\infty, 1)$-category of compactly generated, presentable $(\infty, 1)$-categories and colimit-preserving functors which preserve compact objects. Finally we will also denote by $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the full subcategory spanned by the compactly generated stable and presentable stable $(\infty, 1)$-categories.

Rings, Modules and Derived Categories. Given a commutative unitary ring A we will denote by $\mathrm{Mod}_A^{\heartsuit}$ the 1-category of A-modules, and by Mod_A the derived stable $(\infty, 1)$-category of A-modules. We will also the same notation if A is a derived ring or an $\mathbb{E}_\infty$-ring spectrum. We will denote by $\mathrm{Perf}(A)$ and by $\mathrm{Coh}(A) \subseteq \mathrm{PCoh}(A)$ the full subcategories of compact (that is, perefect) objects and coherent and pseudo-coherent objects.

If X is a scheme, stack or spectral Deligne-Mumford stack, we will denote by $\mathrm{QCoh}(X)$ the derived stable $(\infty, 1)$-category of quasi-coherent sheaves on X, and by $\mathrm{QCoh}(X)^{\heartsuit}$ the corresponding abelian 1-category. We will also denote by $\mathrm{Perf}(X)$ the full subcategory spanned by the compact objects (notice that, in the case where X is a stack or a spectral Deligne-Mumford stack, compact objects and perfect objects coincide only if X is perfect). We will also denote by $\mathrm{Coh}(X) \subseteq \mathrm{PCoh}(X)$ the full subcategory of coherent and pseudo-coherent objects.

1 Geometric $(\infty, 1)$-Categories

The goal of this chapter is to introduce the basic language necessary to formulate Neeman's dualities. We begin in Section 1.1 by reviewing the theory of *stable homotopy theories* and *rigidly-compactly generated* $(\infty, 1)$*-categories*, which we will refer to as *geometric* $(\infty, 1)$*-categories* for brevity. These $(\infty, 1)$-categories are characterized by being compactly generated by the dualizable objects. This simple requirement has numerous implications. For example, every *geometric functor* $f^* : \mathcal{B} \to \mathcal{C}$, that is, every colimit-preserving and symmetric monoidal functor between geometric $(\infty, 1)$-categories, possesses a right adjoint f_*, which itself possesses a right adjoint $f^{(1)}$. Moreover, these functors are related by a *projection formula* and a set of *internal realizations*, which we will prove in Proposition 1.1.9. However, the main result of this section (proved by Balmer, Dell'Ambrogio, and Sanders) is Theorem 1.1.11, an abstract formulation of Grothendieck-Neeman duality.

In Section 1.2, we incorporate the input of a t-structure. We focus on two types of t-structures: *geometric* t-structures and *tensor* t-structures, as well as their combination. Tensor t-structures are those for which the 0-connective objects inherit the structure of a symmetric monoidal $(\infty, 1)$-category, while geometric t-structures are defined as accessible t-structures that are compatible with filtered colimits and right complete. We will also introduce the concepts of t*-geometric* $(\infty, 1)$*-categories* and t*-geometric functors* by adding a geometric tensor t-structure to geometric categories and functors.

With the added datum of a (geometric tensor) t-structure on a geometric $(\infty, 1)$-category, we will introduce in Section 1.3 the third notion of finiteness appearing in Neeman's dualities. We have two choices.

(1) We can consider Lurie's *almost perfect objects* and *bounded almost perfect objects*, $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$.

(2) Alternatively, we can consider Neeman's *pseudo-compact* and *bounded pseudo-compact objects*, $\mathcal{C}^b_c \subseteq \mathcal{C}^-_c$.

We will work with Lurie's almost perfect objects, which we call *pseudo-coherent objects*. If $\mathcal{C}$ is a compactly generated $(\infty, 1)$-category equipped with a t-structure, then the collection of pseudo-coherent objects $\mathrm{PCoh}(\mathcal{C})$ defines a full stable subcategory. Moreover, we will show that connective pseudo-coherent objects are closed under geometric realizations of simplicial objects, a result that will be crucial in the following. Finally, we will study the interaction between compact and pseudo-coherent objects.

However, it is not immediately clear how to compute pseudo-coherent objects explicitly. In Theorem 1.4.12, we will show that under mild assumptions on $\mathcal{C}$, an object $x \in \mathcal{C}$ is pseudo-coherent if and only if each homotopy group $\pi_n(x)$ is a compact object in $\mathcal{C}^\heartsuit$, and $\pi_n(x) \cong 0$ for $n \ll 0$. These mild assumptions on $\mathcal{C}$ will serve as the starting point for Section 1.4. These assumptions boil down to two obstructions. The first obstruction concerns the compact objects in the heart. Since we naively expect the heart $\mathrm{PCoh}(\mathcal{C})^\heartsuit$ to span the subcategory of compact objects in $\mathcal{C}^\heartsuit$, we require the latter to be an abelian subcategory. This occurs if and only if $\mathcal{C}^\heartsuit$ is a *locally coherent abelian* 1*-category*. The second obstruction is more technical. If $\mathcal{C}$ possesses a compact generator G and the t-structure is in the preferred equivalence class, then up to shifting, we may assume the heart $\mathcal{C}^\heartsuit$ to be compactly generated by $\pi_0(G)$. However, the projection $\pi_0 : \mathcal{C} \to \mathcal{C}^\heartsuit$ destroys all the relevant information on the higher homotopy groups $\pi_n(G)$, which are essential for lifting arguments from $\mathcal{C}^\heartsuit$ to $\mathcal{C}$.

Lastly, in Section 1.5, we will compare our pseudo-coherent objects with Neeman's pseudo-compact objects. The main result is Proposition 1.5.7, which shows that if $\mathcal{C}$ is equipped with a single compact generator $G \in \mathcal{C}_c$ such that there exists an integer $N > 0$ for which G is $(-N)$-connective and $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$, then pseudo-coherent objects and Neeman's pseudo-compact objects coincide when computed in the preferred equivalence class.

1.1 Definition and Basic Properties

Lurie [Lur17, Section 4.8.2] showed that the $(\infty, 1)$-category of stable presentable $(\infty, 1)$-categories $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is equipped with a symmetric monoidal structure obtained by localizing Pr^{L} at the idempotent object $(\mathrm{Sp}, \mathbb{S})$ given by the $(\infty, 1)$-category of spectra Sp and the sphere spectrum $\mathbb{S} \in \mathrm{Sp}$. We are interested in commutative algebra object therein.

Definition 1.1.1. A *stable homotopy theory* is a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Explicitly, a stable homotopy theory is a presentable stable $(\infty, 1)$-category $\mathcal{C}$ equipped with a symmetric monoidal structure $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$. Here the tensor product is required to be an exact functor commuting with small colimits in both arguments. The definition has two immediate consequences.

(1) First of all, since for every object $x \in \mathcal{C}$ the functor $x \otimes_{\mathcal{C}} - : \mathcal{C} \to \mathcal{C}$ preserve all colimits, the adjoint functor theorem [Lur09, Corollary 5.5.2.9] implies the existence of an adjunction

$$\mathcal{C} \underset{\underline{\operatorname{Hom}}_{\mathcal{C}}(x,-)}{\overset{x \otimes_{\mathcal{C}} -}{\rightleftarrows}} \mathcal{C}.$$

Moreover, a simple computation with adjoints shows that $\underline{\operatorname{Hom}}_{\mathcal{C}}(x, -)$ as the expected functoriality in x, thus giving a functor $\underline{\operatorname{Hom}}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. We will call it *internal hom functor*.

(2) Secondly, there exists a unique colimit preserving symmetric monoidal functor $H : \mathrm{Sp} \to \mathcal{C}$. This follows from the construction of the symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, which exhibits the $(\infty, 1)$-category of spectra, equipped with the smash product of spectra, as the initial stable and presentable $(\infty, 1)$-category equipped with a symmetric monoidal structure. The adjoint functor theorem produces a right adjoint $\Gamma(\mathcal{C}, -) : \mathcal{C} \to \mathrm{Sp}$, which we call *global sections of* $\mathcal{C}$.

Having said that, the input of a symmetric monoidal structure on a stable and presentable $(\infty, 1)$-category is what allows us introduce a new notion of "finite" object. Given a stable homotopy theory $\mathcal{C}$, we can define a small subcategory $\mathcal{C}_{\mathrm{dual}}$ of dualizable objects in $\mathcal{C}$. Recall that an object x in a symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$ is *dualizable* if there exists an object $x^\vee$ and maps $c : \mathbb{1}_{\mathcal{C}} \to x^\vee \otimes_{\mathcal{C}} x$ and $e : x \otimes_{\mathcal{C}} x^\vee \to \mathbb{1}_{\mathcal{C}}$, called the *coevaluation* and *evalutation*, such that the composites

$$x \simeq x \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \xrightarrow{\mathrm{id}_x \otimes c} x \otimes_{\mathcal{C}} x^\vee \otimes_{\mathcal{C}} x \xrightarrow{e \otimes \mathrm{id}_x} x, \qquad x^\vee \simeq \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} x^\vee \xrightarrow{c \otimes \mathrm{id}_{x^\vee}} x^\vee \otimes_{\mathcal{C}} x \otimes_{\mathcal{C}} x^\vee \xrightarrow{\mathrm{id}_{x^\vee} \otimes e} x^\vee$$

are equivalent to the respective identities. Equivalently, an object $x \in \mathcal{C}$ is dualizable if and only if there is an object $x^\vee$ such that $x^\vee \otimes_{\mathcal{C}} - \dashv x \otimes_{\mathcal{C}} -$ are adjoint functors $\mathcal{C} \to \mathcal{C}$. We will denote by $\mathcal{C}_{\mathrm{dual}}$ the full subcategory of $\mathcal{C}$ spanned by the dualizable objects in $\mathcal{C}$. By [HPS97, Theorem A.2.5], this subcategory is idempotent-complete and closed under the monoidal product. Furthermore, $\mathcal{C}_{\mathrm{dual}}$ is also stable, being $\mathcal{C}$ a stable homotopy theory,.

Remark 1.1.2. Let $\mathcal{C}$ be a stable homotopy theory. Taking compact and dualizable objects produces two idempotent-complete stable $(\infty, 1)$-categories $\mathcal{C}_c$ and $\mathcal{C}_{\mathrm{dual}}$. In general, these two categories do not share any relations. However, if the monoidal unit is compact, then dualizable objects always form a subcategory of compact objects. The reason is simple. Indeed, if $x \in \mathcal{C}$ is dualizable, then we have equivalence

$$\mathrm{Hom}_{\mathcal{C}}(y, z \otimes_{\mathcal{C}} x) \simeq \mathrm{Hom}_{\mathcal{C}}(y \otimes_{\mathcal{C}} x^{\vee}, z)$$

for every $y, z \in \mathcal{C}$. By taking $y = \mathbb{1}_{\mathcal{C}}$ we see that the compactness of the monoidal unit forces the dual $x^{\vee}$ to be compact as well, hence x to be compact. In contrast, no level of compactness is enough to guarantee dualizability.

For this reason, we select a particular class of stable homotopy theories.

Definition 1.1.3. A *geometric* $(\infty, 1)$-*category* is a compactly generated stable $(\infty, 1)$-category $\mathcal{C}$ equipped with a symmetric monoidal structure $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$, compatible[10] with colimits, such that the compact objects coincide with the dualizable objects.

Several names appear in the literature. Classically, geometric $(\infty, 1)$-categories appear in [HPS97, Definition 1.1.4] under the name *unital algebraic stable homotopy categories*. More recently, they have also been called *rigidly-compactly generated categories*. We have preferred a more concise definition. For an $(\infty, 1)$-category, being *geometric* should reminds us that it arises as the standard "big" tensor-stable $(\infty, 1)$-category of common use, regardless if it comes from algebra, geometry or homotopy theory.

Remark 1.1.4. There are two major consequences of the Definition 1.1.3. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category.

(1) Since compact objects coincide with the dualizable ones, every $x \in \mathcal{C}$ determines a canonical equivalence $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -) \simeq x^{\vee} \otimes_{\mathcal{C}} -$ of functors $\mathcal{C} \to \mathcal{C}$. In particular, $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -)$ preserves all small colimits and $x^{\vee} \otimes_{\mathcal{C}} -$ preserves all small limits.
(2) Consider the full subcategory of compact-dualizable objects $\mathcal{C}_c = \mathcal{C}_{\mathrm{dual}}$. This subcategory admits a canonical duality $\Delta_{\mathcal{C}} = \underline{\mathrm{Hom}}_{\mathcal{C}}(-, \mathbb{1}_{\mathcal{C}}) : (\mathcal{C}_c)^{\mathrm{op}} \to \mathcal{C}_c$ satisfying $\Delta_{\mathcal{C}}^2 \simeq \mathrm{id}_{\mathcal{C}}$.

The above duality is natural in $\mathcal{C}$, under geometric functors, introduced in [HPS97, Definition 3.4.1].

Definition 1.1.5. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a functor between geometric $(\infty, 1)$-categories. We say that f^* is *geometric* if it is colimit preserving and symmetric monoidal.

Remark 1.1.6. By applying the adjoint functor theorem, [Lur09, Corollary 5.5.2.9] to the colimit preserving functor $f^* : \mathcal{B} \to \mathcal{C}$ we get a right adjoint $f_* : \mathcal{C} \to \mathcal{B}$. We call f_* the *pushfoward along* f, even if f does not quite have a meaning.

Remark 1.1.7. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a symmetric monoidal functor between geometric $(\infty, 1)$-categories. Then f^* preserves compact objects, since they coincide with the rigid objects, and f^* preserve them, being symmetric monoidal.

Lemma 1.1.8. Let $F \dashv G : \mathcal{B} \to \mathcal{C}$ be an adjoint pair of functors between $(\infty, 1)$-categories. Assume that $\mathcal{B}$ is compactly generated. Then the following are equivalent.

(1) The functor F preserves compact objects.
(2) The functor G preserves filtered colimits.

[10]In Section 2.1 we will show that $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ carries a symmetric monoidal structure. This will imply that geometric $(\infty, 1)$-categories can be identified with particular algebra objects therein.

Proof. Assume that F preserves compact objects and let us prove that G preserves filtered colimits. Let $I \to \mathcal{C}, i \mapsto c_i$ be a filtered diagram in $\mathcal{C}$. We want to show that the canonical map $\operatorname{colim}_{i\in I} G(c_i) \to G(\operatorname{colim}_{i\in I} c_i)$ is an equivalence in $\mathcal{B}$. By Yoneda, it suffices to check that it induces an equivalence

$$\operatorname{Hom}_{\mathcal{B}}(b, \operatorname{colim}_{i\in I} G(c_i)) \to \operatorname{Hom}_{\mathcal{B}}(b, G(\operatorname{colim}_{i\in I} c_i))$$

in Spc for every $b \in \mathcal{B}$. However, being $\mathcal{B}$ compactly generated, every element is a colimit of compact objects, so that we can assume $b \in \mathcal{B}_c$. But then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{B}}(b, \operatorname{colim}_{i\in I} G(c_i)) &\simeq \operatorname{colim}_{i\in I} \operatorname{Hom}_{\mathcal{B}}(b, G(c_i))\\ &\simeq \operatorname{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}(F(b), c_i)\\ &\simeq \operatorname{Hom}_{\mathcal{C}}(F(b), \operatorname{colim}_{i\in I} c_i)\\ &\simeq \operatorname{Hom}_{\mathcal{B}}(b, G(\operatorname{colim}_{i\in I} c_i)).\end{aligned}$$

Here the first equivalence follows by the compactness of $b \in \mathcal{B}_c$, the second one by the adjunction $F \dashv G$, the third one by assumption (1), the fourth one again by adjunction $F \dashv G$. This proves (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (1), assume that G preserves filtered colimits, and consider a compact object $b \in \mathcal{B}_c$. Given a filtered diagram $I \to \mathcal{C}, i \mapsto c_i$ in $\mathcal{C}$ we compute

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(F(b), \operatorname{colim}_{i\in I} c_i) &\simeq \operatorname{Hom}_{\mathcal{B}}(b, G(\operatorname{colim}_{i\in I} c_i))\\ &\simeq \operatorname{Hom}_{\mathcal{B}}(b, \operatorname{colim}_{i\in I} G(c_i))\\ &\simeq \operatorname{colim}_{i\in I} \operatorname{Hom}_{\mathcal{B}}(b, G(c_i))\\ &\simeq \operatorname{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}(F(b), c_i).\end{aligned}$$

Here the first equivalence follows by adjunction $F \dashv G$, the second one by assumption (2), the third one by compactness of $b \in \mathcal{B}_c$ and the fourth one again by adjunction $F \dashv G$. In particular, there is no need to assume $\mathcal{B}$ compactly generated. ☐

Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor. Since Remark 1.1.6 provides us a right adjoint $f_* : \mathcal{C} \to \mathcal{B}$, we can apply Lemma 1.1.8 to discover that f_* must preserve filtered colimits. Since the pushfoward f_* is a limit preserving functor (in particular, it is exact) and preserves filtered colimits, it must preserve all colimits. By applying the adjoint functor theorem [Lur09, Corollary 5.5.2.9], we discover that it fits into adjunctions

$$\mathcal{B} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{C} \underset{f^{(1)}}{\overset{f_*}{\rightleftarrows}} \mathcal{B}.$$

This follows by the basic formalism. Following [LN07], we will call $f^{(1)}$ the *twisted inverse image functor*. In loc. the notation $f^{\times}$ is used; we disregard this choice in order to prefer the notation proposed in [BDS16, Remark 1.11]. The following result (which the reader can find in [BDS16, Proposition 2.15]) shows that the three functors $f^* \dashv f_* \dashv f^{(1)}$ automatically satisfy some basic formulas.

Proposition 1.1.9 (The Projection Formula)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor. Then there is a canonical natural equivalence

$$f_*(x) \otimes_{\mathcal{B}} y \to f_*(x \otimes_{\mathcal{C}} f^*(y)) \tag{2}$$

for all $x \in \mathcal{C}$ and $y \in \mathcal{B}$ obtained by $f^*(f_*(x) \otimes_{\mathcal{B}} y) \simeq f^*(f_*(x)) \otimes_{\mathcal{C}} f^*(y) \to x \otimes_{\mathcal{C}} f^*(y)$ by adjunction. We also have three further canonical equivalences:

(1) $\underline{\operatorname{Hom}}_{\mathcal{B}}(y, f_*(x)) \simeq f_*\underline{\operatorname{Hom}}_{\mathcal{C}}(f^*(y), x)$ for all $y \in \mathcal{B}$ and $x \in \mathcal{C}$.

(2) $\underline{\operatorname{Hom}}_{\mathcal{B}}(f_*(x), y) \simeq f_*\underline{\operatorname{Hom}}_{\mathcal{C}}(x, f^{(1)}(y))$ for all $y \in \mathcal{B}$ and $x \in \mathcal{C}$.

(3) $f^{(1)}\underline{\operatorname{Hom}}_{\mathcal{B}}(y, y') \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(f^*(y), f^{(1)}(y'))$ for all $y, y' \in \mathcal{B}$.

Equation 2 is called the *projection formula*, whereas the other equations appearing in the statement are called *internal realizations*.

Proof. The map $f_*(x) \otimes_{\mathcal{B}} y \to f_*(x \otimes_{\mathcal{C}} f^*(y))$ is well-defined for all y and x, and it is automatically invertible whenever y is dualizable. By fixing an arbitrary $x \in \mathcal{C}$, we see that both sides of equation of the previous equation are exact and respect the colimits in the y-variable. Since $\mathcal{B}$ is generated by its compact (that is, by the rigid) objects, it follows that $f_*(x) \otimes_{\mathcal{B}} y \to f_*(x \otimes_{\mathcal{C}} f^*(y))$ is an equivalence for every $y \in \mathcal{B}$.

By taking adjoints, we now derive the second and the third internal realizations. First, by fixing y, we obtain the composite adjunctions

$$y \otimes_{\mathcal{B}} f_*(-) \dashv f^{(1)}\underline{\mathrm{Hom}}_{\mathcal{B}}(y, -), \qquad f_*(f^*(y) \otimes_{\mathcal{C}} -) \dashv \underline{\mathrm{Hom}}_{\mathcal{C}}(f^*(y), f^{(1)}(-)).$$

However, since the projection formula amounts to an equivalence of left adjoints, the uniqueness of right adjoints implies equivalence on the right adjoints as well, yielding the last internal realization. The naturality in y follows from the fact that the adjunctions above form natural families parametrized by y. Next, fixing x instead leads to

$$(-) \otimes_{\mathcal{B}} f_*(x) \dashv \underline{\mathrm{Hom}}_{\mathcal{B}}(f_*(x), -), \qquad f_*(f^*(-) \otimes_{\mathcal{C}} x) \dashv f_*\underline{\mathrm{Hom}}_{\mathcal{C}}(x, f^{(1)}(-)),$$

from which we can derive the second internal realization. Finally, by fixing y in the equivalence $f^*(y) \otimes_{\mathcal{C}} f^*(x) \simeq f^*(y \otimes_{\mathcal{B}} x)$, which arises from the monoidal structure of f^*, we deduce that

$$f^*(y) \otimes_{\mathcal{C}} f^*(-) \dashv f_*\underline{\mathrm{Hom}}_{\mathcal{C}}(f^*(y), -), \quad f^*(y \otimes_{\mathcal{B}} -) \dashv \underline{\mathrm{Hom}}_{\mathcal{B}}(y, f_*(-)),$$

which gives the remaining internal realization. ☐

We conclude this section by discussing the behaviour of compact objects under f_* (and thus explaining the terminology behind $f^{(1)}$). But first, we need a technical result.

Lemma 1.1.10. Let $F \dashv G : \mathcal{B} \to \mathcal{C}$ be an adjoint pair of exact functors between stable $(\infty, 1)$-categories. Assume that $\mathcal{B}$ is compactly generated and that F preserves compact objects.

(1) If the restriction $F|_{\mathcal{B}_c} : \mathcal{B}_c \to \mathcal{C}_c$ has a right adjoint G_0, then G preserves compact objects and $G|_{\mathcal{C}_c} \simeq G_0$.

(2) If the restriction $F|_{\mathcal{B}_c} : \mathcal{B}_c \to \mathcal{C}_c$ admits a left adjoint E_0 and if $\mathcal{C}$ is compactly generated, then F preserves limits.

Proof. Let us prove (1). First of all, our assumption tells us that for every compact object $x \in \mathcal{B}_c$ and every compact $y \in \mathcal{C}_c$, we have a natural equivalences

$$\mathrm{Hom}_{\mathcal{C}}(y, G_0(x)) \simeq \mathrm{Hom}_{\mathcal{B}}(F|_{\mathcal{B}_c}(y), x) = \mathrm{Hom}_{\mathcal{B}}(F(y), x) \simeq \mathrm{Hom}_{\mathcal{C}}(y, G(x)).$$

We now apply this chain of equivalences to $y := G_0(x)$, so that the identity map of $G_0(x)$ induces a morphism $\gamma_x : G_0(x) \to G(x)$. By letting $x \in \mathcal{B}$ varying, the naturality of the above equivalence induces a natural transformation $\gamma : G_0 \to G|_{\mathcal{B}}$. However, the naturality in y implies that the above chain of equivalences is obtained by composing maps $f \in \mathrm{Hom}_{\mathcal{C}}(y, G_0(x))$ with γ_x. In particular, for any fixed $x \in \mathcal{B}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(-, \gamma_x) : \mathrm{Hom}_{\mathcal{C}}(-, G_0(x)) \to \mathrm{Hom}_{\mathcal{C}}(-, G(x))$ is invertible on all $y \in \mathcal{C}$ by construction. Now, being $\mathcal{C}$ is compactly generated, it follows that $\mathrm{Hom}_{\mathcal{C}}(-, \gamma_x)$ is invertible on all $y \in \mathcal{C}$. In particular, Yoneda lemma implies that γ_x is an equivalence, thus showing the claim.

Let us now prove (2). Let us denote by $\eta : \mathrm{id}_{\mathcal{C}_c} \to F \circ E_0$ the unit of the adjunction $E_0 \dashv F$. Consider the map $\alpha_{x,s} : \mathrm{Hom}_{\mathcal{C}}(x, F(y)) \to \mathrm{Hom}_{\mathcal{B}}(E_0(x), y)$ defined for $x \in \mathcal{C}_c$ compact and $y \in \mathcal{B}$ By adjunction, this morphism is an isomorphism when $y \in \mathcal{B}_c$ is compact. Since both functors $\mathrm{Hom}_{\mathcal{B}}(E_0(x), -)$ and $\mathrm{Hom}_{\mathcal{C}}(x, F(-))$ are homological functors from $\mathcal{B} \to \mathrm{Mod}_{\mathbb{Z}}^{\heartsuit}$ and preserve colimits. This follows because F preserves colimits (as it has a right adjoint), and both x and $E_0(x)$ are compact objects. Now $\alpha_{x,y}$ is an equivalence for every $x \in \mathcal{C}_c$ and every $y \in \mathcal{B}$. This kind of partial adjoint suffices to show that F preserves limits, as usual. ☐

We can now prove the main result of this section.

Theorem 1.1.11 (Abstract Grothendieck-Neeman Duality)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor. Then the following are equivalent.

(1) The functor f^* has a left adjoint $f_{(1)}$.

(2) The functor f_* preserves compact objects.

In this case, we have five adjoints $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ and the *Grothendieck-Neeman duality* equivalence

$$f^{(1)}(-) \simeq \omega_f \otimes_{\mathcal{C}} f^*(-)$$

where $\omega_f = f^{(1)}(\mathbb{1}_{\mathcal{B}})$ is the *dualizing complex of* f.

Proof. Assume first that the functor f^* has a left adjoint $f_{(1)}$. By assumption, f^* preserves all colimits, hence the left adjoint $f_{(1)}$ must preserve compact objects by Lemma 1.1.8. Since f^* is symmetric monoidal, the adjunction $f_{(1)} \dashv f^*$ restricts to an adjunction

$$\mathcal{C}_c \underset{f^*}{\overset{f_{(1)}}{\rightleftarrows}} \mathcal{B}_c. \quad (\perp)$$

Since $\mathcal{B}$ and $\mathcal{C}$ are geometric, the compact objects of both coincide with the rigid objects, and hence duality gives us equivalences $\Delta_{\mathcal{B}} : (\mathcal{B}_c)^{op} \to \mathcal{B}_c$ and $\Delta_{\mathcal{C}} : (\mathcal{C}_c)^{op} \to \mathcal{C}_c$ which are quasi-inverse to themselves. Since f^* preserves compact objects, we have a commutative square

$$\begin{array}{ccc} (\mathcal{B}_c)^{op} & \xrightarrow{\Delta_{\mathcal{B}}} & \mathcal{B}_c \\ {\scriptstyle (f^*)^{op}}\downarrow & & \downarrow{\scriptstyle f^*} \\ (\mathcal{C}_c)^{op} & \xrightarrow[\Delta_{\mathcal{C}}]{} & \mathcal{C}_c \end{array}$$

In particular, the self-duality of $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{C}}$ implies that the composite functor $\Delta_{\mathcal{B}} \circ (f_{(1)})^{op} \circ \Delta_{\mathcal{C}} : \mathcal{C}_c \to \mathcal{B}_c$ is right adjoint to $f^* : \mathcal{B}_c \to \mathcal{C}_c$. Point (1) of Lemma 1.1.10 applied to $F := f^*$ allows us to conclude that the right adjoint f^* must preserve compact objects. The other direction follows easily from point (2) of Lemma 1.1.10 and Brown representability.

Assume now the equivalent conditions (1) and (2). By assumption we have adjunctions $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)}$. The existence of the last adjunction $f^{(1)} \dashv f_{(-1)}$ follows by Lemma 1.1.8: since f_* preserves compact objects, its right adjoint $f^{(1)}$ preserves filtered colimits, and the adjoint functor theorem does the rest. This leaves us to show the Grothendieck duality equivalence. We will actually prove something stronger: given $x, y \in \mathcal{B}$ we will show the existence of a canonical equivalence $f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y) \to f^{(1)}(x \otimes_{\mathcal{B}} y)$. First of all, we can construct the comparison map $f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y) \to f^{(1)}(x \otimes_{\mathcal{B}} y)$ by using the counit $\varepsilon : f_* f^{(1)} \to \mathrm{id}_{\mathcal{B}}$ of the adjunction $f_* \dashv f^{(1)}$. Indeed,

$$\begin{aligned} \varepsilon_x \otimes \mathrm{id}_y \in & \operatorname{Hom}_{\mathcal{B}}((f_* f^{(1)}(x)) \otimes_{\mathcal{B}} y, x \otimes_{\mathcal{B}} y) \\ & \simeq \operatorname{Hom}_{\mathcal{B}}(f_*(f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y)), x \otimes_{\mathcal{B}} y) \\ & \simeq \operatorname{Hom}_{\mathcal{C}}(f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y), f^{(1)}(x \otimes_{\mathcal{B}} y)). \end{aligned}$$

Here the first equivalence follows from the projection formula and the second one from adjunction $f_* \dashv f^{(1)}$. We need to show that this comparison map is an equivalence. Since both sides of the comparison map $f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y) \to f^{(1)}(x \otimes_{\mathcal{B}} y)$ are colimit-preserving exact functors in both variables, we can check that it is an equivalence on compact objects. Assume that $y \in \mathcal{B}_c$ is compact, hence rigid, and that $x \in \mathcal{B}$ is arbitrary. Since for every $z \in \mathcal{C}$ post-composition with the comparison map induces equivalence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(z, f^{(1)}(x) \otimes_{\mathcal{C}} f^*(y)) &\simeq \mathrm{Hom}_{\mathcal{C}}(z \otimes_{\mathcal{C}} \Delta f^*(y), f^{(1)}(x)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(z \otimes_{\mathcal{C}} f^*\Delta(y), f^{(1)}(x)) \\ &\simeq \mathrm{Hom}_{\mathcal{B}}(f_*(z \otimes_{\mathcal{C}} f^*\Delta(y)), x) \\ &\simeq \mathrm{Hom}_{\mathcal{B}}(f_*(z) \otimes_{\mathcal{C}} \Delta(y), x) \\ &\simeq \mathrm{Hom}_{\mathcal{B}}(f_*(z), x \otimes_{\mathcal{B}} y) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(z, f^{(1)}(x \otimes_{\mathcal{B}} y)),\end{aligned}$$

Yoneda's lemma allows us to conclude that the general comparison map is an equivalence whenever y is rigid. Here the first equivalence follows since $f^*(y)$ is rigid, the second one since $f^*\Delta \simeq \Delta f^*$ on rigids, the third one by adjunction $f_* \dashv f^{(1)}$, the fourth one by the projection formula, the fifth one since y is rigid and the last one again by adjunction $f_* \dashv f^{(1)}$. □

Remark 1.1.12. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor and assume that Grothendieck duality holds. Then $f^{(1)}$ preserves colimits (since f^* does). Lemma 1.1.8 implies then that its left adjoint f_* preserves compact objects. In particular, we deduce that Grothendieck duality is not only necessary for conditions (1) and (2) of Theorem 1.1.11, but it is also sufficient!

Remark 1.1.13. We can now explain our choice of terminology and notation behind the functor $f^\times = f^{(1)}$. We call it twisted inverse image since, under one of the assumptions of Theorem 1.1.11, it can be computed as by "twisting" f^* by a dualizing complex $\omega_f \in \mathcal{C}$. We denote it by $f^{(1)}$ since we take ω_f tensored with itself just one time. Indeed, one can define the functors $f^{(n)} = \omega_f^{\otimes n} \otimes_{\mathcal{C}} f^*$ and $f_{(n)} = f_*(\omega_f^{\otimes n} \otimes_{\mathcal{C}} -)$ for every $n \in \mathbb{Z}$. Without further assumptions, these functors are unrelated. However, as soon as ω_f is a compact object of $\mathcal{C}$, they fit into adjunctions $f^{(n)} \dashv f_{(-n)} \dashv f^{(n+1)}$. See [BDS16].

1.2 Geometric and Tensor t-Structures

In the previous section we have introduced two notion of finiteness by means of compact and dualizable objects. To introduce the third notion we need the additional input of a t-structure.

Definition 1.2.1. Let $\mathcal{C}$ be a presentable[11] stable $(\infty, 1)$-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We will say that the t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is *geometric* if

(1) The t-structure is accessible[12].

(2) The t-structure is compatible with filtered colimits. That is, $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$.

(3) The t-structure is right complete.

We will furthermore say that the t-structure is *excellent* if it is also left complete.

The following result explains the various assumptions in the definition of geometric t-structures.

Lemma 1.2.2. Suppose $\mathcal{C}$ is a presentable stable $(\infty, 1)$-category with accessible t-structure. Then the following are equivalent.

(1) $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$.

(2) $i_{\leq 0} : \mathcal{C}_{\leq 0} \to \mathcal{C}$ preserves filtered colimits.

(3) $i_{\leq 0}\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}$ preserves filtered colimits.

(4) $i_{\geq 0}\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}$ preserves filtered colimits.

(5) $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}_{\geq 0}$ preserves filtered colimits.

These equivalent conditions imply the following.

[11] For this definition to make sense we only need $\mathcal{C}$ to have filtered colimits.

[12] Recall that this means that the $(\infty, 1)$-category $\mathcal{C}_{\geq 0}$ is presentable. See [Lur17, Proposition 1.4.4.13] for some equivalent conditions.

(6) $i_{\geq 0} : \mathcal{C}_{\geq 0} \to \mathcal{C}$ preserves compact objects.

(7) $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}_{\leq 0}$ preserves compact objects.

Furthermore,

(a) If $\mathcal{C}$ is compactly-generated, then so is $\mathcal{C}_{\leq 0}$ (with compact objects retracts of objects of the form $\tau_{\leq 0}c$ for $c \in \mathcal{C}_c$). In this case, the above conditions are equivalent to $\tau_{\leq 0}$ preserving compact objects.

(b) If $\mathcal{C}$ and $\mathcal{C}_{\geq 0}$ are compactly-generated, then the above conditions are equivalent to $i_{\geq 0}$ preserving compact objects.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is by definition. The implication (2) $\Rightarrow$ (3) follows since $\tau_{\leq 0}$ is a left adjoint whereas the converse (3) $\Rightarrow$ (2) follows since the t-structure is accessible. A similar argument (coupled with the fact that also $\mathcal{C}_{\geq 0}$ is presentable) shows the equivalence (4) $\Leftrightarrow$ (5). To conclude, let us note that (2) $\Leftrightarrow$ (5) follows by considering the cofibre sequence of functors $\tau_{\geq 0} \to \mathrm{id} \to \tau_{\leq -1} \simeq \Sigma^{-1}\tau_{\leq 0}$ on $\mathcal{C}$. Assume now the equivalent conditions (1)-(5). Direction ($\Rightarrow$) of Lemma 1.1.8 shows that (2) implies (7) and that (5) implies (6). This leaves us to prove the "furthermore" part. Point (a) follows since a left localization of a compactly generated $(\infty, 1)$-category is again compactly generated; see [Lur09, Corollary 5.5.7.3]. For (b), if $\mathcal{C}$ and $\mathcal{C}_{\geq 0}$ are compactly generated, then Lemma 1.1.8 shows that (6) implies (5). ☐

We can also add the datum of a symmetric monoidal structure.

Definition 1.2.3. Let $\mathcal{C}$ be a stable and symmetric monoidal $(\infty, 1)$-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We will say that the $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a *tensor* t-*structure* if the connective objects $\mathcal{C}_{\geq 0}$ inherits the symmetric monoidal structure of $\mathcal{C}$. If $\mathcal{C}$ is furthermore presentable, then we will say that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a *geometric tensor* t-*structure* if it is a geometric and tensor t-structure. We give a similar definition for *excellent tensor* t-*structures*.

More explicitly, for a stable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$ with tensor product $\otimes_{\mathcal{C}}$ and unit $\mathbb{1}_{\mathcal{C}}$, a tensor t-structure is a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that $\otimes_{\mathcal{C}}$ restricts to a functor $- \otimes_{\mathcal{C}} - : \mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$ and $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}_{\geq 0}$. In particular, the inclusion functor $i_{\geq 0} : \mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ is symmetric monoidal.

Let us note that the definition of tensor t-structure is rather awkward. Indeed, it gives to the 0-connective aisle a special role among all connective aisles. To explain the problem, we need equivalent t-structures.

Definition 1.2.4. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category equipped with a pair of t–structures $(\mathcal{C}^1_{\geq 0}, \mathcal{C}^1_{\leq 0})$ and $(\mathcal{C}^2_{\geq 0}, \mathcal{C}^2_{\leq 0})$. We will say that the t–structures are *equivalent* if there exists an integer $A > 0$ such that $\mathcal{C}^1_{\geq A} \subseteq \mathcal{C}^2_{\geq 0} \subseteq \mathcal{C}^1_{\geq -A}$.

Remark 1.2.5. The previous definition can be stated also in terms of the coconnective aisle. Indeed, since the 0-connective and 0-coconnective aisles of a t-structure completely determine each other, it follows that two t-structures $(\mathcal{C}^1_{\geq 0}, \mathcal{C}^1_{\leq 0})$ and $(\mathcal{C}^2_{\geq 0}, \mathcal{C}^2_{\leq 0})$ are equivalent if and only if there exists an integer $A > 0$ such that $\mathcal{C}^1_{\leq -A} \subseteq \mathcal{C}^2_{\leq 0} \subseteq \mathcal{C}^1_{\leq A}$.

The notion of equivalence for t-structure is close to be an equivalence relation on the set of t-structures on a stable $(\infty, 1)$-category. However, as simple as it may seem, this is not the case: the collection of t-structures does not form a set. Nonetheless, to capture the geometry of a stable $(\infty, 1)$-category we should always use constructions which only depend on the "equivalence class" of the t-structure. We have already encountered examples of such constructions, since the full subcategories $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{C}^b$ are do not feel equivalent t-structures.

Remark 1.2.6. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then equivalent t-structures do not share the same categorical properties of $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. For example, if $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is accessible (or compatible with filtered colimits) then an equivalent t-structure need not to be accessible (or compatible with filtered colimits).

Standing to the above principle, it is clear that the notion of tensor t-structure is *not* stable under equivalence. For this reason, it is convenient to fix a tensor t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and work with those t-structures whose 0-connective aisle is closed under tensoring with $\mathcal{C}_{\geq 0}$.

Remark 1.2.7. Let us recall some terminology. Keller and Vossieck in [KV88] observed that it is possible to characterize the connective objects of a t-structure in terms of certain subcategories. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category. First of all, recall that a full subcategory $\mathcal{U} \subseteq \mathcal{C}$ is called a *preaisle* of $\mathcal{C}$ if

(1) $\mathcal{U}$ is closed under positive shifts, that is $\Sigma\mathcal{U} \subseteq \mathcal{U}$.

(2) $\mathcal{U}$ is closed under extensions, that is, given a cofibre sequence $x \to y \to z$ in $\mathcal{C}$ with $x, z \in \mathcal{U}$, then also $y \in \mathcal{U}$.

Given a preaisle $\mathcal{U}$ we will denote by $\mathcal{U}^{\perp}$ the right orthogonal of $\mathcal{U}$, that this the full subcategory of $\mathcal{C}$ spanned by those $y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(x, y) \simeq 0$ for all $x \in \mathcal{U}$. Finally, we will say that a preaisle $\mathcal{U}$ of $\mathcal{C}$ is an *aisle* if

(3) The inclusion $i_{\leq \mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{C}$ admits a left adjoint $\tau_{\leq \mathcal{U}}$. We call this functor the *truncation functor associated to* $\mathcal{U}$.

Keller and Vossieck proved in [KV88, Proposition 1.1] that the assignments $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \mapsto \mathcal{C}_{\geq 0}$ and $\mathcal{U} \mapsto (\mathcal{U}, \Sigma\mathcal{U}^{\perp})$ realize a correspondence between the aisles of $\mathcal{C}$ and the t-structures on $\mathcal{C}$, thus giving equivalent approach to the theory of t-structures.

Remark 1.2.8. The aisle-point of view has an advantage: it allows us to construct t-structures in reasonable stable $(\infty, 1)$-categories. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category with small colimits. Recall that a thick subcategory is a *localizing subcategory* if it is closed under small colimits in $\mathcal{C}$. Given a class of objects $S \subseteq \mathcal{C}$, recall that $\langle S \rangle$ denotes the smallest localizing subcategory of $\mathcal{C}$ containing S; similarly, we denote by $\langle S \rangle_{\geq 0}$ the smallest cocomplete preaisle containing S and call it the *cocomplete preaisle generated by* S. In particular, we will say that a preaisle $\mathcal{U}$ is *compactly generated* if there exists a set of compact objects $S \subseteq \mathcal{C}_c$ such that $\mathcal{U} = \langle S \rangle_{\geq 0}$. We then extend this definition to aisle and t-structures by requiring the underlying preaisle to be compactly generated. The result to keep in mind is [KN13, Theorem A.7]. Keller and Nicolás showed that, if $\mathcal{C}$ has small colimits, then the preaisle $\langle S \rangle_{\geq 0}$ associated to a set of compact objects S is actually an aisle, and hence the associated t-structure $(\mathcal{U}, \Sigma\mathcal{U}^{\perp})$ is compactly generated.

We can now formulate a refinement of Definition 1.2.1.

Definition 1.2.9. Let $\mathcal{C}$ be a stable and symmetric monoidal $(\infty, 1)$-category equipped with a tensor t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We will say that a preaisle $\mathcal{U}$ of $\mathcal{C}$ is a *$\otimes_{\mathcal{C}}$-preaisle with respect to* $\mathcal{C}_{\geq 0}$ if $\mathcal{C}_{\geq 0} \otimes_{\mathcal{C}} \mathcal{U} \subseteq \mathcal{U}$. We then extend this definition to aisles. In particular, a t-structure $(\mathcal{U}, \Sigma\mathcal{U}^{\perp})$ is a *tensor* t*-structure with respect to* $\mathcal{C}_{\geq 0}$ if the aisle $\mathcal{U}$ is a $\otimes_{\mathcal{C}}$-aisle with respect to $\mathcal{C}_{\geq 0}$. We use all the other adjectives with the obvious meaning.

Our next goal is to show that geometric $(\infty, 1)$-categories have a sensible theory of tensor t-structures.

Proposition 1.2.10. Let $\mathcal{C}$ be a stable homotopy theory equipped with a tensor t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ (in the sense of Definition 1.2.1). If $\mathcal{C}_{\geq 0} = \langle \mathbb{1}_{\mathcal{C}} \rangle_{\geq 0}$ then every cocomplete preaisle of $\mathcal{C}$ is a $\otimes_{\mathcal{C}}$-preaisle. In particular, every t-structure whose aisle is cocomplete is a tensor t-structures with respect to $\mathcal{C}_{\geq 0}$.

Proof. First of all, let us clarify that we are not assuming the monoidal unit $\mathbb{1}_{\mathcal{C}}$ to be compact. We are instead assuming that we are provided with a tensor t-structure whose connective aisle is generated by $\mathbb{1}_{\mathcal{C}}$. Now the proposition is a consequence of the following claim.

(*) Assume that $\mathcal{C}_{\geq 0}$ is generated by a set of objects S, that is, $\mathcal{C}_{\geq 0} = \langle S \rangle_{\geq 0}$. Then a cocomplete preaisle $\mathcal{U}$ of $\mathcal{C}$ is a $\otimes_{\mathcal{C}}$-preaisle with respect to $\mathcal{C}_{\geq 0}$ if and only if $S \otimes_{\mathcal{C}} \mathcal{U} \subseteq \mathcal{U}$.

Since the "only if" direction is immediate, let us prove the "if" direction. Assume that $S \otimes_{\mathcal{C}} \mathcal{U} \subseteq \mathcal{U}$ and consider the full subcategory $\mathcal{U}'$ of $\mathcal{C}$ spanned by those $x \in \mathcal{C}$ such that $x \otimes_{\mathcal{C}} \mathcal{U} \subseteq \mathcal{U}$. By assumption we have that $S \subseteq \mathcal{U}'$. Moreover, since $\mathcal{U}$ is a preaisle, it follows that $\mathcal{U}'$ is closed under positive shifts and extensions (here we are using that $\mathcal{C}$ is a stable homotopy theory). Since $\mathcal{U}$ is cocomplete, it follows that also $\mathcal{U}'$ is, thus showing $\mathcal{C}_{\geq 0} \otimes_{\mathcal{C}} \mathcal{U} \subseteq \mathcal{U}$. ☐

Corollary 1.2.11. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category. Then:

(1) The monoidal unit determines a tensor t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ (in the sense of Definition 1.2.1) such that[13] $\mathcal{C}_{\geq 0} = \langle \mathbb{1}_{\mathcal{C}} \rangle_{\geq 0}$.

[13] We stress that this t-structure in *not* geometric, since it is not right complete.

(2) Every t-structure with cocomplete aisle is a tensor t-structure with respect to $\mathcal{C}_{\geq 0}$.

Proof. Since $\mathcal{C}$ is geometric, the monoidal unit is compact. Hence [KN13, Theorem A.7] shows that $\langle \mathbb{1}_{\mathcal{C}} \rangle$ is the aisle of a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Since $\otimes_{\mathcal{C}}$ is exact and commutes with small colimits in both arguments, it follows that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a tensor t-structure. This proves (1). For (2), Proposition 1.2.10 shows that every t-structure with cocomplete aisle is tensor with respect to $\mathcal{C}_{\geq 0}$. □

Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ denote the tensor t-structure generated by the monoidal unit. We will refer to $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ as the *standard* t*-structure*. Standing to our discussion On Finiteness, we regard the standard t-structure as a *trivial geometry*. Non-trivial geometries may be obtained in presence of compact generators.

Lemma 1.2.12. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category, and let Let $\mathcal{G}$ be a collection of compact objects of $\mathcal{C}$ that generates[14] $\mathcal{C}$. Then there exists a geometric t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that:

(1) The coconnective objects are given by $\mathcal{C}_{\leq 0} = \{x \in \mathcal{C} \,|\, \pi_n \operatorname{Hom}_{\mathcal{C}}(g, x) = 0 \text{ for all } g \in \mathcal{G}, n > 0\}$.

(2) Let $\mathcal{E}$ be the smallest full subcategory which contains $\mathcal{G}$ and is closed under finite colimits and extensions. Then the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}$ extends to an equivalence of $(\infty, 1)$-categories $\operatorname{Ind}(\mathcal{E}) \to \mathcal{C}_{\geq 0}$.

Proof. First of all, given $\mathcal{C}_{\leq 0}$ as in (1), [Lur17, Proposition 1.4.4.11] implies the existence of t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Moreover, since $\mathcal{G}$ consists of compact objects it follows that the full subcategory $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under filtered colimits. Since it is also closed under coproducts, [Lur17, Proposition 1.2.1.19] tells us that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is right complete (and hence geometric) if and only if $\cup_{n \geq 0} \mathcal{C}_{\leq -n} \simeq 0$ consists of zero objects. Now if $x \in \mathcal{C}_{\leq -n}$ for every $n \geq 0$, then $\pi_0 \operatorname{Hom}_{\mathcal{C}}(\Sigma^n g, x) \cong 0$ for every $n \in \mathbb{Z}$ and $g \in \mathcal{G}$, thus implying that $x \simeq 0$.

We are left to prove (2). First of all, since the compact objects of $\mathcal{C}$ are closed under finite colimits and extensions, every object of $\mathcal{E}$ is compact in $\mathcal{C}$. Secondly, the full subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is also closed under finite colimits and extensions and therefore contains $\mathcal{E}$. It follows that every object of $\mathcal{E}$ is compact when viewed as an object of $\mathcal{C}_{\geq 0}$, so the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}_{\geq 0}$ extends to a fully faithful embedding $\theta : \operatorname{Ind}(\mathcal{E}) \hookrightarrow \mathcal{C}_{\geq 0}$ which commutes with small colimits. We wish to show that the essential image of θ, say $\mathcal{C}'$, is actually $\mathcal{C}' = \mathcal{C}_{\geq 0}$, that is, $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}'$. Since the proof of [Lur17, Proposition 1.4.4.11] shows that $\mathcal{C}_{\geq 0}$ is the smallest full subcategory of $\mathcal{C}$ which contains $\mathcal{G}$ and is closed under small colimits and extensions, it will suffice to show that the $\mathcal{C}'$ is closed under extensions in $\mathcal{C}$. Suppose we are given a fiber sequence $x \to y \to z$ where $x, z \in \mathcal{C}'$. We wish to prove that $y \in \mathcal{C}'$. Now z can be written as a filtered colimit $\operatorname{colim}_{i \in I} z_i$, where $z_i \in \mathcal{E}$ for every $i \in I$. In particular, it follows that y can be written as a filtered colimit of objects of the form $y \times_x z_i$. Since $\mathcal{C}'$ is closed under filtered colimits in $\mathcal{C}$, it will suffice to show that each of the $y \times_x z_i$'s belongs to $\mathcal{C}'$. Replacing z by z_i, we may reduce to the case where $z_i \in \mathcal{E}$. In this case, we can realize y as the fiber of a map $f : z \to \Sigma x$. By writing x as a filtered colimit $\operatorname{colim}_{i \in I} x_i$ with $x_i \in \mathcal{E}$, the compactness of z in $\mathcal{C}$ implies that f factors through Σx_i for some index $i \in I$. We may therefore write y as a filtered colimit of objects of the form $\operatorname{fib}(z \to \Sigma x_i)$, which are extensions of objects of $\mathcal{E}$ and therefore belong to $\mathcal{E}$. □

To go back to the discussion, assume now that the geometric $(\infty, 1)$-category $\mathcal{C}$ is compactly generated by a single compact object $G \in \mathcal{C}_c$. The above lemma implies then the existence of a geometric t-structure $(\mathcal{C}^G_{\geq 0}, \mathcal{C}^G_{\leq 0})$. We will refer to the class determined by this t-structure as the *preferred equivalence class*. Notice that this equivalence class is well defined (see for example [Nee18b, Example 0.13]). Corollary 1.2.11 implies now that $(\mathcal{C}^G_{\geq 0}, \mathcal{C}^G_{\leq 0})$ is a tensor t-structure with respect to $\mathcal{C}_{\geq 0}$. If now $G \in \mathcal{C}_{\geq -N}$ is $(-N)$-connective for some integer $N \geq 0$, it follows that the the standard t-structure is in the preferred equivalence class:

$$\mathcal{C}_{\geq N} \subseteq \mathcal{C}^G_{\geq 0} \subseteq \mathcal{C}_{\geq -N}.$$

Indeed, the inclusion $\mathcal{C}^G_{\geq 0} \subseteq \mathcal{C}_{\geq -N}$ follows since the latter contains G and both categories are closed under colimits and extension; the first inclusion $\mathcal{C}_{\geq N} \subseteq \mathcal{C}^G_{\geq 0}$ follows since G is a compact generator: it implies that the monoidal unit is $(-n)$-connective $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^G_{\geq 0}$, and after a change of index (if necessary), it implies the required inclusion. In particular, we have the following result.

[14] That is, for every nonzero object $x \in \mathcal{C}$ there exists $g \in \mathcal{G}$ and $n \in \mathbb{Z}$ for which the graded abelian group $\pi_0 \operatorname{Hom}_{\mathcal{C}}(\Sigma^n g, x)$ is nonzero.

Corollary 1.2.13. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category equipped with a compact generator $G \in \mathcal{C}_c$ and let $(\mathcal{C}^G_{\geq 0}, \mathcal{C}^G_{\leq 0})$ be the t-structure generated by G. Assume that G is $(-N)$-connective for $N \in \mathbb{N}$ in the standard t-structure. Then $\mathcal{C}^G_{\geq 0} \otimes_{\mathcal{C}} \mathcal{C}^G_{\geq 0} \subseteq \mathcal{C}^G_{\geq -N}$.

Proof. We have

$$\mathcal{C}^G_{\geq 0} \otimes_{\mathcal{C}} \mathcal{C}^G_{\geq 0} \subseteq \mathcal{C}^G_{\geq 0} \otimes_{\mathcal{C}} \mathcal{C}_{\geq -N} = \mathcal{C}^G_{\geq 0} \otimes_{\mathcal{C}} (\Sigma^{-N}\mathcal{C}_{\geq 0}) = \Sigma^{-N}(\mathcal{C}^G_{\geq 0} \otimes_{\mathcal{C}} \mathcal{C}_{\geq 0}) \subseteq \Sigma^{-N}\mathcal{C}^G_{\geq 0} = \mathcal{C}^G_{\geq -N}.$$

Here the first inclusion (or equality) follows since G is $(-N)$-connective, the second equality is by definition, the third one since $\otimes_{\mathcal{C}}$ is an exact functor, the fourth inclusion by Corollary 1.2.11 and the last equality by definition. □

Lemma 1.2.14. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category equipped with a compact generator $G \in \mathcal{C}_c$ such that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \Sigma^i G) = 0$ for $i >> 0$. Then every t-structure in the preferred equivalence class is left and right complete.

Proof. Since being left and right complete is clearly preserved by equivalent t-structures, we can prove the claim by choosing any t-structure in the preferred equivalence class. We choose the t-structure of Lemma 1.2.12 since we already know that it is right complete. To show that it is left complete we can apply [Lur17, Proposition 1.2.1.19] and show that $\cap_{n>0}\mathcal{C}_{\geq n} \simeq 0$. Let $x \in \mathcal{C}$ be non zero so that there exists a non zero morphism $\Sigma^n G \to x$ for some $n \in \mathbb{Z}$. In particular, x does not belong to $\mathcal{C}_{\leq n-1}$. Now our assumption ensures that there exists $N > 0$ such that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$. In particular, $\pi_0 \operatorname{Hom}_{\mathcal{C}}(\Sigma^n G, \mathcal{C}_{\geq N+n}) = 0$, and we deduce that x does not belong to $\mathcal{C}_{\geq N+n}$. By increasing N we deduce that x does not belong to $\cap_{n>0}\mathcal{C}_{\geq n}$. □

The next result analyses the interaction between internal homs and the tensor t-structures.

Lemma 1.2.15. Let $\mathcal{C}$ be a stable homotopy theory equipped with a tensor t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ (in the sense of Definition 1.2.1), and let $(\mathcal{U}, \Sigma\mathcal{U}^{\perp})$ be a t-structure. Then the following are equivalent:

(1) $(\mathcal{U}, \Sigma\mathcal{U}^{\perp})$ is a tensor t-structure with respect to $\mathcal{C}_{\geq 0}$.

(2) $\mathcal{U}$ is a $\otimes_{\mathcal{C}}$-aisle with respect to $\mathcal{C}_{\geq 0}$.

(3) We have $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}_{\geq 0}, \mathcal{U}^{\perp}) \subseteq \mathcal{U}^{\perp}$.

Proof. First of all, the equivalence (1) ⇔ (2) is by definition. The equivalence (2) ⇔ (3) follows instead from the equivalence $\operatorname{Hom}_{\mathcal{C}}(x \otimes_{\mathcal{C}} z, y) \simeq \operatorname{Hom}_{\mathcal{C}}(z, \underline{\operatorname{Hom}}_{\mathcal{C}}(x, y))$ valid for every $x \in \mathcal{C}_{\geq 0}, y \in \mathcal{U}^{\perp}$ and $z \in \mathcal{U}$. □

We conclude this section by adding to geometric $(\infty, 1)$-categories and functors the input of a geometric tensor t-structure.

Definition 1.2.16. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category. We will say that $\mathcal{C}$ is a t-*geometric* $(\infty, 1)$-*category* if it is equipped with a geometric tensor t-structure. Similarly, a geometric functor $f^* : \mathcal{B} \to \mathcal{C}$ between t-geometric $(\infty, 1)$-categories is called t-*geometric* if it is right t-exact.

To be precise, a geometric functor $f^* : \mathcal{B} \to \mathcal{C}$ is t-geometric if $f^*(\mathcal{B}_{\geq 0}) \subseteq \mathcal{C}_{\geq 0}$, or equivalently $f_*(\mathcal{C}_{\leq 0}) \subseteq \mathcal{B}_{\leq 0}$. Our interest in t-geometric functors lies in their behaviour at the level of the hearts.

Remark 1.2.17. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor and let $f_* : \mathcal{C} \to \mathcal{B}$ denote its right adjoint. Let $i_{\mathcal{B}} : \mathcal{B}^{\heartsuit} \hookrightarrow \mathcal{B}$ and $i_{\mathcal{C}} : \mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}$ be the inclusions of the heart. We denote by ${}^p f^*$ and ${}^p f_*$ the compositions

$${}^p f^* : \mathcal{B}^{\heartsuit} \xrightarrow{i_{\mathcal{B}}} \mathcal{B} \xrightarrow{f^*} \mathcal{C} \xrightarrow{\pi_0^{\mathcal{C}}} \mathcal{C}^{\heartsuit}, \qquad {}^p f_* : \mathcal{C}^{\heartsuit} \xrightarrow{i_{\mathcal{C}}} \mathcal{C} \xrightarrow{f_*} \mathcal{B} \xrightarrow{\pi_0^{\mathcal{B}}} \mathcal{B}^{\heartsuit}$$

and call them the *induced functors of* $f^* \dashv f_*$.

Lemma 1.2.18. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a functor between stable $(\infty, 1)$-categories equipped with t-structures. Assume that f^* is left t-exact and that is has a right adjoint f_*. Then

(1) Given $x \in \mathcal{B}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, there are a natural isomorphisms

$$ {}^{p}f^{*}(\pi_0^{\mathcal{B}}x) \to \pi_0^{\mathcal{C}}(f^{*}(x)), \qquad \pi_0^{\mathcal{B}}(f_{*}(y)) \to {}^{p}f_{*}(\pi_0^{\mathcal{C}}(y)) $$

in $\mathcal{C}^{\heartsuit}$ and $\mathcal{B}^{\heartsuit}$, respectively.

(2) The adjunction $f^{*} \dashv f_{*}$ determines an adjunction ${}^{p}f^{*} \dashv {}^{p}f_{*}$.

Proof. We now prove (1). We only prove the first isomorphism, since the second one follows by duality. Consider $x \in \mathcal{B}_{\geq 0}$ and consider the cofibre sequence $\tau_{\geq 1}^{\mathcal{B}}x \to x \to \tau_{\leq 0}^{\mathcal{B}}x$ in $\mathcal{B}$. Since x is 0-connective, we can identify $\tau_{\leq 0}^{\mathcal{B}}x$ with $\pi_0^{\mathcal{B}}(x)$. Apply f^{*} to obtain the cofibre sequence $f^{*}(\tau_{\geq 1}^{\mathcal{B}}x) \to f^{*}(x) \to f^{*}(\pi_0^{\mathcal{B}}x)$ in $\mathcal{C}$. Since f^{*} is left t-exact, it preserves connective objects. In particular, the natural map $f^{*}(\pi_0^{\mathcal{B}}x) \to \pi_0^{\mathcal{C}}f^{*}(x)$ has to be an equivalence.

The proof of point (2) is a chain of equivalences. Take $x \in \mathcal{B}^{\heartsuit}$ and $y \in \mathcal{C}^{\heartsuit}$ and compute

$$ \begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\heartsuit}}({}^{p}f^{*}(x), y) &= \mathrm{Hom}_{\mathcal{C}^{\heartsuit}}(\pi_0^{\mathcal{C}}f^{*}i_{\mathcal{B}}(x), y) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(f^{*}i_{\mathcal{B}}(x), i_{\mathcal{C}}(y)) \\ &\cong \mathrm{Hom}_{\mathcal{B}}(i_{\mathcal{B}}(x), f_{*}i_{\mathcal{C}}(y)) \\ &\cong \mathrm{Hom}_{\mathcal{B}^{\heartsuit}}(x, \pi_0^{\mathcal{B}}f_{*}i_{\mathcal{C}}(y)) = \mathrm{Hom}_{\mathcal{B}^{\heartsuit}}(x, {}^{p}f_{*}(y)). \end{aligned} $$

Here the first equality is the definition of ${}^{p}f^{*}$, the second by our assumption (since $f^{*}(\mathcal{B}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$ and since $i_{\mathcal{C}}(y)$ is 0-coconnective, the space of maps $f^{*}i_{\mathcal{B}}(x) \to i_{\mathcal{C}}(y)$ is discrete by using the standard argument with the cofibre sequence give by truncating), the third one by adjunction $f^{*} \dashv f_{*}$, and the fourth one again by assumption (since $f_{*}(\mathcal{C}_{\leq 0}) \subseteq \mathcal{B}_{\leq 0}$ and since $i_{\mathcal{B}}(x) \in \mathcal{B}_{\geq 0}$ is 0-connective, the space of maps $i_{\mathcal{B}}(x) \to f_{*}i_{\mathcal{C}}(y)$ is discrete, again by the same argument above). The last equality is the definition of ${}^{p}f_{*}$. ☐

1.3 Pseudo-Coherent and Coherent Objects

Let $\mathcal{C}$ be a presentable $(\infty, 1)$-category. Recall that and object $x \in \mathcal{C}$ is said to be *almost compact* if for every integer $n \geq 0$ the truncation $\tau_{\leq n}x$ is a compact object of $\tau_{\leq n}\mathcal{C}$. Here $\tau_{\leq n}\mathcal{C}$ denotes the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects, that is, by those objects $x \in \mathcal{C}$ such that the mapping space $\mathrm{Hom}_{\mathcal{C}}(y, x)$ is n-truncated for all $y \in \mathcal{C}$.

Definition 1.3.1. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. An object $x \in \mathcal{C}$ is called:

(1) *Pseudo-coherent* if it is connective $x \in \mathcal{C}_{\geq n}$ and almost compact as an object of $\mathcal{C}_{\geq n}$;

(2) *Coherent* if pseudo-coherent and coconnective, that is $x \in \mathcal{C}_{\leq N}$ for some $N \in \mathbb{Z}$.

The nomenclature we have chosen here comes from algebraic geometry. Pseudo-coherent objects were first introduced by Illusie in [BJG+71] via a slightly different definition. There, pseudo-coherent complexes on a scheme are defined as complexes which, locally, are (quasi-)isomorphic to bounded above complexes which admit projective resolutions by finitely generated projectives. If the scheme appears to be noetherian, then pseudo-coherent complexes are precisely the bounded-above complexes whose cohomology is coherent. To see how our definition fits in this picture we need to wait Theorem 1.4.12 and Chapter 4.

We let $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ denote the full subcategories of $\mathcal{C}$ spanned by the coherent and pseudo-coherent objects, respectively.

Remark 1.3.2. Note that the definition of $\mathrm{PCoh}(\mathcal{C})$ and $\mathrm{Coh}(\mathcal{C})$ depends on the choice of a t–structure. However, equivalent geometric t–structures lead to the same $\mathrm{PCoh}(\mathcal{C})$ and $\mathrm{Coh}(\mathcal{C})$.

The first result of this section shows some basic properties of these subcategories.

Lemma 1.3.3. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. Then $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are stable subcategories of $\mathcal{C}$. Moreover, they are closed under retracts.

Proof. To show that $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are stable subcategories of $\mathcal{C}$ it suffices to prove that they are closed under finite colimits. This follows immediately since almost compact are closed under finite colimits (because compact obejcts are), together with the stability of connective and bounded objects. A similar argument shows also that $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are closed under retracts, since compact objects are. □

Remark 1.3.4. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. Then $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are idempotent-complete.

Indeed, since $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are thick, their homotopy category are again thick. By [Lur17, Lemma 1.2.4.6] we deduce that $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are idempotent-complete if and only if $h\mathrm{Coh}(\mathcal{C}) \subseteq h\mathrm{PCoh}(\mathcal{C})$ are Karoubian. However, a thick subcategory of a Karoubian triangulated category is Karoubian. We therefore conclude by remembering that a triangulated category closed under countable direct sums is Karoubian and noting that $h\mathcal{C}$ is such an example.

Lemma 1.3.5. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. Then the full subcategory $\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is closed under the formation of geometric realization of simplicial objects.

Proof. Let $x_\bullet$ be a simplicial object of $\tau_{\leq n}(\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0})$ such that each x_k is a compact $\tau_{\leq n}(\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0})$. We wish to show that the geometric realization $|x_\bullet|$ can be computed in $\tau_{\leq n}(\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0})$ and that it is preserved by the inclusion $\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$. This follows from the fact that $\tau_{\leq n}(\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0})$ is equivalent to an $(n+1)$-category, so that the equivalence

$$|x_\bullet| \simeq \mathrm{colim}_{[k] \in \Delta^{op}_{\leq n+1}} x_k$$

exhibits the geometric realization $|x_\bullet|$ as a finite colimit, which is then preserved by the inclusion $\mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$. □

We can also analyse the interaction between (pseudo)-coherent objects and compact objects.

Remark 1.3.6. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. Assume that $\mathcal{C}$ has a single compact generator G.

(1) If G is connective, that is $G \in \mathcal{C}_{\geq -N}$ for some integer $N \geq 0$, then $\mathcal{C}_c \subseteq \mathrm{PCoh}(\mathcal{C})$.

(2) If G is bounded, then $\mathcal{C}_c \subseteq \mathrm{Coh}(\mathcal{C})$.

Indeed, being the t-structure compatible with filtered colimits, Lemma 1.2.2 implies that each truncation functor $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$ preserves compact objects, leaving us to prove that compact objects are bounded below for point (1) and bounded for point (2). This follows by the a more general assertion regarding thickness and $\mathcal{C}_c$. Indeed, if $\mathcal{C}^-$ (respectively $\mathcal{C}^+$, respectively $\mathcal{C}^b$) contains a compact generator $G \in \mathcal{C}_c$, then $\mathcal{C}^-$ (respectively $\mathcal{C}^+$, respectively $\mathcal{C}^b$) contains all of $\mathcal{C}_c$.

In order to further analyse the structure of $\mathrm{PCoh}(\mathcal{C})$ and $\mathrm{Coh}(\mathcal{C})$ we need one more definition.

Definition 1.3.7. Let $\mathcal{C}$ be a stable symmetric monoidal $(\infty, 1)$-category. A $\mathcal{C}_c$*-submodule* is a stable thick subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ closed under tensor product by compact objects. That is, $c \otimes_{\mathcal{C}} x \in \mathcal{C}_0$ for all $x \in \mathcal{C}_0$ and all compact $c \in \mathcal{C}_c$.

Then we have the following.

Lemma 1.3.8. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category equipped with a geometric tensor t-structure. Then

(1) Pseudo-coherent objects are closed under tensor product by pseudo-coherent.

(2) Coherent objects are closed under tensor product by coherent.

(3) Assume that $\mathcal{C}_c \subseteq \mathrm{PCoh}(\mathcal{C})$. Then $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ are $\mathcal{C}_c$-submodules.

Proof. We begin with point (1). Let $x, y \in \mathrm{PCoh}(\mathcal{C})$ be pseudo-coherent. Since $\mathcal{C}$ is compactly generated, we can write them as filtered colimits of compact objects, say $x \simeq \mathrm{colim}_{i \in I} x_i$ and $y \simeq \mathrm{colim}_{j \in J} y_j$ for $x_i, y_j \in \mathcal{C}_c$. Being x and y pseudo-coherent, each truncation $\tau_{\leq n} x$ and $\tau_{\leq n} y$ is a compact object of $\mathcal{C}_{\leq n}$.

Since $\tau_{\leq n}$ commutes with filtered colimits, we get that $\tau_{\leq n}x$ and $\tau_{\leq n}y$ are filtered colimits of compact objects of $\mathcal{C}_{\leq n}$. This implies that $\tau_{\leq n}x$ and $\tau_{\leq n}y$ are retract of $\tau_{\leq n}x_i$ and $\tau_{\leq n}y_j$ for some $i \in I$ and $j \in J$. Now the t-structure is a tensor t-structure, so that

$$\tau_{\leq n}(x \otimes_{\mathcal{C}} y) \simeq \tau_{\leq n}(\tau_{\leq n}x \otimes_{\mathcal{C}} \tau_{\leq n}y).$$

In particular, $\tau_{\leq n}(x \otimes_{\mathcal{C}} y)$ is a retract of

$$\tau_{\leq n}(\tau_{\leq n}x_i \otimes_{\mathcal{C}} \tau_{\leq n}y_j) \simeq \tau_{\leq n}(x_i \otimes_{\mathcal{C}} y_j).$$

But now $x_i, y_j \in \mathcal{C}_c$ are compact, so that, being $\mathcal{C}$ geometric, $x_i \otimes_{\mathcal{C}} y_j$ is compact. Lemma 1.2.2 showed that the truncation of compact objects is compact, thus proving that $\tau_{\leq n}(x \otimes_{\mathcal{C}} y)$ is compact, being a retract of a compact object. We now conclude by noting that the tensor product of $x \in \mathcal{C}_{\geq N}$ and $y \in \mathcal{C}_{\geq M}$ lies in $\mathcal{C}_{\geq N} \otimes_{\mathcal{C}} \mathcal{C}_{\geq M} \subseteq \mathcal{C}_{\geq NM}$. Point (2) follows by a similar argument. The only new input here is that if x and y are bounded above, say $x \in \mathcal{C}_{\leq N}$ and $y \in \mathcal{C}_{\leq M}$, then we can tensor the cofibre sequences $0 \to x \to \tau_{\leq N}x$ and $0 \to y \to \tau_{\leq M}y$ to get equivalence $x \otimes_{\mathcal{C}} y \simeq \tau_{\leq N}x \otimes_{\mathcal{C}} \tau_{\leq M}y$. Since

$$x \otimes_{\mathcal{C}} y \simeq \tau_{\leq N}x \otimes_{\mathcal{C}} \tau_{\leq M}y \simeq \tau_{\leq NM}(\tau_{\leq N}x \otimes_{\mathcal{C}} \tau_{\leq M}y) \simeq \tau_{\leq NM}(x \otimes_{\mathcal{C}} y)$$

we are done. Point (3) is now a formal consequence of point (1) in the case of pseudo-coherent objects and a special case of the above argument for coherent objects. □

Remark 1.3.9. We will use the previous lemma in the following situation. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category and consider the standard t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ introduced in Corollary 1.2.11. As we pointed out, this t-structure is tensor but not geometric (since not right complete in general). Assume now that $\mathcal{C}$ comes equipped with a compact generator $G \in \mathcal{C}_c$ with is $(-N)$-connective for some integer $N \geq 0$. In this case the standard t-structure is in the preferred equivalence class. If now $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \Sigma^n G) = 0$ for[15] $n >> 0$, then we can apply Lemma 1.2.14 and deduce that the standard t-structure is geometric. If we compute now the (pseudo)-coherent objects defined by this t-structure, the previous lemma implies that $\operatorname{Coh}(\mathcal{C})$ and $\operatorname{PCoh}(\mathcal{C})$ are $\mathcal{C}_c$-submodules.

1.4 Coherent $(\infty, 1)$-Categories

Even if pseudo-coherent and coherent objects behaved well categorically, it is not entirely clear how to compute them explicitly. The goal of this section is to fill this gap by showing that, under some coherentness assumption, $\operatorname{PCoh}(\mathcal{C})$ can be computed in terms of the homotopy groups of the t-structure.

The starting point is quite simple. For a t-geometric $(\infty, 1)$-category $\mathcal{C}$ we might naively expect the heart $\operatorname{PCoh}(\mathcal{C})^{\heartsuit}$ to be equivalent to the full subcategory of $\mathcal{C}^{\heartsuit}$ spanned by the compact objects. However, it is not clear that why the t-structure of $\mathcal{C}$ should restrict to pseudo-coherent objects and why $\operatorname{PCoh}(\mathcal{C})^{\heartsuit}$ should be an abelian 1-category.

Remark 1.4.1. Abelian categories whose compact objects form an abelian subcategory have already been studied in the literature. Popescu [Pop73] introduced *locally coherent abelian 1-categories*. These are compactly generated Grothendieck abelian 1-category whose compact objects form an abelian category. Examples include modules over a coherent ring and quasi-coherent sheaves over a coherent scheme.

With locally coherent abelian 1-categories in our hand, we give the following.

Definition 1.4.2. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a compact generator $G \in \mathcal{C}_c$. We will say that a geometric t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is *coherent* if

(1) There exists an integer $N > 0$ such that the compact generator $G \in \mathcal{C}_{\geq -N}$ is $(-N)$-connective and $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$.

(2) The t-structure is in the preferred equivalence class.

[15] Notice that this assumption is equivalent to $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for the (possible different) integer $N \geq 0$.

(3) For every 0-connective object $x \in \mathcal{C}_{\geq 0}$ with $\pi_0(x) \in (\mathcal{C}^\heartsuit)_c$, there exists a compact and connective object $p \in \mathcal{C}$ with a π_0-epimorphism $p \to x$ such that $\pi_n(p) \in (\mathcal{C}^\heartsuit)_c$ is compact for every $n \in \mathbb{Z}$.

(4) The heart $\mathcal{C}^\heartsuit$ is a locally coherent abelian 1-category.

Before exploiting all the properties of coherent t-structures we point out a few discrepancies between our definition and the one already available in the literature.

Warning 1.4.3. Our definition of coherent $(\infty, 1)$-category is *not* the same of Ben-Zvi, Nadler and Preygel [BZNP17, Definition 6.2.7]. Lemma 6.2.5 in loc. shows how to compute pseudo-coherent objects in terms of their homotopy groups, proving the same statement of Theorem 1.4.12. The autoher believes that this claim is actually false. Indeed, if R is a connective $\mathbb{E}_\infty$-ring spectrum with $\pi_0(R)$ coherent, then Mod_R is coherent in the sense of Definition 6.2.7. In particular, since $R \in (\mathrm{Mod}_R)_c \subseteq \mathrm{PCoh}(\mathrm{Mod}_{\mathcal{R}})$, Lemma 6.2.5 implies that every $\pi_n(R)$ is finitely presented as a $\pi_0(R)$-module. This is true if and only if R is coherent in the sense of [Lur17, Definition 7.2.4.16].

Warning 1.4.4. Lurie has a notion of *coherent Grothendieck prestable* $(\infty, 1)$*-categories*, which the reader can find in [Lur18, Section C.6.5].

Remark 1.4.5. Let $\mathcal{C}$ be a weakly approximable stable $(\infty, 1)$-category. Accordingly to [Nee18b, Proposition 2.4], the t-structure used to prove that $\mathcal{C}$ is weakly approximable must be in the same equivalence class of the t-structure generated by G. In this case $\mathcal{C}$ is coherent if and only if conditions (3) and (4) are satisfied.

Remark 1.4.6. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a geometric t-structure. Assume that $\mathcal{C}$ comes equipped with a compact generator $G \in \mathcal{C}_c$ such that $\pi_n(G) \in (\mathcal{C}^\heartsuit)_c$ is compact for every $n \in \mathbb{Z}$. Since the full subcategory spanned by objects with compact homotopy groups is stable, thick and contains the compact generator G, it follows that every compact object $c \in \mathcal{C}_c$ has compact homotopy groups $\pi_n(c) \in (\mathcal{C}^\heartsuit)_c$.

Lemma 1.4.7. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with geometric t-structure. Then the following conditions are equivalent.

(1) The t-structure on $\mathcal{C}$ restricts to one on $\mathrm{PCoh}(\mathcal{C})$.

(2) The inclusion $i_{\leq 0} : \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq 1}$ preserves compact objects.

(3) Desuspending $\Sigma^{-1} : \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq 0}$ preserves compact objects.

In this case $\mathrm{Coh}(\mathcal{C})^\heartsuit = \mathrm{PCoh}(\mathcal{C})^\heartsuit = (\mathcal{C}^\heartsuit)_c$. Furthermore, conditions (1), (2) and (3) imply– and in case $\mathcal{C}$ is right complete, are equivalent to–

(4) The subcategory of compact objects in the heart $(\mathcal{C}^\heartsuit)_c \subseteq \mathcal{C}^\heartsuit$ is abelian.

Proof. We begin by proving (1) $\Leftrightarrow$ (2). First of all, recall that the t-structure restricts to $\mathrm{PCoh}(\mathcal{C})$ if and only if for every $x \in \mathrm{PCoh}(\mathcal{C})$ the truncation $\tau_{\leq n} x \in \mathcal{C}_{\leq n}$ is again pseudo-coherent. Since $\tau_{\leq m} \circ \tau_{\leq n} \simeq \tau_{\leq \min(n,m)}$, the claim then follows by noting that $\tau_{\leq n} x \in \mathcal{C}_{\leq n}$ is compact if and only if $\tau_{\leq m} x \in \mathcal{C}_{\leq m}$ is compact for all $m \leq n$ and $i_{\leq m} : \mathcal{C}_{\leq m} \to \mathcal{C}_{\leq n}$ preserves compact objects for $m \geq n$. The equivalence (2) $\Leftrightarrow$ (3) follows instead from the fact that the two functors are equivalent once we remember that $\mathcal{C}_{\leq 1} = \Sigma^{-1}\mathcal{C}_{\leq 0}$.

Assume now the equivalent conditions (1)-(2) and (3). First of all, note that the t-structure also restricts to $\mathrm{Coh}(\mathcal{C})$, allowing us to consider the hearts $\mathrm{Coh}(\mathcal{C})^\heartsuit$ and $\mathrm{PCoh}(\mathcal{C})^\heartsuit$. In particular, we have

$$\mathrm{Coh}(\mathcal{C})^\heartsuit = \mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^\heartsuit, \qquad \mathrm{PCoh}(\mathcal{C})^\heartsuit = \mathrm{PCoh}(\mathcal{C}) \cap \mathcal{C}^\heartsuit.$$

This shows the equality $\mathrm{Coh}(\mathcal{C})^\heartsuit = \mathrm{PCoh}(\mathcal{C})^\heartsuit$. We now note that $\mathrm{Coh}(\mathcal{C})^\heartsuit = (\mathcal{C}^\heartsuit)_c$ by definition. In particular, since the heart of a t-structure is always abelian, the above equalities allow us to prove (4).

If $\mathcal{C}$ is right complete, the compact objects of $\mathcal{C}_{\leq 0}$ are bounded, giving the converse. □

In order to prove the main result of this section we need to recall some easy results.

Remark 1.4.8 ([Nee18b, Lemma 1.2])**.** Let $\mathcal{C}$ be a stable $(\infty, 1)$-category and equipped with a t–structure. If $x \in \mathcal{C}^-$ and $\pi_l(x) = 0$ for $l < i$, then $x \in \mathcal{C}_{\geq i}$. Indeed, since x belongs to $\mathcal{C}^-$, it belongs to some $\mathcal{C}_{\geq -n}$ for some $n > 0$. Thus the canonical map $\tau_{\geq -n}x \to x$ is an equivalence. Now, since $\pi_l(x) = 0$ for every $l < i$, the cofibre sequence $\tau_{\geq 1+l} \to \tau_l x \to \Sigma^{-l}\pi_l(x)$ informs us that, as long as $l < i$, the map $\tau_{\geq 1+l} \to \tau_l x$ is an equivalence. The chain of equivalences $\tau_{\geq i}x \to \tau_{i-1}x \to \cdots \to \tau_{\geq -n}x \to x$ shows then the claim.

Remark 1.4.9 ([Nee18b, Lemma 1.3])**.** Let $\mathcal{C}$ be a stable $(\infty, 1)$-category and equipped with a t–structure. Assume also that $\mathcal{C}$ is equipped with a generator G such that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some $N \in \mathbb{N}$. Then:

(1) Every object $x \in \mathcal{C}^-$, with $\pi_n(x) \cong 0$ for all $n \in \mathbb{Z}$, must vanish.

(2) If $f : x \to y$ is a morphism in $\mathcal{C}^-$ such that $\pi_n(f)$ is an isomorphism for every $n \in \mathbb{Z}$, then f is an equivalence.

Indeed, x belongs to $\cap_l \mathcal{C}_{\geq l}$ by Remark 1.4.8, so that point (1) follows by noting that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(\Sigma^i G, x) = 0$ for all $i \in \mathbb{Z}$ implies $x = 0$, being G a generator. Point (2) follows by applying (1) to the cofibre of f.

We can also prove a converse to Lemma 1.3.5.

Lemma 1.4.10. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with geometric t-structure. Assume that $\mathcal{C}$ comes equipped with a compact generator $G \in \mathcal{C}_c$ such that the t-structure is in the preferred equivalence class and there exists an integer $N > 0$ such that

(1) G is $(-N)$-connective, $G \in \mathcal{C}_{\geq -N}$.

(2) $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$.

Let $x \in \operatorname{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$ be a connective pseudo-coherent object. Then x can be obtained as the geometric realization of a simplicial object $x_\bullet$ such that each x_n is a compact object.

Proof. We may assume that the given t-structure is the one generated by G. In particular, every compact object is connective and Remark 1.3.6 implies that compact objects are pseudo-coherent. Moreover, Lemma 1.2.2 implies that the heart $\mathcal{C}^\heartsuit$ is a compactly generated 1-category, with generator $\pi_0 G$.

Standing to the $(\infty, 1)$-Dold-Kan correspondence [Lur17, Theorem 1.2.4.1 and Remark 1.2.4.3], it suffices to show that x can be written as filtered colimit over a diagram

$$D(0) \xrightarrow{f_1} D(1) \to \dots$$

where each $\Sigma^{-n}\operatorname{cofib}(f_n)$ is compact. We construct the sequence by induction; we agree by convention that f_0 denotes the zero map $0 \to D(0)$. Assume that we have constructed

$$D(0) \xrightarrow{f_1} D(1) \to \cdots \to D(n) \xrightarrow{g} x$$

with $D(i)$ in $\mathcal{C}_c$ for $0 \leq i \leq n$ and $\operatorname{fib}(g) \in \mathcal{C}_{\geq n}$. Since x is pseudo-coherent, the fibre $\operatorname{fib}(g)$ is pseudo-coherent. In particular, bottom homotopy group $\pi_n \operatorname{fib}(g) \in (\mathcal{C}^\heartsuit)_c$ is compact in the heart. Since $\mathcal{C}^\heartsuit$ is compactly generated by $\pi_0 G$, we can pick a π_0-epimorphism $\Sigma^n Q \to \operatorname{fib}(g)$ where Q is compact and connective. We now construct $D(n+1)$ as the cofibre

$$\begin{array}{ccccc} \Sigma^n Q & \longrightarrow & \operatorname{fib}(g) & \longrightarrow & D(n) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ 0 & \longrightarrow & \operatorname{fib}(g)/D(n) & \longrightarrow & D(n+1) \end{array}$$

Since both $\Sigma^n Q$ and $D(n)$ are compact, also $D(n+1)$ is compact. Moreover, $\operatorname{cofib}(f_{n+1}) \simeq \Sigma^{n+1}Q$ is a compact object. Now the universal property of pushouts implies the existence of a unique $g' : D(n+1) \to x$ such that $g \simeq g' \circ f_{n+1}$. Since the octahedral axiom provides a cofibre sequence $\Sigma^n Q \to \operatorname{fib}(g) \to \operatorname{fib}(g')$, the long exact sequence of homotopy groups in $\mathcal{C}^\heartsuit$ implies that $\operatorname{fib}(g') \in \mathcal{C}_{\geq n+1}$. This concludes the inductive construction. Since for fixed i the maps $\pi_i D(n) \to \pi_i x$ are isomorphisms for $n >> 0$, we conclude that the canonical map $\operatorname{colim}_n D(n) \to x$ is an equivalence by applying Remark 1.4.9. □

In order to prove the main result of this section we need some terminology.

Remark 1.4.11 (Reminds on Grothendieck Abelian Categories)**.** Abelian categories furnish different finiteness conditions, which are modelled on what happens in categories of modules. Let $\mathcal{A}$ be an abelian category satisfying Grothendieck axiom AB5.

(1) An object $x \in \mathcal{A}$ is called *of finite type* or also *finitely generated* if, for any filtered colimit $\operatorname{colim}_{i\in I} y_i$ of monomorphisms, the natural map $\operatorname{colim}_{i\in I} \operatorname{Hom}_{\mathcal{A}}(x, y_i) \to \operatorname{Hom}_{\mathcal{A}}(x, \operatorname{colim}_{i\in I} y_i)$ is an isomorphisms.

(2) An object $x \in \mathcal{A}$ is called *finitely presented* if it is of finite type and for every epimorphism $p : y \to x$ from an object of finite type the kernel $\ker(p)$ is also of finite type.

(3) An object $x \in \mathcal{A}$ is called *coherent* if it is of finite type and for every morphism $p : y \to x$ from an object of finite type the kernel $\ker(p)$ is also of finite type.

(4) An object $x \in \mathcal{A}$ is called *compact* if it satisfies the usual definition, that is, $\operatorname{Hom}_{\mathcal{A}}(x, -)$ preserves filtered colimits.

Clearly, every coherent object is finitely presented and every finitely presented object is always finitely generated. If $\mathcal{A}$ is also Grothendieck abelian, then an object is compact if and only if it is finitely presented, see [Ste12, Proposition V.3.4]. The other missing implications require further assumptions on $\mathcal{A}$. If $\mathcal{A}$ is locally coherent, then every finitely presented object is coherent, implying equivalence between (2), (3) and (4). If $\mathcal{A}$ is locally noetherian, then every finitely generated object is finitely presented, implying equivalence between all the above assertions.

Then we have the following.

Theorem 1.4.12. Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$-category equipped with a coherent t-structure. Then:

(1) $\operatorname{Coh}(\mathcal{C})^\heartsuit = \operatorname{Coh}(\mathcal{C}) \cap \mathcal{C}^\heartsuit$ consists precisely of the compact objects of $\mathcal{C}^\heartsuit$.

(2) $x \in \operatorname{PCoh}(\mathcal{C})$ if and only if $\pi_n x \in \operatorname{Coh}(\mathcal{C})^\heartsuit$ and $\pi_n x = 0$ for $n << 0$.

(3) $x \in \operatorname{Coh}(\mathcal{C})$ if and only if $\pi_n x \in \operatorname{Coh}(\mathcal{C})^\heartsuit$ and $\pi_n x = 0$ for all but finitely many n.

In particular, $\operatorname{PCoh}(\mathcal{C})$ is the left t-completion of $\operatorname{Coh}(\mathcal{C})$.

Proof. Point (1) one follows by Lemma 1.4.7. Indeed, the t-structure on $\mathcal{C}$ is right complete, so that, being $\mathcal{C}^\heartsuit$ a locally abelian 1-category, one of the equivalent conditions (1), (2) and (3) hold. Hence $\operatorname{Coh}(\mathcal{C})^\heartsuit = (\mathcal{C}^\heartsuit)_c$ consists precisely of the compact objects of $\mathcal{C}^\heartsuit$.

Let us prove (2). Let $x \in \mathcal{C}$. Since both conditions imply that x is n-connective for some $n \in \mathbb{Z}$, we will assume that $x \in \mathcal{C}_{\geq 0}$ is connective. We wish to show that $x \in \operatorname{PCoh}(\mathcal{C})$ if and only if $\pi_n x \in \operatorname{Coh}(\mathcal{C})^\heartsuit$ for every $n \in \mathbb{Z}$.

Suppose first that x is pseudo-coherent. We will argue by induction on $n \in \mathbb{N}$ that $\pi_n x$ is compact in $\mathcal{C}^\heartsuit$. Since for $n = 0$ the claim is trivial (it simply reduces to note that $\pi_0 x \simeq \tau_{\leq 0} x$ is compact, being $x \in \mathcal{C}_{\geq 0}$ pseudo-coherent), we can directly proceed with the inductive step. Since $\mathcal{C}$ is coherent and $\pi_0 x \in (\mathcal{C}^\heartsuit)_c$ we can find a π_0-epimorphism $p \to x$ from a compact and 0-connective object $p \in \mathcal{C}_c$ such that $\pi_n p \in (\mathcal{C}^\heartsuit)_c$. Since the compact generator G is bounded above, we can apply Remark 1.3.6 to discover that every compact object is pseudo-coherent. Thus $p \in \mathcal{C}_c$ is pseudo-coherent so that the fibre f of $p \to x$ is pseudo-coherent, and 0-connective. The inductive hypothesis then ensures that $\pi_i f \in (\mathcal{C}^\heartsuit)_c$ is compact for $0 \leq i \leq n-1$. Consider then the exact sequence

$$0 \to \operatorname{coker}(\pi_n f \to \pi_n p) \to \pi_n x \to \ker(\pi_{n-1} f \to \pi_{n-1} p) \to 0.$$

Now $\pi_{n-1} f$ and $\pi_{n-1} p$ are compact in $\mathcal{C}^\heartsuit$, the first one by the inductive hypothesis and the second one by construction. Since $(\mathcal{C}^\heartsuit)_c$ is abelian, being $\mathcal{C}$ coherent, it follows that $\ker(\pi_{n-1} f \to \pi_{n-1} p)$ is compact, and, in particular, finitely generated. On the other side, since $\pi_n p$ is compact, hence finitely generated, it follows that $\operatorname{coker}(\pi_n f \to \pi_n p)$ is again finitely generated. In particular, $\pi_n x$ is finitely generated since it fits in a short exact sequence between two finitely generated objects. The exact sequence

$$0 \to \operatorname{coker}(\pi_n p \to \pi_n x) \to \pi_n f \to \ker(\pi_{n-1} p \to \pi_{n-1} x) \to 0$$

shows then that $\pi_n f$ is finitely generated. Indeed, the cokernel of $\pi_n p \to \pi_n x$ is finitely generated being $\pi_n x$ finitely generated, and the kernel of $\pi_{n-1} p \to \pi_{n-1} x$ is compact, being $\pi_{n-1} p$ and $\pi_{n-1} x$ compact in the locally coherent abelian 1-category $\mathcal{C}^\heartsuit$. By looking again at the exact sequence

$$0 \to \operatorname{coker}(\pi_n f \to \pi_n p) \to \pi_n x \to \ker(\pi_{n-1} f \to \pi_{n-1} p) \to 0$$

we see that $\pi_n x$ is compact, since it fits into a short exact sequence between two compact objects. The compactness of the first term follows since it is the cokernel of a map from a finitely generated to a compact object. This concludes the proof: $\pi_n x$ is compact in $\mathcal{C}^\heartsuit$ for every $n \in \mathbb{Z}$.

Conversely, assume that $\pi_n x \in (\mathcal{C}^\heartsuit)_c$ for every $n \in \mathbb{N}$. We will prove that x can be obtained as geometric realization of a simplicial object $x_\bullet$ such that every x_i is compact. As in the proof of Lemma 1.4.10, it will suffice to show that x can be obtained as the colimit of a sequence

$$D(0) \xrightarrow{f_1} D(1) \xrightarrow{f_1} D(2) \to \dots$$

such that $\Sigma^{-n}\operatorname{cofib}(f_n)$ is compact. Here we agree by convention that f_0 denotes the zero map $0 \to D(0)$. We now proceed by induction. Assume that we have constructed a diagram

$$D(0) \to \cdots \to D(n) \xrightarrow{g} x$$

with $D(i) \in \operatorname{PCoh}(\mathcal{C})$ pseudo-coherent for $0 \leq i \leq n$ that $\operatorname{fib}(g)$ is n-connective. The difference with the proof of Lemma 1.4.10 appears now. We cannot deduce that $\operatorname{fib}(g)$ is pseudo-coherent, since we do not know if x is pseudo-coherent. However, the short exact sequence

$$0 \to \operatorname{coker}(\pi_{n+1} D(n) \to \pi_{n+1} x) \to \pi_n \operatorname{fib}(g) \to \ker(\pi_n D(n) \to \pi_n x) \to 0$$

shows that $\pi_n \operatorname{fib}(g)$ is compact, since it fits between two compact objects. Here we have used our assumption on x and the first part of the proof to deduce that every $\pi_i D(n)$ and $\pi_i x$ is compact in the locally coherent abelian 1-category $\mathcal{C}^\heartsuit$. In particular, since $\Sigma^{-n}\operatorname{fib}(g)$ is connective with compact π_0, the coherentness of $\mathcal{C}$ allows us to pick a π_0-epimorphism $\beta : \Sigma^n p \to \operatorname{fib}(g_n)$ from a compact and 0-connective object $p \in \mathcal{C}$ such that $\pi_n p \in (\mathcal{C}^\heartsuit)_c$. We now define $D(n+1)$ to be the cofibre

$$\begin{array}{ccccc} \Sigma^n p & \xrightarrow{\beta} & \operatorname{fib}(g_n) & \longrightarrow & D(n) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ 0 & \longrightarrow & \operatorname{cofib}(\beta) & \longrightarrow & D(n+1) \end{array}$$

of the composite map $\Sigma^n p \to \operatorname{fib}(g_n) \to D(n)$. By the universal property of pushouts we also obtain a map $g' : D(n+1) \to x$ such that $g \simeq g' \circ f_{n+1}$. By means of the octahedral axiom, we obtain a cofibre sequence $\Sigma^n p \to \operatorname{fib}(g) \to \operatorname{fib}(g')$, and the associated long exact sequence of homotopy groups proves that $\operatorname{fib}(g')$ is $(n+1)$-connective. Since the maps $\pi_i D(n) \to \pi_i x$ are isomorphisms for $n >> 0$ and fixed i, Remark 1.4.9 implies that the natural map $\operatorname{colim} D(n) \to x$ is an equivalence, concluding the proof of point (2).

Point (3) follows immediately from point (2) by noticing that coherent objects are defined to be bounded above. ☐

1.5 Pseudo-Compact Objects

We now introduce Neeman's pseudo-compact objects.

Definition 1.5.1. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. An object $x \in \mathcal{C}$ is called *pseudo-compact* if for every $n > 0$ there exists a cofibre sequence $c \to x \to d$ where $c \in \mathcal{C}_c$ is compact and $d \in \mathcal{C}_{\geq n}$ is n-connective.

We will denote by $\mathcal{C}_c^-$ the full subcategory of $\mathcal{C}$ spanned by the pseudo-compact objects. It can be easily shown that $\mathcal{C}_c^-$ is a stable subcategory of $\mathcal{C}$, by adjusting the argument of [Nee18b, Lemma 2.9]. Moreover, if $\mathcal{C}$ is compactly generated and equipped with a compact generator $G \in \mathcal{C}$ such that $G \in \mathcal{C}_{\geq -N}$ and $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some integer $N > 0$, then [Nee18b, Proposition 2.10] shows that $\mathcal{C}_c^-$ is also thick.

Remark 1.5.2. Let $\mathcal{C}$ be geometric $(\infty,1)$-category equipped with a tensor t-structure. Assume that $\mathcal{C}$ is generated by a single compact object G which is bounded below, say $G \in \mathcal{C}_{\geq -N}$ for $N > 0$. Then $\mathcal{C}_c^-$ is a $\mathcal{C}_c$-submodule.

Consider $x \in \mathcal{C}_c^-$ and $c \in \mathcal{C}_c$. Since $\mathcal{C}$ is generated by a single compact object which is bounded below, we have that $c \in \mathcal{C}_{\geq -m}$ for some $m > 0$. Fix $n > 0$. Being $x \in \mathcal{C}_c^-$ pseudo-compact, we can apply the definition with $n + m > 0$ and get a cofibre sequence $c' \to x \to d$ with $c' \in \mathcal{C}_c$ and $d \in \mathcal{C}_{\geq n+m}$. By tensoring with c we get the cofibre sequence $c' \otimes_{\mathcal{C}} c \to x \otimes_{\mathcal{C}} c \to d \otimes_{\mathcal{C}} c$ where $c' \otimes_{\mathcal{C}} c$ is again compact and $d \otimes_{\mathcal{C}} c \in \mathcal{C}_{\geq n+m} \otimes_{\mathcal{C}} \mathcal{C}_{\geq -m} \subseteq \mathcal{C}_{\geq n}$.

Definition 1.5.3. [Nee18b, Definition 7.3] Let $\mathcal{C}$ be a geometric $(\infty,1)$-category equipped with a single compact generator $G \in \mathcal{C}_c$. Assume that $\mathcal{C}$ comes equipped with the t-structure generated by G introduced in Lemma 1.2.12[16]. Assume furthermore that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some integer $N > 0$. A *strong* $\langle G \rangle_n$*–approximating system* is a sequence of objects and morphisms $D(1) \to D(2) \to \dots$ such that:

(1) Each $D(i)$ belongs to $\langle G \rangle_n$.
(2) The map $\pi_l(D(i)) \to \pi_l(D(i+1))$ is an isomorphism in $\mathcal{C}^\heartsuit$ whenever $l \leq i$.

In this definition we also allow $n = \infty$ by declaring $\langle G \rangle_\infty = \mathcal{C}_c$. Given an object $x \in \mathcal{C}$, a *a strong* $\langle G \rangle_n$*-approximating system for* x is a strong $\langle G \rangle_n$–approximating system $D(1) \to D(2) \to \dots$ with a map to x that induces isomorphism $\pi_l(D(i)) \to \pi_l(x)$ whenever $i \geq l$.

The notation $\langle G \rangle_n$ is borrowed from [Nee18b, Reminder 0.8]. It denotes the full thick subcategory of $\mathcal{C}$ whose objects are finite coproducts of at most n-extensions of the objects $\Sigma^i G$, for $i \in \mathbb{Z}$.

Our next goal is to show that every pseudo-compact object admits a $\mathcal{C}_c$-strong approximating system. This is a key result in Neeman's theorem, and we now present his argument for completeness. But first we recall a couple of technical results.

Remark 1.5.4 ([Nee18b, Lemma 1.4 and Remark 1.5])**.** Let $\mathcal{C}$ be a presentable stable $(\infty,1)$-category. Assume that $\mathcal{C}$ comes equipped with a t-structure with both $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ closed under the small colimits in $\mathcal{C}$. Let $D(1) \to D(2) \to \dots$ be a sequence of objects and morphisms in $\mathcal{C}$. Then, for every $l \in \mathbb{Z}$, there is an exact sequence

$$0 \to \operatorname{colim}_i \pi_l(D(i)) \to \pi_l(\operatorname{colim}_i D(i)) \to \operatorname{colim}_i^1 \pi_{l-1}(D(i)) \to 0$$

in the heart $\mathcal{C}^\heartsuit$. Here colim^1 is the derived functor of the colimit. In particular, if the sequences $\pi_l(D(1)) \to \pi_l(D(2)) \to \dots$ eventually stabilize for every l, then the colim^1 terms all vanish, and the natural map $\operatorname{colim}_i \pi_l(D(i)) \to \pi_l(\operatorname{colim}_i D(i))$ is an isomorphism.

Remark 1.5.5 ([Nee18b, Lemma 2.8])**.** Let $\mathcal{C}$ be a presentable stable $(\infty,1)$-category. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t–structure on $\mathcal{C}$. Assume that $\mathcal{C}$ comes equipped with a compact generator $G \in \mathcal{C}_c$ and an integer $N > 0$ such that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$. Then for any compact object $c \in \mathcal{C}_c$ there exists an integer $n > 0$, depending on c, with $\pi_0 \operatorname{Hom}_{\mathcal{C}}(c, \mathcal{C}_{\geq n}) = 0$. Indeed, since G is a compact generator, it follows that $c \in \mathcal{C}_c = \langle G \rangle$ must belong to some $\langle G \rangle^{[-m,m]}$. By picking $n = m + N$ the claim follows.

We can now prove the following.

Lemma 1.5.6 ([Nee18b, Lemma 7.5])**.** Let $\mathcal{C}$ be a presentable stable $(\infty,1)$-category equipped with a single compact generator $G \in \mathcal{C}_c$. Assume that $\mathcal{C}$ is equipped with the t-structure generated by G introduced in Lemma 1.2.12[17]. Assume furthermore that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some $N \in \mathbb{N}$. Then:

(1) Every $\langle G \rangle_n$-strong approximating system $D(1) \to D(2) \to \dots$ is a strong $\langle G \rangle_n$-approximating system for the filtered colimit $\operatorname{colim}_i D(i)$. Moreover $\operatorname{colim}_i D(i)$ belongs to $\mathcal{C}_c^-$.
(2) Given $x \in \mathcal{C}^-$ and a strong $\langle G \rangle_n$-approximating system D for x, then the canonical map $\operatorname{colim}_i D(i) \to x$ is an equivalence.

[16] But every t-structure in the preferred equivalence class with presentable aisle will work.
[17] But every t-structure in the preferred equivalence class with presentable aisle will work.

(3) Every object $x \in \mathcal{C}_c^-$ admits a strong $\mathcal{C}_c$–approximating system.

Proof. Let us prove (1) first. Take a $\langle G\rangle_n$-strong approximating system $D(1) \to D(2) \to \dots$ and let $d = \operatorname{colim}_i D(i)$ be the colimit. We first show that d is in $\mathcal{C}_{\geq -n}$ for some $n > 0$. Since every factor $D(i)$ in is $\langle G\rangle_n$, it is also in $\mathcal{C}_c$. In particular, being $G \in \mathcal{C}_{\geq 0}$, every $D(i)$ in $\mathcal{C}^-$. (Note that if we had picked a t-structure the preferred equivalence class the argument would have been the same). Chose now $n > 0$ such that $D(1) \in \mathcal{C}_{\geq -n}$. Since the map $\pi_l(D(1)) \to \pi_l(D(m))$ is an isomorphism for all $l \leq 1$, it follows that $\pi_l(D(m)) = 0$ for all $l > -n$ and all m. Remark 1.4.8 implies now that the $D(m)$ all lie in $\mathcal{C}_{\geq -n}$. Hence the colimit d also belongs to $\mathcal{C}_{\geq -n}$. Now Remark 1.5.4 tells us that the map $\operatorname{colim}_i \pi_l(D(i)) \to \pi_l(d)$ is an isomorphism for every $l \in \mathbb{Z}$. On the other side, the cofibre sequence $D(m) \to D \to \operatorname{cofib}_m$ lies in $\mathcal{C}^-$, and since the map pi$_l$($D(m)) \to \pi_l(d)$ is an isomorphism for $m \geq l$ we deduce that $\pi_l(\operatorname{cofib}_m) = 0$ for all $m \geq l$. By applying Remark 1.4.8 once again, we discover that $\operatorname{cofib}_m \in \mathcal{C}_{\geq m+1}$, and as $D(m) \in \langle G\rangle_n \subseteq \mathcal{C}_c$ it follows that $d \in \mathcal{C}_c^-$

Having completed the proof of (1), we now use it to prove (2). Since the map $\operatorname{colim}_i D(i) \to x$ is a morphism from an object of $\mathcal{C}_c^-$ to and object in $\mathcal{C}^-$, it must be a morphism in $\mathcal{C}^-$. Now our assumption coupled with Remark 1.5.4 tells us that the natural map $\pi_l(\operatorname{colim}_i D(i)) \to \pi_l(x)$ is an isomorphism in $\mathcal{C}^\heartsuit$. Apply now Remark 1.4.9 to deduce the claim.

Let us conclude the proof by showing (3). We go by induction. Since $x \in \mathcal{C}_c^-$, there exists a cofibre sequence $D(1) \to x \to \operatorname{cofib}_1$ with $D(1) \in \mathcal{C}_c$ and $\operatorname{cofib}_1 \in \mathcal{C}_{\geq 3}$. When $l \leq 1$, the exact sequence

$$\pi_{l+1}(\operatorname{cofib}_1) \to \pi_l(D(1)) \to \pi_l(x) \to \pi_l(\operatorname{cofib}_1)$$

has $\pi_{l+1}(\operatorname{cofib}_1) = 0 = \pi_l(\operatorname{cofib}_1)$, starting the construction of $D(n)$. Suppose now that we have constructed the sequence up to an integer $n > 0$, that is we have a map $f_m : D(m) \to x$, with $D(m) \in \mathcal{C}_c$, and so that $\pi_l(f_m)$ is an isomorphism for all $m \leq l$. Remark 1.5.5 allows us to choose an integer $N > 0$ so that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(D(m), \mathcal{C}_{\geq N}) = 0$. Because x belongs to $\mathcal{C}_c^-$, we may choose a cofibre sequence $D(m+1) \to x \to \operatorname{cofib}_{m+1}$ with $D(m+1) \in \mathcal{C}_c$ and $\operatorname{cofib}_{m+1} \in \mathcal{C}_{N+m+3}$. As in the paragraph above we get that the map $\pi_l(D(m+1)) \to \pi_l(x)$ is an isomorphism for all $m+1 \geq l$. Since the composite $D(m) \to x \to D(m+1)$ is null-homotopic, the map f_n must factor as $D(m) \to D(m+1) \to x$, concluding the proof. ☐

We can now prove our comparison result between pseudo-coherent and pseudo-compact objects.

Proposition 1.5.7. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category equipped with a single compact generator $G \in \mathcal{C}_c$. Assume that the t-structure is in the preferred equivalence class Assume furthermore that there exists an integer $N > 0$ such that

(1) G is $(-N)$-connective, $G \in \mathcal{C}_{\geq -N}$.

(2) $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$.

Then $\operatorname{PCoh}(\mathcal{C}) = \mathcal{C}_c^-$.

Proof. Let $x \in \mathcal{C}$. We want to show that $x \in \operatorname{PCoh}(\mathcal{C})$ if and only if $x \in \mathcal{C}_c^-$. First of all, since $\mathcal{C}$ has a single compact generator $G \in \mathcal{C}_c$ which is $(-N)$-connective for some integer $N > 0$, Remark 1.3.6 tells us that every compact object is pseudo-coherent, hence bounded below. In particular, every object in $\mathcal{C}_c^-$ is an extension of bounded below objects, and hence bounded below. Since every object in $\operatorname{PCoh}(\mathcal{C})$ is bounded below, we can freely assume that $x \in \mathcal{C}_{\geq 0}$ is 0-connective.

Assume first that $x \in \operatorname{PCoh}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$. The proof of Lemma 1.4.10 showed that x can be written as a filtered colimit $x \simeq \operatorname{colim}_i D(i)$ where each $D(i)$ is compact. We can now apply [Lur17, Proposition 1.2.4.5]. We deduce the existence of a spectral sequence $\{E^r_{p,q}, r\}_{r\geq 1}$ in $\mathcal{C}^\heartsuit$ with the following properties.

(a) For each $r \geq 1$, the objects $E^r_{p,q}$ vanish unless $p, q \geq 0$.

(b) Fix $p, q \geq 0$. For $r > p, q+1$, we have canonical isomorphisms $E^r_{p,q} \cong E^{r+1}_{p,q} \cong \dots$ in the abelian category $\mathcal{C}^\heartsuit$. We let $E^\infty_{p,q} \in \mathcal{C}^\heartsuit$ denote the colimit of this sequence of isomorphisms, so that $E^\infty_{p,q} \cong E^{r'}_{p,q}$ for all $r' \geq r$.

(c) For $0 \leq m \leq n$, we have $\operatorname{cofib}(D(m) \to D(n)) \in \mathcal{C}_{\geq m+1}$.

(d) Fix an integer $n \in \mathbb{Z}$. The map $\pi_n D(k) \to \pi_n D(k+1)$ is an epimorphism for $k = n$ and an isomorphism for $k > n$. In particular, we have isomorphisms $\pi_n D(n+1) \cong \pi_n D(n+2) \cong \dots$. We let A_n denote the colimit of this sequence of isomorphisms, so that we have isomorphisms $A_n \cong \pi_n D(k)$ for $k > n$.

(e) For each integer $n \geq 0$, the object $A_n \in \mathcal{C}^\heartsuit$ admits a finite filtration

$$0 = F^{-1}A_n \subseteq F^0 A_n \subseteq \cdots \subseteq F^n A_n = A_n,$$

where $F^p A_n$ is the image of the map $\pi_n D(p) \to \pi_n D(n+1) \cong A_n$. We have canonical isomorphisms $F^p A_{p+q}/F^{p-1}A_{p+q} \cong E^\infty_{p,q}$.

(f) Since $\mathcal{C}$ has small limits, we have for every $n \in \mathbb{Z}$ canonical isomorphisms $\pi_n x \cong A_n$ in the abelian category $\mathcal{C}^\heartsuit$.

Fix now $n > 0$. Then the map $D(n+1) \to x$ is a morphism from a compact object to x, and it is a π_i-equivalence for every $i \leq n$, hence its cofibre lies in $\mathcal{C}_{\geq n}$, thus showing $x \in \mathcal{C}^-_c$.

Assume now that $x \in \mathcal{C}^-_c \cap \mathcal{C}_{\geq 0}$. We will prove that x is pseudo-coherent by showing that it can be written as filtered colimit $x \simeq \operatorname{colim}_i D(i)$ of 0-connective and pseudo-coherent objects $D(i)$. Then the $(\infty,1)$-Dold-Kan correspondence [Lur17, Theorem 1.2.4.1] coupled with Lemma 1.3.5 will conclude. Now we can easily modify the proof of point (3) of Lemma 1.5.6 and pick the $D(i)$'s in $\mathcal{C}_{\geq 0} \cap \mathcal{C}_c$. Since x is 0-connective (hence in $\mathcal{C}^-$), point (2) allows us to apply Remark 1.4.9. We deduce that the canonical map $\operatorname{colim}_i D(i) \to x$ is an equivalence, thus concluding the proof. □

2 Duality for Stable $(\infty, 1)$-Categories

The goal of this chapter is to discuss some categorical properties of the categories we have introduces so far. In particular, in Section 2.1 we will compare the theory of stable homotopy theories with the theory of small stable idempotent-complete $(\infty,1)$-categories. We will also discuss the presentability of the two and we will show that geometric $(\infty,1)$-categories can be identified with a subcategory of compactly generated stable homotopy theories.

The core of this chapter is Section 2.2. In particular, we will prove that every geometric functor $f^* : \mathcal{B} \to \mathcal{C}$ contains enough information to exhibit $\mathcal{C}$ as a Frobenius algebra object of $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{L,\omega}_{st})$. This will be Theorem 2.2.5.

2.1 Morita Theory

Recall that $\mathrm{Cat}^{st}_{(\infty,1)}$ denotes the $(\infty,1)$-category of of small stable $(\infty,1)$-categories and exact functor. From now on, we will denote by $\mathrm{Cat}^{perf}_{(\infty,1)}$ the full subcategory of $\mathrm{Cat}^{st}_{(\infty,1)}$ spanned by the stable idempotent-complete $(\infty,1)$-categories. By [Lur09, Section 5.1.4], any $(\infty,1)$-categories admits an idempotent completion, defining then a Bousfield localization

$$\mathrm{Cat}^{st}_{(\infty,1)} \underset{\longleftarrow}{\overset{\mathrm{Idem}}{\longrightarrow}} \mathrm{Cat}^{perf}_{(\infty,1)}, \quad \perp$$

We call Idem the *idempotent-completion functor*. It is defined by $\mathrm{Idem}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C})_c$, that is, taking compact objects in the Ind-completion.

By [Lur09, Proposition 5.5.7.8], the Ind-category construction provides an equivalence between small idempotent-complete $(\infty,1)$-categories and compactly generated $(\infty,1)$-categories. Since the Ind-category of a stable $(\infty,1)$-category is again stable by [Lur17, Proposition 1.1.3.6], the above correspondence refines to an equivalence

$$\mathrm{Ind} : \mathrm{Cat}^{perf}_{(\infty,1)} \to \mathrm{Pr}^{L,\omega}_{st}. \tag{Ind}$$

The inverse is given by taking compact objects.

Remark 2.1.1. Is it possible to study the Ind-completion as a functor $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{st}}_{(\infty,1)} \to \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$. Taking compact objects defines a functor in the other way. However, these two functors are not in general inverse to each other, since taking compact objects always produce a small idempotent-complete (stable) $(\infty,1)$-category.

We now define a symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ and $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ that makes the Ind-completion symmetric monoidal. By [Lur17, Proposition 4.8.2.18], the $(\infty,1)$-category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ inherits from Pr^{L} a symmetric monoidal structure whose monoidal unit is given by the $(\infty,1)$-category of spectra Sp. This structure is also closed, since the internal object is given by the presentable stable $(\infty,1)$-category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})$ of colimit-preserving functors. This tensor product induces a tensor product on $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$. Indeed, for every $\mathcal{C}_1, \mathcal{C}_2$ in $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ the $(\infty,1)$-category

$$\mathcal{C}_1 \otimes \mathcal{C}_2 := (\mathrm{Ind}(\mathcal{C}_1) \otimes \mathrm{Ind}(\mathcal{C}_2))_c,$$

obtained taking compact objects of the tensor product in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of the Ind-completions, is still a small stable and idempotent-complete $(\infty,1)$-category. Ben-Zvi, Francis and Nadler have studied this construction.

Lemma 2.1.2 ([BZFN10, Proposition 4.4])**.** There is a symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ such that for every small stable idempotent-complete $(\infty,1)$-categories $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$, the $(\infty,1)$-category of exact functors $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D})$ is equivalent to the full subcategory of all functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ that preserve finite colimits in $\mathcal{C}_1$ and $\mathcal{C}_2$ separately. The monoidal unit is given by compact spectra.

Furthermore, passing to the corresponding stable presentable $(\infty,1)$-categories of Ind-objects is naturally a symmetric monoidal functor $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Remark 2.1.3. The equivalence Equation Ind induces a symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$, making the Ind-construction a symmetric monoidal equivalence. Note that is is also possible to induce a symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ from $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ via the criterion [Lur17, Remark 2.2.1.2]. By means of the Ind-completion, a symmetric monoidal structure is induced on $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$. Fortunately, these two construction are exactly the same.

We can now describe commutative algebra objects in all the symmetric monoidal structures we have introduced so far. Commutative algebra objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ are the stable homotopy theories introduced in Definition 1.1.1, whereas the $(\infty,1)$-category of commutative algebra objects in $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ defines the $(\infty,1)$-category

$$2\text{-Ring} = \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)})$$

of *2-rings*, studied in [Mat16, Section 2.2]. The equivalence Equation Ind shows that the theory of 2-rings is equivalent to the one of commutative algebra objects in

$$2\text{-Ring} = \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}) \simeq \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}),$$

that is, to the theory of symmetric monoidal, compactly generated and stable $(\infty,1)$-categories. Finally, geometric $(\infty,1)$-categories are obtained by asking compact objects to be dualizable.

Definition 2.1.4. We will denote by Geom the full subcategory of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ spanned by geometric $(\infty,1)$-categories.

Phrased in these terms, the theory of geometric $(\infty,1)$-categories exhibits several drawbacks. For example, even if it is (remarkably) known that $\mathrm{Pr}^{\mathrm{L},\omega}$ and the equivalent $(\infty,1)$-categories $\mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)} \simeq \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}}$ are presentable (see [Mat16, Proposition 2.4 and Corollary 2.9]), it is not clear if Geom has limits or colimits, and the obstruction lies in understanding the behaviour of compact and dualizable objects. Let us carefully explain the situation.

Remark 2.1.5. Let us first analyze the case of 2-rings. By [Lur17, Proposition 3.2.2.1], small limits in $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st})$ are computed in $\mathrm{Pr}^{L,\omega}_{st}$. That is, the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st}) \to \mathrm{Pr}^{L,\omega}_{st}$ detects limits. For colimits, the picture is much more complicated. Even if [Lur17, Corollary 3.2.3.3] shows that $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st})$ has small colimits, the forgetful functor does not detect all shape of colimits. An exception is given by sifted colimits, see [Lur17, Corollary 3.2.3.2].

Remark 2.1.6. Consider now $\mathrm{Geom} \subseteq \mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st})$ and consider a limit $\lim_{i\in I} \mathcal{C}_i$ of geometric $(\infty,1)$-categories. for every index $i \in I$ we have an adjunction

$$\lim_{i\in I}\mathcal{C}_i \underset{(p_i)_*}{\overset{p_i^*}{\rightleftarrows}} \mathcal{C}_i \qquad (p_i^* \dashv (p_i)_*)$$

in $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st})$ which determines compact and dualizable objects of the limit. To be precise:

(1) By [Lur09, Proposition 5.5.7.6], or better, its dual, the compact objects of $\lim_{i\in I} \mathcal{C}_i$ are obtained by the image of the compact objects of $\mathcal{C}_i$ along $(p_i)_*$.

(2) By [Lur17, Proposition 4.6.1.1], an object of $\lim_{i\in I} \mathcal{C}_i$ is dualizable if and only if its image along p_i^* is dualizable for every $i \in I$.

From these two general observations it is not possible to deduce that $\lim_{i\in I} \mathcal{C}_i$ is geometric.

Example 2.1.7. Actually, we can also construct a counterexample. Let X be a stack and let $\mathrm{QCoh}(X)$ be the derived stable $(\infty,1)$-category of quasi-coherent sheaves. By writing X as a colimit of its affine pieces, we deduce the existence of an equivalence of $(\infty,1)$-categories

$$\mathrm{QCoh}(X) \simeq \lim_{\mathrm{Spec}(R)\to X} \mathrm{Mod}_R.$$

Here Mod_R denotes the stable $(\infty,1)$-category of modules over R. Now, even if this is a limit of geometric $(\infty,1)$-categories, $\mathrm{QCoh}(X)$ is geometric if and only if X is perfect. See [BZFN10, Definition 3.2].

2.2 Duality Theory

Through this section $\mathcal{B}$ will denote a commutative algebra object in $\mathrm{Pr}^{L,\omega}_{st}$. Let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{L,\omega}_{st})$ be a compactly generated stable $(\infty,1)$-category with an action of $\mathcal{B}$, and let us denote it by $a : \mathcal{B}\times\mathcal{C} \to \mathcal{C}$. Since for every $x \in \mathcal{C}$ the functor $a(-,x) : \mathcal{B} \to \mathcal{C}$ preserves small colimits, the adjoint functor theorem implies the existence of a right adjoint

$$\mathcal{B} \underset{\mathcal{C}(x,-)}{\overset{a(-,x)}{\rightleftarrows}} \mathcal{C} \qquad (a(-,x) \dashv \mathcal{C}(x,-))$$

Notice that the notation $\mathcal{C}(x,y)$ is not optimal, since it depends not only on $x, y \in \mathcal{C}$, but also on the action of $\mathcal{B}$ on $\mathcal{C}$. Nonetheless, by varying x it is possible to verify that the construction $(x,y) \mapsto \mathcal{C}(x,y)$ assembles into a functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{B}$ which we call the *$\mathcal{B}$-graph of $\mathcal{C}$*.

Remark 2.2.1. Under the above assumptions we can apply the machinery of enriched $(\infty,1)$-category theory. Indeed, the action of $\mathcal{B}$ on $\mathcal{C}$ exhibits $\mathcal{C}$ as a *tensored $\mathcal{B}$-enriched $(\infty,1)$-category*, see Equation 1. In particular, we can identify the $\mathcal{B}$-graph of $\mathcal{C}$ with the $\mathcal{B}$-enrichment $\mathcal{C}(-,-)$. This allows us to apply [Hei23, Theorem 10.11].

We can now curry the $\mathcal{B}$-graph of $\mathcal{C}$ to obtain a functor

$$u : \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C},\mathcal{B}).$$

Let us denote by u_c the restriction of u to $\mathcal{C}^{op}_c$. Since u preserves small limits, u_c is left exact. Since the domain and codomain of u_c are stable, it is also right exact. By the universal property of the Ind-completion we get a unique extension

$$\begin{array}{ccc} \mathcal{C}_c^{op} & \xrightarrow{u_c} & \mathrm{Fun}(\mathcal{C}, \mathcal{B}) \\ \downarrow & \nearrow_{\overline{u}_c} & \\ \mathrm{Ind}(\mathcal{C}_c^{op}) & & \end{array}$$

which preserves small colimits. By uncurrying we get a functor

$$\Phi_{\mathcal{C}}^{\mathcal{B}} : \mathrm{Ind}(\mathcal{C}_c^{op}) \times \mathcal{C} \to \mathcal{B}.$$

We have then the following result.

Lemma 2.2.2. Let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$ be a compactly generated stable $(\infty, 1)$-category tensored over $\mathcal{B}$. Then the functor $\Phi_{\mathcal{C}}^{\mathcal{B}}$ constructed above preserves small colimits in each variable.

Proof. Since the construction of $\Phi_{\mathcal{C}}^{\mathcal{B}}$ shows that it preserves small colimits in the first variable, it will suffice to show that for every $x \in \mathrm{Ind}(\mathcal{C}_c^{op})$ the functor $\Phi_{\mathcal{C}}^{\mathcal{B}}(x, -) : \mathcal{C} \to \mathcal{B}$ preserves small colimits. Since the collection of those $x \in \mathrm{Ind}(\mathcal{C}_c^{op})$ such that $\Phi_{\mathcal{C}}^{\mathcal{B}}(x, -) : \mathcal{C} \to \mathcal{B}$ preserves small colimits is closed under small colimits, we may assume $x \in \mathcal{C}_c^{op}$. In particular, we are left to show that $u_c(x) : \mathcal{C} \to \mathcal{B}$ preserves small colimits, that is, the assignment $y \mapsto u(x)(y) = \mathcal{C}(x, y)$ does that. Since $\mathcal{C}(x, -)$ is clearly exact and small colimits are constructed via finite colimits and filtered colimits, it suffices to show that it preserves filtered colimits. This follows since $x \in \mathcal{C}_c^{op}$ is compact. □

We now combine this result with the monoidal structure on $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$ given by the relative tensor product (see [Lur17, Section 4.4]). What matters to us is that the functor $\Phi_{\mathcal{C}}^{\mathcal{B}}$ induces a colimit preserving functor $e_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{op}) \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{B}$ thanks to the universal property of the relative tensor product.

Proposition 2.2.3. Let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$ be a compactly generated stable $(\infty, 1)$-category tensored over $\mathcal{B}$. Then the functor $e_{\mathcal{C}}$ constructed above is a duality datum in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$. That is, $\mathcal{C}$ is a dualizable object in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$ and its dual can be identified with $\mathrm{Ind}(\mathcal{C}_c^{op})$.

Proof. By [Lur17, Lemma 4.6.1.6] it suffices to show that for every $\mathcal{D}, \mathcal{E} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})$ the composite map

$$\begin{array}{c} \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathcal{D}, \mathcal{C} \otimes_{\mathcal{B}} \mathcal{E}) \\ \downarrow \\ \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathrm{Ind}(\mathcal{C}_c^{op}) \otimes_{\mathcal{B}} \mathcal{D}, \mathrm{Ind}(\mathcal{C}_c^{op}) \otimes_{\mathcal{B}} \mathcal{C} \otimes_{\mathcal{B}} \mathcal{E}) \\ \downarrow \\ \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathrm{Ind}(\mathcal{C}_c^{op}) \otimes_{\mathcal{B}} \mathcal{D}, \mathcal{E}), \end{array}$$

which we call θ, is an equivalence. Here the first map is induced by tensoring with $\mathrm{Ind}(\mathcal{C}_c^{op})$ and the second by post-composition with $e_{\mathcal{C}}$. Now, the definition of the relative tensor product implies that we can identify $\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathrm{Ind}(\mathcal{C}_c^{op}) \otimes_{\mathcal{B}} \mathcal{D}, \mathcal{E})$ with the subcategory of $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C}_c^{op} \times \mathcal{D}, \mathcal{E})$ whose objects are $\mathcal{B}$-linear functors which are exact in the first argument and colimit preserving in the second, and whose morphisms are equivalences. In particular, we can identify θ with the map

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathcal{D}, \mathcal{C} \otimes_{\mathcal{B}} \mathcal{E}) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}_{st}^{L,\omega})}(\mathcal{D}, \mathrm{Fun}_{\mathcal{B}_c}^{ex}(\mathcal{C}_c^{op}, \mathcal{E}))$$

given by post-composition with the equivalence

$$\mathcal{C} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathrm{Fun}_{\mathcal{B}}^{L}(\mathcal{C}^{op}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathcal{B}_c}^{ex}(\mathcal{C}_c^{op}, \mathcal{E}).$$

Here the first equivalence is given by definition of relative tensor product whereas the second one follows since $\mathcal{C}$ and $\mathcal{B}$ are compactly generated. □

Let us now specialise these constructions in the case where $\mathcal{B}$ is a geometric $(\infty, 1)$-category and $\mathcal{C}$ is not just a module over $\mathcal{B}$ but actually a geometric $(\infty, 1)$-category equipped with a geometric functor $f^* : \mathcal{B} \to \mathcal{C}$. As we have discussed in Section 1.1, the tensor product of $\mathcal{C}$ allows us to construct the internal hom functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. We can use this functor to construct the $\mathcal{B}$-graph of $\mathcal{C}$, so that the geometric functor f^* exhibits $\mathcal{C}$ as a $\mathcal{B}$-enriched $(\infty, 1)$-category. Indeed, the functor f^* equips $\mathcal{C}$ with the structure of a commutative algebra object in $\mathrm{Pr}^{L,\omega}_{st}$ under $\mathcal{B}$, so that we can interpret this structure as a module action

$$a : \mathcal{B} \times \mathcal{C} \to \mathcal{C}, \qquad (b, x) \mapsto f^*(b) \otimes_{\mathcal{C}} x.$$

By fixing $x \in \mathcal{C}$ we see that the right adjoint to $a(-, x) = f^*(-) \otimes_{\mathcal{C}} x$ is given by $\mathcal{C}(x, -) \simeq f_*\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -)$.

Now the previous proposition implies then that $\mathcal{C}$ is a dualizable object in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{L,\omega}_{st})$ with dual $\mathrm{Ind}(\mathcal{C}_c^{op})$. However, the geometric nature of $\mathcal{C}$ allows us to sharpen this result by better identifying the dual. But first we need to recall the notion of *Frobenius algebra object* in a symmetric monoidal $(\infty, 1)$-category.

Remark 2.2.4. Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 1)$-category. Let $x \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object with multiplication map $m : x \otimes_{\mathcal{C}} x \to x$. We will say that x is *Frobenius algebra object of* $\mathcal{C}$ if there exists a morphism $\Gamma : x \to \mathbb{1}_{\mathcal{C}}$ such that the composite $\Gamma \circ m : x \otimes_{\mathcal{C}} x \to x \to \mathbb{1}_{\mathcal{C}}$ is a duality datum in $\mathcal{C}$. By [Lur17, Proposition 4.6.5.2] an algebra object is a Frobenius algebra object if and only if it is dualizable and equivalent to its dual.

We can now prove the following.

Theorem 2.2.5. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor, and let $\Gamma : \mathcal{C} \to \mathcal{B}$ denote the functor given by $\Gamma(x) = \mathcal{C}(\mathbb{1}_{\mathcal{C}}, x)$. Then $(\mathcal{C}, \Gamma)$ is a Frobenius algebra object of $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{L,\omega}_{st})$. In other words, the composite morphism

$$u : \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{C} \xrightarrow{\Gamma} \mathcal{B}$$

is a duality datum in the symmetric monoidal $(\infty, 1)$-category $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{L,\omega}_{st})$.

Proof. We wish to that the composition $u : \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{B}$ given by the tensor product on $\mathcal{C}$ and Γ is a duality datum. First of all, u classifies a functor $\beta : \mathcal{C} \times \mathcal{C} \to \mathcal{B}$ given by $\beta(x, y) = \mathcal{C}(\mathbb{1}_{\mathcal{C}}, x \otimes_{\mathcal{C}} y)$. Secondly, Proposition 2.2.3 implies that there exists a unique functor $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_c^{op})$ such that the triangle

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times \mathrm{id}_{\mathcal{C}}} & \mathrm{Ind}(\mathcal{C}_c^{op}) \times \mathcal{C} \\ & \searrow^{\beta} & \downarrow \Phi^{\mathcal{B}}_{\mathcal{C}} \\ & & \mathcal{B} \end{array}$$

commutes. Here $\Phi^{\mathcal{B}}_{\mathcal{C}}$ is the functor constructed at the beginning of this section. Our goal is to show that F is an equivalence, and to do that we will show that it is fully-faithful and essentially surjective. Let us first note that F is defined by taking duals on compact objects. Indeed, since for every $x \in \mathcal{C}_c$ we have

$$\beta(x, -) = \mathcal{C}(\mathbb{1}_{\mathcal{C}}, x \otimes_{\mathcal{C}} -) \simeq \mathcal{C}(x^\vee, -)$$

the above diagram implies the claim. In particular, $F(\mathcal{C}_c) \subseteq \mathcal{C}_c^{op}$. Fully-faitfhulness of F now follows by the fully-faithfulness of $F|_{\mathcal{C}_c}$ combined with the fact that F preserves filtered colimits by construction. The essential surjectivity follows since the essential image of F contains $\mathcal{C}_c^{op}$ and it is closed under filtered colimits. □

3 Neeman Dualities

The goal of this chapter is to prove Functors out of $\mathcal{C}_c^{op}$ and Functors out of $\mathrm{Coh}(\mathcal{C})$. We will begin with Section 3.1 by introducing *quasi-perfect* and *quasi-proper* functors. These notions are modelled on quasi-perfect and quasi-proper scheme maps. To be precise, a t-geometric functor $f^* : \mathcal{B} \to \mathcal{C}$ will be called quasi-perfect (respectively, quasi-proper) if the t-structures on $\mathcal{B}$ and $\mathcal{C}$ are the one in the preferred equivalence classes and the right adjoint f_*, which is required to be right t-exact up to a finite shift, preserves compact (respectively pseudo-coherent) objects. Our main result is then Corollary 3.1.7, which shows that every quasi-perfect functor is quasi-proper. In particular, the abstract Grothendieck-Neeman duality will (almost) imply the first abstract Neeman duality.

We will prove Functors out of $\mathcal{C}_c^{op}$ in Section 3.2 by carefully playing with the enriched Yoneda embedding. Since we have already explained our strategy in the introduction, let us just give the most important details. Our proof relies on the equivalence $よ : \mathcal{C} \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B})$ induced by the enriched Yoneda embedding. Our strategy consists in computing the kernel of the enriched Yoneda embedding when the target is restricted to a $\mathcal{B}_c$-submodule $\mathcal{B}_0 \subseteq \mathcal{B}$. The result will be Theorem 3.2.2. The first abstract Neeman duality (that is, Theorem 3.2.4) consists in a further computation of the kernel when the $\mathcal{B}_c$-submodule is given by $\mathrm{Coh}(\mathcal{B}) \subseteq \mathrm{PCoh}(\mathcal{B})$. The main ingredient will be Lemma 1.3.5.

Our work towards the second abstract Neeman duality begins in Section 3.3 by introducing the theory of *morphisms of universal descent*. Since the theory is fairly technical, we refer the reader to the principal result of the section. The first one, that is Proposition 3.3.8, shows that for a morphism of universal descent $h^* : \mathcal{C} \to \mathcal{R}$ the compact objects $\mathcal{C}_c$ can be reconstructed in terms of a cosimplicial diagram $(\mathcal{R})^{\otimes_{\mathcal{C}_c} \bullet}$ constructed in terms of the compact objects of the target. The second main result, that is Lemma 3.3.12, shows that morphisms of universal descent preserve and detect both compact and pseudo-compact objects, and that, under finite tor-dimension, the same happens for bounded pseudo-compact objects.

Finally, in Section 3.4 we will prove Functors out of $\mathrm{Coh}(\mathcal{C})$ by descent via a *relative notion* of morphism of universal descent whose target is a *regular* $(\infty, 1)$*-category*. Regular $(\infty, 1)$-categories are defined as those geometric $(\infty, 1)$-category for which compact and coherent objects coincide (so that, up to a duality, the second abstract Neeman duality follows immediately from the first one). The existence of a (relative) morphisms of universal descent to a regular $(\infty, 1)$-category is not known in general and constitutes the main obstruction of the second abstract Neeman duality.

3.1 Quasi-Perfect and Quasi-Proper Functors

We begin with some terminology.

Definition 3.1.1. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor. We will say that:

(1) f^* is of *finite tor-dimension* if there exists $N \in \mathbb{N}$ such that $f^*(\mathcal{B}_{\leq 0}) \subseteq \mathcal{C}_{\leq N}$.

(2) f^* is of *finite cohomological dimension* if there exists $N \in \mathbb{N}$ such that $f_*(\mathcal{C}_{\geq 0}) \subseteq \mathcal{B}_{\geq -N}$.

In the limit cases where $N = 0$, we will say that f^* is *flat* and *affine*, respectively.

That is, f^* is of finite tor-dimension if it is right t-exact and left t-exact, up to a finite shift, and of finite cohomological dimension if its right adjoint f_* is left t-exact and right t-exact up to a finite shift. More generally, we will say that a functor between geometric categories is t-*bounded* if it it is right and left t-exact, up to a finite shift. Hence, if f^* has finite tor-dimension, then it is t-bounded, and if it is of finite cohomological dimension, then f_* is t-bounded.

Remark 3.1.2. Since f^* preserves compact objects and is right t-exact, it preserves pseudo-compact objects. Indeed, if $x \in \mathcal{B}_c^-$, then for every $m > 0$ we can find a cofibre sequence $b \to x \to e$ where $b \in \mathcal{B}_c$ is compact and $e \in \mathcal{B}_{\geq m}$. By applying f^*, we see that the cofibre sequence $f^*(b) \to f^*(x) \to f^*(e)$ exhibits $f^*(x)$ as a pseudo-compact in $\mathcal{C}$. If now f^* is also of finite tor-dimension, then f^* preserves coconnective pseudo-compact objects.

We are more interested in understanding when the right adjoint $f_* : \mathcal{C} \to \mathcal{B}$ preserves pseudo-coherent objects. Note that, when f_* sends pseudo-coherent to pseudo-coherent, then it also preserves coherent objects, being left t-exact. Anyway, let us give a name to these functors.

Definition 3.1.3. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor of finite cohomological dimension. We will say that f^* is *quasi-proper* if:

(1) Both $\mathcal{B}$ and $\mathcal{C}$ are equipped with a single compact generator.

(2) The t-structures are in the preferred equivalence classes.

(3) The right adjoint $f_* : \mathcal{C} \to \mathcal{B}$ preserves pseudo-coherent objects.

The nomenclature comes form algebraic geometry. Indeed, a map $f : X \to Y$ between quasi-compact quasi-separated schemes is called quasi-proper if the (derived) pushfoward f_* preserves pseudo-coherent complexes. We suggest Lipman and Neeman's article [LN07] for a nice review of all the categorical properties of the (derived) pushfoward. For us what matters is that they also review the notion of quasi-perfect maps, that is those maps of quasi-compact quasi-separated schemes whose (derived) pushfoward preserves perfect complexes. This notion has an immediate generalization in our context.

Definition 3.1.4. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor of finite cohomological dimension. We will say that f^* is *quasi-perfect* if:

(1) Both $\mathcal{B}$ and $\mathcal{C}$ are equipped with a single compact generator.
(2) The t-structures are in the preferred equivalence classes.
(3) The right adjoint $f_* : \mathcal{C} \to \mathcal{B}$ preserves compact objects.

Remark 3.1.5. Note that when f^* is quasi-perfect, then the abstract Grothendieck-Neeman duality, that is Theorem 1.1.11, applies.

The main result of the above mentioned article is [LN07, Theorem 1.2]. It says that a quasi-compact quasi-separated map $f : X \to Y$ between quasi-compact quasi-separated schemes is quasi-perfect if and only if it is quasi-proper and the twisted inverse image $f^{(!)}$ is t-bounded (or, equivalently, that f is quasi-proper and of finite tor-dimension). We can prove a (partial) analogue of this statement in the abstract setting we are developing.

Lemma 3.1.6. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor of finite cohomological dimension between t-geometric $(\infty, 1)$-categories equipped with compact generators $F \in \mathcal{B}_c$ and $G \in \mathcal{C}_c$. Assume there exist integers $N, M > 0$ such that:

(1) $F \in \mathcal{B}_{\geq -N}$ and $G \in \mathcal{C}_{\geq -M}$.
(2) $\pi_0 \operatorname{Hom}_{\mathcal{B}}(F, \mathcal{B}_{\geq N}) = 0$ and $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq M}) = 0$.

If the t-structures on $\mathcal{B}$ and $\mathcal{C}$ are in the preferred equivalence class, then the following are equivalent.

(1) f^* is quasi-proper.
(2) f_* carries compact objects to pseudo-coherent objects.
(3) f_* carries the compact generator G of $\mathcal{C}$ to a pseudo-coherent object.

Proof. The implications (1) ⇒ (2) ⇒ (3) are clear. Let us prove that (2) ⇒ (1). Standing to our assumption on the compact generators, Proposition 1.5.7 implies that pseudo-coherent objects in $\mathcal{B}$ and $\mathcal{C}$ are exactly the pseudo-compact objects. In particular, we are left to prove that f_* preserves pseudo-compact objects. Let $K \in \mathbb{N}$ be the cohomological dimension of f^*, so that $f_*(\mathcal{C}_{\geq 0}) \subseteq \mathcal{B}_{\geq -K}$. Let $x \in \mathcal{C}$ be pseudo-compact, and fix an integer $n > 0$. By assumption, there exists a cofibre sequence $c \to x \to c'$ with $c \in \mathcal{C}_c$ and $c' \in \mathcal{C}_{\geq n+K}$. By applying f_* we get the cofibre sequence $f_*(c) \to f_*(x) \to f_*(c')$, where $f_*(c)$ is now pseudo-compact and $f_*(c') \in \mathcal{C}_{\geq n}$. In particular, we can find a cofibre sequence $b \to f_*(c) \to b'$ with $b \in \mathcal{B}_c$ and $b' \in \mathcal{B}_{\geq n}$. By pasting, the diagram

$$\begin{array}{ccccc} b & \longrightarrow & f_*(c) & \longrightarrow & f_*(x) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & b' & \longrightarrow & \operatorname{cofib}(b \to f_*(x)) \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & f_*(c') \end{array}$$

shows that $\operatorname{cofib}(b \to f_*(x)) \in \mathcal{B}_{\geq n}$ is n-connective, and thus that $f_*(x)$ is pseudo-compact.

Let us conclude by proving (3) ⇒ (2). By Lemma 1.3.3, the subcategory $\operatorname{PCoh}(\mathcal{C}) \subseteq \mathcal{C}$ is thick. Therefore, the subcategory $\{x \in \mathcal{C} \mid f_*(x) \in \operatorname{PCoh}(\mathcal{B})\}$ is a thick subcategory of $\mathcal{C}$. But since $\mathcal{C}_c$ is the smallest thick full subcategory containing G, we can conclude if G is sent by f_* to a pseudo-coherent object. This is exactly our assumption. □

In particular, we deduce the following result.

Corollary 3.1.7. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor, and assume that $\mathcal{C}$ has a compact generator. If f^* is quasi-perfect, then it is quasi-proper.

Proof. If f^* is quasi-perfect, then its right adjoint f_* preserves compact objects. Condition (3) of Lemma 3.1.6 applies. □

We can also prove that, under suitable assumptions on the monoidal structure of the categories involved, every quasi-proper of finite tor-dimension is quasi-perfect. To do that, we need first to identify the compact objects as the pseudo-coherent objects of *finite tor-amplitude*.

Definition 3.1.8. Let $\mathcal{C}$ be a stable symmetric monoidal $(\infty,1)$-category equipped with a t-structure. An object $d \in \mathcal{C}$ has *tor-amplitude in* $[a,b]$ if $\pi_i(d \otimes_{\mathcal{C}} x) = 0$ for every discrete object $x \in \mathcal{C}^\heartsuit$ and $i \notin [a,b]$. We will say that d has *finite tor-dimension* if it has tor-amplitude in $[a,b]$ for some $a,b \in \mathbb{Z}$.

Let $\mathrm{Tor}^{<\infty}(\mathcal{C})$ denote the full subcategory of $\mathcal{C}$ spanned by the objects of finite tor-dimension.

Lemma 3.1.9. Let $\mathcal{C}$ be a stable symmetric monoidal $(\infty,1)$-category equipped with a t-structure. Then $\mathrm{Tor}^{<\infty}(\mathcal{C})$ is a stable subcategory, closed under retracts.

Proof. We use [Lur17, Lemma 1.1.3.3]. Clearly, $\mathrm{Tor}^{<\infty}(\mathcal{C})$ contains the zero object of $\mathcal{C}$. Let us consider a map $f : d \to d'$ between objects of finite tor-dimension. Without loss of generality, assume that they both have tor-amplitude in $[a,b]$. Compute the cofibre $d \to d' \to \mathrm{cofib}(f)$ and consider a discrete object $x \in \mathcal{C}^\heartsuit$. Since the tensor product is exact, we get the cofibre sequence $d \otimes_{\mathcal{C}} x \to d' \otimes_{\mathcal{C}} x \to \mathrm{cofib}(f) \otimes_{\mathcal{C}} x$. This gets us the long exact sequence

$$\cdots \to \pi_{i+1}(\mathrm{cofib}(f) \otimes_{\mathcal{C}} x) \to \pi_i(d \otimes_{\mathcal{C}} x) \to \pi_i(d' \otimes_{\mathcal{C}} x) \to \pi_i(\mathrm{cofib}(f) \otimes_{\mathcal{C}} x) \to \pi_{i-1}(d \otimes_{\mathcal{C}} x) \to \dots$$

in the heart in $\mathcal{C}^\heartsuit$. Since $\pi_i(d \otimes_{\mathcal{C}} x)$ and $\pi_i(d' \otimes_{\mathcal{C}} x)$ vanish for $i \notin [a,b]$, the same happens for $\pi_i(\mathrm{cofib}(f) \otimes_{\mathcal{C}} x)$, showing that $\mathrm{cofib}(f)$ is of finite tor-dimension. Since $\pi_i \Sigma^{-1} \simeq \pi_{i+1}$ it is immediate to check that if d has tor-amplitude in $[a,b]$ then $\Sigma^{-1} d$ has tor-amplitude in $[a-1, b-1]$. This makes $\mathrm{Tor}^{<\infty}(\mathcal{D})$ a stable subcategory of $\mathcal{C}$.

Finally, if $d \to d' \to d$ is a retract of an object d' of finite tor-dimension, then, for every $x \in \mathcal{C}^\heartsuit$, the retract $d \otimes_{\mathcal{C}} x \to d' \otimes_{\mathcal{C}} x \to d \otimes_{\mathcal{C}} x$ shows that the homotopy groups of $d \otimes_{\mathcal{C}} x$ vanish whenever the homotopy groups of $d' \otimes_{\mathcal{C}} x$ vanish. □

Lemma 3.1.10. Let $\mathcal{C}$ be a stable homotopy theory equipped with a geometric tensor t-structure. Assume that $G \in \mathcal{C}$ is a compact generator and that the unit of the monoidal structure is discrete. Then $G \in \mathrm{Coh}(\mathcal{C})$ if and only if $\mathcal{C}_c = \mathrm{PCoh}(\mathcal{C}) \cap \mathrm{Tor}^{<\infty}(\mathcal{C})$.

Proof. Assume first that that the compact objects coincide with the pseudo-coherent objects of finite tor-dimension. Since the compact generator G is, well, compact, it must be pseudo-coherent. Hence we have to check that it is also bounded above. Since it is also of finite tor-amplitude and the unit $\mathbb{1}_{\mathcal{C}}$ is discrete, we have $\pi_i(G) \cong \pi_i(G \otimes \mathbb{1}_{\mathcal{C}}) = 0$ for $i \notin [a,b]$. Hence $G \in \mathcal{C}_{\leq b}$.

Conversely, assume that the compact generator $G \in \mathrm{Coh}(\mathcal{C})$ is coherent. We would like to show that $\mathrm{PCoh}(\mathcal{C}) \cap \mathrm{Tor}^{<\infty}(\mathcal{C})$ is a thick subcategory containing G. By Lemma 1.3.3 and Lemma 3.1.9, we see that the intersection $\mathrm{PCoh}(\mathcal{C}) \cap \mathrm{Tor}^{<\infty}(\mathcal{C})$ is thick. Since $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$, it suffices to show that G is of finite tor-dimension. This is immediate, since G is bounded above and below. □

Example 3.1.11. Let R be a connective $\mathbb{E}_1$-ring spectrum and let Mod_R denote the stable presentable $(\infty,1)$-category of R-module spectra equipped with the usual symmetric monoidal structure. Since the unit R is a compact generator, we can apply Lemma 1.2.12 to deduce that Mod_R comes equipped with a geometric tensor t-structure. In particular, even if compact objects coincide with pseudo-coherent objects of finite tor-dimension by [Lur17, Proposition 7.2.4.23], we cannot argue that $R \in \mathrm{Coh}(\mathrm{Mod}_{\mathcal{R}})$: the unit is not bounded above in general! The next remark fixes this issue.

Remark 3.1.12. We can improve the assumptions of Lemma 3.1.10 by requiring the monoidal unit to be bounded above (hence bounded). Indeed, if the compact generator $G \in \mathcal{C}_c$ has tor-amplitude $[a, b]$ and $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}_{\leq N}$ for some $n \in \mathbb{Z}$, then we can argue by induction on the cofibre sequence

$$\tau_{\leq n} \mathbb{1}_{\mathcal{C}} \to \tau_{\leq n-1} \mathbb{1}_{\mathcal{C}} \to \Sigma^{n+1} \pi_n \mathbb{1}_{\mathcal{C}}$$

that $\pi_i G \cong \pi_i(G \otimes \mathbb{1}_{\mathcal{C}}) = 0$ vanishes for $i \notin [a - (N+1), b + N + 1]$.

Remark 3.1.13. Let $\mathcal{C}$ be a stable symmetric monoidal $(\infty, 1)$-category equipped with a t-geometric t-structure. Lemma 3.1.10 is tacitly imposing an identification between the geometry and and categorical structure of $\mathcal{C}$. Define $\mathrm{Perf}(\mathcal{C})$ to be the intersection $\mathrm{PCoh}(\mathcal{C}) \cap \mathrm{Tor}^{<\infty}(\mathcal{C})$, and call its objects *perfect objects*. This subcategory is defined only in terms of the t-structure, hence it feels only the *geometry* of $\mathcal{C}$. The above lemma says that, in the presence of a coherent compact generator, the geometry $\mathrm{Perf}(\mathcal{C})$ coincides the categorical structure $\mathcal{C}_c$.

Remark 3.1.14. It is still possible that the geometry $\mathrm{Perf}(\mathcal{C})$ coincides with the categorical structure $\mathcal{C}_c$ without $\mathcal{C}$ having a coherent compact generator. The category of modules Mod_R over a connective but not bounded $\mathbb{E}_\infty$-ring R is such an example. Schemes, on the other side, produce examples of $(\infty, 1)$-categories satisfying the assumption of Lemma 3.1.10.

Following the argument proposed in [BJG+71, Corollary 3.7.2] we can also prove a converse to Corollary 3.1.7.

Lemma 3.1.15. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor and assume that both $\mathcal{B}$ and $\mathcal{C}$ are compactly generated by a coherent object, and that their unit are bounded. If f^* is quasi-proper and of finite tor-dimension then it is quasi-perfect.

Proof. Lemma 3.1.10 gives us a complete description of the compact objects in $\mathcal{B}$ and $\mathcal{C}$: they are precisely the pseudo-coherent objects of finite tor-amplitude. In particular, since f_* preserves pseudo-coherent objects (being f^* quasi-proper), to show that f^* is quasi-perfect it suffices to show that f_* preserves finite tor-amplitude. Let $x \in \mathrm{Tor}^{<\infty}(\mathcal{C})$ be of finite tor-amplitude $[a, b]$ and let $y \in \mathcal{B}^{\heartsuit}$ be discrete. We wish to show that $\pi_i^{\mathcal{B}}(f_*(x) \otimes_{\mathcal{B}} y)$ vanishes for i outside some interval $[A, B]$. Now the projection formula of Proposition 1.1.9 gives us an isomorphism

$$\pi_i^{\mathcal{B}}(f_*(x) \otimes_{\mathcal{B}} y) \cong \pi_i^{\mathcal{B}}(f_*(x \otimes_{\mathcal{C}} f^*(y)).$$

Since f^* is of finite tor-dimension, we have that $f^*(y) \in \mathcal{C}_{\leq N}$ for some integer $N \geq 0$. This allows us to make two reductions. First of all, by arguing via induction on the cofibre sequence $\tau^{\mathcal{C}}_{\leq n} x \otimes_{\mathcal{C}} f^*(y) \to \tau^{\mathcal{C}}_{\leq n-1} x \otimes_{\mathcal{C}} f^*(y) \to \Sigma^{n+1} \pi_n^{\mathcal{C}}(x \otimes_{\mathcal{C}} f^*(y))$ we can assume that $x \otimes_{\mathcal{C}} f^*(y)$ is in the heart $\mathcal{C}^{\heartsuit}$. In particular, this implies that

$$\pi_i^{\mathcal{B}}(f_*(x \otimes_{\mathcal{C}} f^*(y)) \cong {}^{p}f_* \pi_i^{\mathcal{C}}(x \otimes_{\mathcal{C}} f^*(y))$$

where ${}^{p}f_*$ is the functor defined in Remark 1.2.17. Secondly, we can assume that $f^*(y) \in \mathcal{C}^{\heartsuit}$ is discrete, by arguing via induction on the cofibre sequence $\tau^{\mathcal{C}}_{\leq n} f^*(y) \to \tau^{\mathcal{C}}_{\leq n-1} f^*(y) \to \Sigma^{n+1} \pi_n^{\mathcal{C}}(f^*(y))$ as before. Now the claim follows since $x \in \mathrm{Tor}^{<\infty}(\mathcal{C})$ is of finite tor-amplitude. □

3.2 Functors out of $\mathcal{C}_c^{\mathrm{op}}$

Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor and let f_* be its right adjoint. In Section 2.2 we learned that the restricted Yoneda embedding induces equivalence

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{B}), \qquad x \mapsto \mathcal{C}(-, x).$$

Our goal is now to study the above Yoneda embedding when the source is a $\mathcal{C}_c$-submodule $\mathcal{C}_0$, such as $\mathrm{Coh}(\mathcal{C}) \subseteq \mathrm{PCoh}(\mathcal{C})$ under suitable assumptions on $\mathcal{C}$. In general, by restricting the source to a $\mathcal{C}_c$-submodule $\mathcal{C}_0$, the Yoneda embedding will still be fully-faithful. However, it will cease to be an equivalence, since not every exact and $\mathcal{B}_c$-enriched functor $\mathcal{C}_c^{\mathrm{op}} \to \mathcal{B}$ arises from an object of $\mathcal{C}_0$. For this reason, it is helpful to approach the problem from the other side. Instead of restricting the source of the Yondea embedding, it is easier to restrict its target, and then identify its kernel.

Definition 3.2.1. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor and let $\mathcal{B}_0$ be a $\mathcal{B}_c$-submodule of $\mathcal{B}$. The *compact pullback along* f_* *of* $\mathcal{B}_0$ is the full subcategory of $\mathcal{C}$ defined as

$$f^\#(\mathcal{B}_0) = \{x \in \mathcal{C} \mid f_*(c \otimes_{\mathcal{C}} x) \in \mathcal{B}_0 \text{ for all } c \in \mathcal{C}_c\}.$$

By definition, it is clear that compact pullback $f^\#(\mathcal{B}_0)$ along f_* of a $\mathcal{B}_c$-submodule $\mathcal{B}_0$ is $\mathcal{C}_c$-submodule. The next result identifies the compact pullback $f^\#(\mathcal{B}_0)$ as the kernel of the target-restricted Yondea.

Theorem 3.2.2 (Functors out of $\mathcal{C}_c^{op}$)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a geometric functor and let $\mathcal{B}_0$ a $\mathcal{B}_c$-submodule. Then there is an equivalence of $(\infty, 1)$-categories

$$f^\#(\mathcal{B}_0) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B}_0)$$

induced by the restrict Yoneda embedding.

Proof. Let $i : \mathcal{B}_0 \hookrightarrow \mathcal{B}$ be the inclusion. We begin by showing that the functor

$$i_* : \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B}_0) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B})$$

is fully-faithful. First of all, the enrichment condition is vacuous since i is the inclusion of a $\mathcal{B}_c$-submodule. Futhermore, the exactness of i ensures that i_* preserves exact functors. Therefore we are reduced to show that, the post-composition $i_* : \mathrm{Fun}(\mathcal{C}_c^{op}, \mathcal{B}_0) \to \mathrm{Fun}(\mathcal{C}_c^{op}, \mathcal{B})$ is fully-faithful. The claim now follows by [CCNW24, Theorem 6.4.7]: it is a general fact that the post-composition i_* is fully-faithful if and only if i is fully-faithful. Consider now the following commutative diagram

$$\begin{array}{ccc} f^\#(\mathcal{B}_0) & \longrightarrow & \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B}_0) \\ \big\downarrow & & \big\downarrow i_* \\ \mathcal{C} & \xrightarrow{\simeq} & \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathcal{B}). \end{array}$$

where we have depicted the equivalence given by the Yoneda embedding and the fully-faithfulness determined before. Now the restricted Yoneda embedding is still fully-faithful by the 2-out of-3 property. Its essential surjectivity, instead, follows by diagram chasing. □

We now discuss the case where $\mathcal{B}_0$ is the subcategory of (pseudo-)coherent objects. Assume that $\mathcal{B}$ comes equipped witha geometric t-structure and that it is generated by a single compact object. If this generator is connective, then Remark 1.3.6 implies that $\mathcal{B}_c \subseteq \mathrm{PCoh}(\mathcal{B})$. If the t-structure is furthermore tensor, then Lemma 1.3.8 applies. We deduce that $\mathrm{Coh}(\mathcal{B})$ and $\mathrm{PCoh}(\mathcal{B})$ are $\mathcal{B}_c$-submodules.

Lemma 3.2.3. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor. Then $\mathrm{PCoh}(\mathcal{C}) \subseteq f^\#(\mathrm{PCoh}(\mathcal{B}))$ and $\mathrm{Coh}(\mathcal{C}) \subseteq f^\#(\mathrm{Coh}(\mathcal{B}))$.

Proof. Indeed, being $\mathrm{PCoh}(\mathcal{C})$ a $\mathcal{C}_c$-submodule, we have that the tensor product of $\mathcal{C}$ restricts to a functor $\mathrm{PCoh}(\mathcal{C}) \times \mathcal{C}_c \to \mathrm{PCoh}(\mathcal{C})$. Since f^* is quasi-proper, the right adjoint f_* sends $\mathrm{PCoh}(\mathcal{C})$ to $\mathrm{PCoh}(\mathcal{B})$. In particular, every pseudo-coherent object of $\mathcal{C}$ must be contained in the compact pullback along f_* of $\mathrm{PCoh}(\mathcal{B})$. The second inclusion follows since coherent objects are the coconnective pseudo-coherent objects, and f_* is left t-exact. □

In particular, we can combine this observation with Theorem 3.2.2 to deduce that for a quasi-proper functor $f^* : \mathcal{B} \to \mathcal{C}$ there are fully-faithful functors

$$\mathrm{PCoh}(\mathcal{C}) \subseteq f^\#(\mathrm{PCoh}(\mathcal{B})) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathrm{PCoh}(\mathcal{B})), \qquad \mathrm{Coh}(\mathcal{C}) \subseteq f^\#(\mathrm{Coh}(\mathcal{B})) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathrm{Coh}(\mathcal{B})).$$

In order to deduce the other inclusions we need to exploit the compact generators.

Theorem 3.2.4 (Functors out of $\mathcal{C}_c^{op}$)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor. Assume that $\mathcal{B}$ is coherent. Assume furthermore that the compact generator G of $\mathcal{C}$ is such that $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ detects connective and coconnective objects and that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(G, \mathcal{C}_{\geq N}) = 0$ for some integer $N > 0$. Then there are equivalences of $(\infty, 1)$-categories

$$\mathrm{PCoh}(\mathcal{C}) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathrm{PCoh}(\mathcal{B})), \qquad \mathrm{Coh}(\mathcal{C}) \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{C}_c^{op}, \mathrm{Coh}(\mathcal{B}))$$

induced by the restricted Yoneda embedding.

Proof. We only need to show the inclusions $f^\#(\mathrm{PCoh}(\mathcal{B})) \subseteq \mathrm{PCoh}(\mathcal{C})$ and $f^\#(\mathrm{Coh}(\mathcal{B})) \subseteq \mathrm{Coh}(\mathcal{C})$. First of all, note that if we prove the first inclusion, then the second will follow since the Yoneda embedding $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ detects coconnective objects. Let $N \in \mathbb{N}$ be the cohomological dimension of f^*, so that $f_*(\mathcal{C}_{\geq 0}) \subseteq \mathcal{B}_{\geq -N}$. Let $x \in f^\#(\mathrm{PCoh}(\mathcal{B}))$, so that $f_*(x \otimes_{\mathcal{C}} c)$ is in $\mathrm{PCoh}(\mathcal{B})$ for every compact object $c \in \mathcal{C}_c$. Since $\mathcal{C}(G, x) = f_* \underline{\mathrm{Hom}}_{\mathcal{C}}(G, x) \simeq f_*(x \otimes_{\mathcal{C}} G^\vee)$ is pseudo-coherent, hence connective, it follows that x must be connective. Without loss of generality, assume $x \in \mathcal{C}_{\geq 0}$. Thanks to Lemma 1.3.5 (and Lemma 1.4.10), to show that x is pseudo-coherent it will suffice to obtaining it as geometric realization of a simplicial object $x_\bullet$ where each x_n is compact and 0-connective. As usual, it suffices to show that x can be written as filtered colimit over a diagram

$$D(0) \xrightarrow{f_1} D(1) \to \dots$$

where each $\Sigma^{-n}\mathrm{cofib}(f_n)$ is compact. By agreeing that f_0 denotes the zero map $0 \to D(0)$ we can argue by induction. Assume that we have constructed

$$D(0) \xrightarrow{f_1} D(1) \to \dots \to D(n) \xrightarrow{g} x$$

with $D(i)$ compact in $\mathcal{C}$ for $0 \leq i \leq n$ and[18] $\mathrm{fib}(g) \in \mathcal{C}_{\geq n-N}$. Since $D(n)$ is pseudo-coherent and f_* preserves pseudo-coherent objects, it follows that $f_*(\mathrm{fib}(g))$ is pseudo-coherent, being $f_*(x)$ pseudo-coherent by assumption. Since the bottom homotopy group $\pi_{n-N} f_*(\mathrm{fib}(g)) \in (\mathcal{B}^\heartsuit)_c$ is compact in the heart, we can pick a π_0-epimorphism $\Sigma^{n-N} q \to f_*(\mathrm{fib}(g))$ where q is compact and 0-connective. Here we have used the fact that $\mathcal{B}$ is coherent. By adjunction, we get a morphism $f^*(\Sigma^{n-N} q) \to \mathrm{fib}(g)$ from a compact and $(n-N)$-connective object. We now construct $D(n+1)$ as the cofibre

$$\begin{array}{ccccc}
f^*(\Sigma^{n-N} q) & \longrightarrow & \mathrm{fib}(g) & \longrightarrow & D(n) \\
\downarrow & & \downarrow & & \downarrow f_{n+1} \\
0 & \longrightarrow & \mathrm{fib}(g)/f^*(\Sigma^n q) & \longrightarrow & D(n+1)
\end{array}$$

By applying the argument of Lemma 1.4.10 we conclude that x is pseudo-coherent, thus showing the inclusion $f^\#(\mathrm{PCoh}(\mathcal{B})) \subseteq \mathrm{PCoh}(\mathcal{C})$. □

This concludes the proof of Functors out of $\mathcal{C}_c^{op}$. We end this section by proving a couple of properties of the Yoneda embedding $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$.

Lemma 3.2.5. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor of finite cohomological dimension and assume that $\mathcal{C}$ comes equipped with a compact generator G which is $(-N)$-connective for some integer $N \geq 0$. Assume also that the t-structure on $\mathcal{C}$ is in the preferred equivalence class. Then the Yoneda embedding $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ is t-bounded.

Proof. Let $M \geq 0$ be the cohomological dimension of f^*, so that $f_*(\mathcal{C}_{\geq 0}) \subseteq \mathcal{B}_{\geq -M}$. The left texactness up to a shift of $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ is immediate. Indeed, since G is compact, hence dualizable, we have an equivalence

$$\mathcal{C}(G, -) = f_* \underline{\mathrm{Hom}}_{\mathcal{C}}(G, -) \simeq f_*(G^\vee \otimes_{\mathcal{C}} -).$$

Since $G^\vee \in \mathcal{C}_{\geq -n}$ for some integer $n > 0$, being $G^\vee$ compact, it follows that $f_*(\mathcal{C}_{\geq -n} \otimes_{\mathcal{C}} \mathcal{C}_{\geq 0}) \subseteq \mathcal{B}_{\geq -M-n}$, thus showing $\mathcal{C}(G, \mathcal{C}_{\geq 0}) \in \mathcal{B}_{\geq -M-n}$. The left texactness up to a shift of $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ is just as easy.

[18]Compare with Lemma 1.4.10. We have changed the induction hypothesis! Notice that the fibre of the zero map $0 \to x$ is x and it is in $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq -N}$ since $N \geq 0$.

Indeed, Lemma 1.2.15 implies that $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}_{\geq -N}, \mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\leq N}$, so that $\mathcal{C}(G, \mathcal{C}_{\leq 0}) = f_* \underline{\mathrm{Hom}}_{\mathcal{C}}(G, \mathcal{C}_{\leq 0}) \subseteq \mathcal{B}_{\leq N}$. □

Notice that, despite its fully-faithfulness, this result is not sufficient to deduce that it detects connective and coconnective objects. The right and left error of t-exactness are independent on each other!

Lemma 3.2.6. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor of finite cohomological dimension and assume that $\mathcal{C}$ comes equipped with a compact generator G. Assume also that the t-structure on $\mathcal{C}$ is in the preferred equivalence class.

(1) Then the Yoneda embedding $\mathcal{C}(c, -) : \mathcal{C} \to \mathcal{B}$ is t-bounded for every compact object $c \in \mathcal{C}$.

(2) If $\mathcal{C}(G, -) : \mathcal{C} \to \mathcal{B}$ detects connective and coconnective objects, then $\mathcal{C}(c, -) : \mathcal{C} \to \mathcal{B}$ detects connective and coconnective objects for every compact object $c \in \mathcal{C}$.

Proof. Let us begin with (1). Consider the full subcategory $\mathcal{C}'$ of $\mathcal{C}$ spanned by those $c \in \mathcal{C}$ such that the Yoneda embedding $\mathcal{C}(c, -) : \mathcal{C} \to \mathcal{B}$ is t-bounded. By the previous lemma it contains the compact generator G. Being it thick, it follows that $\mathcal{C}_c \subseteq \mathcal{C}'$, that is, $\mathcal{C}(c, -) : \mathcal{C} \to \mathcal{B}$ is t-bounded for every compact object $c \in \mathcal{C}$. Point (2) follows by an analogue argument. □

3.3 Morphism of Universal Descent

We now introduce the theory of universal descent associated to a geometric functor. But first let us fix some terminology.

Remark 3.3.1. In the following we will denote by Δ the simplex category, and by $\Delta_{\leq n}$ be the full subcategory of Δ spanned by those objects $[m] \in \Delta$ such that $m \leq n$. We will also denote by Δ^+ the augmented simplex category, and we will denote by $[-1]$ the augmented object. Let $\mathcal{D}$ be an $(\infty, 1)$-category. An *augmented cosimplicial object* in $\mathcal{D}$ is a functor $\mathcal{C}^\bullet : \Delta^+ \to \mathcal{D}$.

Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor. We define two different augmented cosimplicial objects.

(1) First of all, we can regard $\mathcal{R}$ as $\mathcal{C}$-module in $\mathrm{Pr}^{L,\omega}_{st}$. In particular, we can form the Cech nerve of h^* inside $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{L,\omega}_{st})$. Objectwise, this augmented cosimplicial object is given by

$$[n] \mapsto \mathcal{R} \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \mathcal{R},$$

where the tensor product is taken $(n+1)$-times if $n \geq 0$. If $n = -1$ we set $[-1] \mapsto \mathcal{C}$. The augmentation map is given by $h^* : \mathcal{C} \to \mathcal{R}$. We will generally regard it as a cosimplicial object in $\mathrm{Pr}^{L,\omega}_{st}$ and we will denote it by $\mathcal{C}^\bullet : \Delta^+ \to \mathrm{Pr}^{L,\omega}_{st}$.

(2) Since h^* is geometric, it restricts to a functor $h^* : \mathcal{C}_c \to \mathcal{R}_c$. Since $\mathcal{R}_c$ is a $\mathcal{C}_c$-module in $\mathrm{Cat}^{perf}_{(\infty,1)}$, the same construction as before produces an augmented cosimplicial object $\mathcal{C}^\bullet_c : \Delta^+ \to \mathrm{Cat}^{perf}_{(\infty,1)}$.

Since the Ind-completion furnishes a symmetric monoidal equivalence $\mathrm{Ind} : \mathrm{Cat}^{perf}_{(\infty,1)} \to \mathrm{Pr}^{L,\omega}_{st}$, the two augmented cosimplicial objects contains the same amount of data. However, it is easier to impose assumptions on the second augmented cosimplicial object. To explain what we mean, let us give the following.

Definition 3.3.2. Let $\mathcal{C}^\bullet_c : \Delta^+ \to \mathrm{Cat}^{perf}_{(\infty,1)}$ be an augmented cosimplicial object. We will say that $\mathcal{C}^\bullet_c$ satisfies the *Beck–Chevalley condition* if for any morphism $\alpha : [m] \to [n]$ in Δ^+ the square

$$\begin{array}{ccc} \mathcal{C}^m_c & \xrightarrow{d^0} & \mathcal{C}^{m+1}_c \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathcal{C}^n_c & \xrightarrow[d^0]{} & \mathcal{C}^{n+1}_c \end{array}$$

is horizontal right adjointable, that is, the horizontal maps admits right adjoints and the canonical morphism $\alpha \circ d^0_* \to d^0_* \circ \alpha$ is an equivalence.

The following result is an instance of the claim made above.

Lemma 3.3.3. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor, and $\mathcal{C}^\bullet_c : \Delta^+ \to \mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ the associated augmented cosimplicial object. If $\mathcal{C}^\bullet_c$ satisfies the Beck–Chevalley condition, then the right adjoint restricts to a functor $h_* : \mathcal{R}_c \to \mathcal{C}_c$.

Proof. Apply the definition to $\alpha = \mathrm{id}_{[-1]}$ and use $d^0 = h^*$ to deduce the existence of a right adjoint $r : \mathcal{R}_c \to \mathcal{C}_c$. By Ind-completing it follows that r must be equivalence to h_* restricted to the compact objects. ☐

Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor. We now construct a filtration

$$\cdots \to \varphi_n \to \cdots \to \varphi_1 \to \varphi_0$$

of small colimit (and limit) preserving functors $\mathcal{C} \to \mathcal{C}$. For every integer $n \geq 0$ we define $\varphi_n : \mathcal{C} \to \mathcal{C}$ to be the limit

$$\varphi_n = \lim_{\Delta_{\leq n}} (h_*h^* \rightrightarrows h_*h^*h_*h^* \Rrightarrow \dots)$$

where the morphisms in the diagram are obtained via pre- and post-composition with the unit of the adjunction $h^* \dashv h_*$. The natural transformations $\varphi_n \to \varphi_{n-1}$ are then given by the universal property of the limit, by projecting from the pieces of φ_n and forgetting the last term in the limit. In particular, a careful analysis of the natural transformations involved, together with the triangle identities, shows the following result.

Lemma 3.3.4. For every integer $n \geq 1$ the cofibre of $\varphi_n \to \varphi_{n-1}$ is equivalent to $(h_*h^*)^n$.

Morphism of universal descent are defined to be the one for which the filtration becomes uninteresting for larger indexes.

Definition 3.3.5. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor. We will say that h^* is of *universal descent* if:

(1) The associated augmented cosimplicial object $\mathcal{C}^\bullet_c$ satisfies the Beck–Chevalley condition.

(2) There exists an integer $e \geq 0$ such that the identity on $\mathcal{C}$ is a retract of φ_e.

We will call e the *exponent of* h^*.

Remark 3.3.6. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor of universal descent. Since the associated augmented cosimplicial object $\mathcal{C}^\bullet_c : \Delta^+ \to \mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$ satisfies the Beck–Chevalley condition, we get that h_* preserves compact objects. In particular, then the filtration $\cdots \to \varphi_n \to \cdots \to \varphi_1 \to \varphi_0$ restrict to a filtration of exact functors $\mathcal{C}_c \to \mathcal{C}_c$. We conclude that the identity on $\mathcal{C}_c$ is a retract of $\varphi_e|_{\mathcal{C}_c}$ for some exponent $e \geq 0$.

Our next goal is to show that for morphism of universal descent $h^* : \mathcal{C} \to \mathcal{R}$ the $(\infty,1)$-category $\mathcal{C}_c$ can be reconstructed from the augmented cosimplicial object $\mathcal{C}^\bullet_c : \Delta^+ \to \mathrm{Cat}^{\mathrm{perf}}_{(\infty,1)}$. By Ind-completing, the same result holds for $\mathcal{C}^\bullet : \Delta^+ \to \mathrm{Pr}^{L,\omega}_{st}$. Since the forgetful functor $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{L,\omega}_{st}) \to \mathrm{Pr}^{L,\omega}_{st}$ preserves and detects limits, we deduce that, for morphism of universal descent $h^* : \mathcal{C} \to \mathcal{R}$, the source can be reconstructed as a geometric $(\infty,1)$-category from the target (see [Lur17, Proposition 4.6.1.1]).

In order to prove our main result we need first a technical lemma.

Lemma 3.3.7. Let $h^* : \mathcal{C} \to \mathcal{R}$ be of universal descent. Then h^* is conservative.

Proof. Let $\alpha : x \to x'$ in $\mathcal{C}$ be a morphism such that $h^*(\alpha)$ is an equivalence in $\mathcal{R}$. We wish to show that α is an equivalence. By assumption there exists some integer $e \geq 0$ such that the identity $\mathrm{id}_{\mathcal{C}}$ is a retract φ_e. Since equivalences are stable under retracts, it suffices to show that $\varphi_e(\alpha)$ is an equivalence. This follows by the explicit definition of the φ_n's. ☐

We then deduce the following.

Proposition 3.3.8. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a geometric functor of universal descent. Then $\mathcal{C}_c^\bullet$ is a limit diagram. In other words, the canonical map

$$\mathcal{C}_c \to \lim_{\Delta} (\mathcal{R}_c)^{\otimes_{\mathcal{C}_c} \bullet}$$

is an equivalence of $(\infty, 1)$-categories.

Proof. We wish to apply [Lur17, Corollary 4.7.5.3]. Since $\mathcal{C}_c^\bullet$ satisfies the Beck-Chevalley condition and since the augmentation map h^* is conservative by Lemma 3.3.7, we are left to prove that the $(\infty, 1)$-category $\mathcal{C}_c$ admits geometric realizations of h^*-split simplicial objects, and those geometric realizations are preserved by h^*.

Let $\mathcal{U} \subseteq \mathrm{Fun}(\Delta, \mathcal{C}_c)$ be the full subcategory spanned by those cosimplicial objects which admit limit in $\mathcal{C}_c$ and which is preserved by h^*. Notice that $\mathcal{U}$ is a thick subcategory (since $\mathcal{C}_c$ is idempotent-complete). Moreover, it contains those cosimplicial objects which admit splittings (since they admit a limit, and since applying h^* furnishes cosimplicial objects which admit splittings, hence that have a limit). Let now $x^\bullet : \Delta \to \mathcal{C}$ be an h^*-split cosimplicial object. Since $x^\bullet \in \mathcal{U}$ by assumption, we are done. □

We can also add the input of a t-structure.

Definition 3.3.9. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a t-geometric functor. We will say that h^* is of *universal descent* if:

(1) It is of universal descent in the sense of Definition 3.3.5.

(2) It is of finite cohomological dimension.

(3) It is of finite tor-dimension.

Remark 3.3.10. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a t-geometric functor of universal descent. Assume that $\mathcal{C}$ and $\mathcal{R}$ come equipped with single compact generators and assume that the t-structures are in the preferred equivalence class. Since Lemma 3.3.3 implies that $h_* : \mathcal{R} \to \mathcal{C}$ preserves compact objects, it follows that h^* is quasi-perfect. In particular, it is quasi-proper.

Remark 3.3.11. In the following we will also need a relative situation. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a t-geometric functor of universal descent, and let $R \in \mathcal{R}$ be a compact generator. If we are given a t-geometric functor $f^* : \mathcal{B} \to \mathcal{C}$ satisfying the assumption of the first abstract Neeman duality, we will say that $h^* : \mathcal{C} \to \mathcal{R}$ is of $\mathcal{B}$-*universal descent* if the composite functor $h^* \circ f^*$ satisfies the assumption of the first abstract Neeman duality. More explicitly, this condition boils down to show that the Yoneda $\mathcal{R}(R, -) : \mathcal{R} \to \mathcal{B}$ detects being connective and coconnective.

We conclude this section with one last observation regarding t-geometric functors of universal descent.

Lemma 3.3.12. Let $h^* : \mathcal{C} \to \mathcal{R}$ be t-geometric functor of universal descent. Then:

(1) An object $x \in \mathcal{C}_c$ is compact if and only if $h^*(x) \in \mathcal{R}_c$ is compact.

(2) An object $x \in \mathcal{C}^-$ is connective if and only if $h^*(x) \in \mathcal{R}^-$ is connective.

(3) Assume that both $\mathcal{C}$ and $\mathcal{R}$ are equipped with compact generators satisfying the assumption of Proposition 1.5.7. Then an object $x \in \mathrm{PCoh}(\mathcal{C})$ is pseudo-coherent if and only if $h^*(x) \in \mathrm{PCoh}(\mathcal{R})$ is pseudo-coherent. Then the same is true for coherent objects.

Proof. First of all, since $h^* : \mathcal{C} \to \mathcal{R}$ is of universal descent, we can find an exponent $e \geq 0$ such that the identity on $\mathcal{C}$ is a retract of $\varphi = \varphi_e : \mathcal{C} \to \mathcal{C}$ defined as above. With that being said, point (1) is exactly Proposition 3.3.8 so let us prove (2). Since h^* is left t-exact the implication $(\Rightarrow)$ is always true, so let us prove $(\Leftarrow)$. Assume that $x \in \mathcal{C}$ is such that $h^*(x) \in \mathcal{R}_{\geq n}$ for some integer $n \in \mathbb{Z}$. Since x is retract of $\varphi(x)$, it suffices to prove that this object is connective. However, $\varphi(x)$ is obtained as a finite limit

$$\varphi(x) = \lim_{\Delta_{\leq e}} (h_* h^*(x) \rightrightarrows h_* h^* h_* h^*(x) \Rrightarrow \dots)$$

so it suffices to show that each term is connective. This follows since h^* is right t-exact and of finite cohomological dimension (hence h_* is right t-exact up to a shift).

For (3) note that our assumption guarantees that the pseudo-coherent objects of $\mathcal{C}$ and $\mathcal{R}$ coincide with the pseudo-compact. In particular, the implication $(\Rightarrow)$ is true by Remark 3.1.2. Let us show the converse $(\Leftarrow)$. Let $x \in \mathcal{C}$ such that $h^*(x) \in \mathrm{PCoh}(\mathcal{R})$. Since $\mathrm{PCoh}(\mathcal{C})$ is stable under retracts, it suffices to show that $\varphi(x)$ is pseudo-coherent. This is obvious, since $\varphi(x)$ is a finite limit involving powers of h_*h^* and both h_* and h^* preserves pseudo-coherent objects. Notice that in this last claim we are using Remark 3.3.10, since we need h^* to be quasi-proper.

Since h^* is of finite tor-dimension by assumption, it clearly preserves coherent object. On the other side, if $x \in \mathcal{C}$ is such that $h^*(x) \in \mathrm{Coh}(\mathcal{R})$ is coherent then the same argument above shows that x is a retract of $\varphi(x)$, which is a finite limit of coherent objects. □

3.4 Functors out of Coh($\mathcal{C}$)

We now prove our generalization of Neeman, Functors out of $\mathcal{T}_c^b$. First of all, let us introduce a class of geometric $(\infty,1)$-categories for which Functors out of Coh($\mathcal{C}$) is automatically satisfied.

Definition 3.4.1. Let $\mathcal{R}$ be a t-geometric $(\infty,1)$-category. We will say that $\mathcal{R}$ is *regular* if the compact objects coincide with the coherent ones, that is $\mathcal{R}_c = \mathrm{Coh}(\mathcal{R})$.

Let $\mathcal{R}$ be a regular $(\infty,1)$-category and let $R \in \mathcal{R}_c$ be a compact generator. Then R must be bounded. In general, this is a strong restriction on the class of regular $(\infty,1)$-categories.

Example 3.4.2. Let A be a connective $\mathbb{E}_\infty$-ring and consider the $(\infty,1)$-category of modules Mod_A. In general, every perfect objects of Mod_A, that is, every the compact object, is pseudo-coherent, thus providing an inclusion $\mathrm{Perf}(A) \subseteq \mathrm{PCoh}(A)$. If A is also coconnective, then every perfect object is coconnective, thus proving refining the previous inclusion to $\mathrm{Perf}(A) \subseteq \mathrm{Coh}(A)$. If Mod_A is regular then the other inclusion holds, and by [Lur18, Lemma 11.3.3.3] it follows that A must be discrete.

For this reason, regular $\mathbb{E}_\infty$-rings are defined by just asking $\mathrm{Coh}(A) \subseteq \mathrm{Perf}(A)$.

This implies that our definition of regular $(\infty,1)$-categories is not optimal, and therefore that all the arguments which we now propose suffer of the same problem. Anyway, the our next goal is to show that for regular $(\infty,1)$-categories the second abstract Neeman duality is a consequence of the first one.

Lemma 3.4.3. Let $f^* : \mathcal{B} \to \mathcal{R}$ be a quasi-proper functor to a regular $(\infty,1)$-category, and assume that the compact generator R of $\mathcal{R}$ is such that $\mathcal{R}(R,-) : \mathcal{R} \to \mathcal{B}$ detects connective and coconnective objects and that $\pi_0 \operatorname{Hom}_{\mathcal{R}}(R, \mathcal{R}_{\geq N}) = 0$ for some integer $N > 0$. Then there exist an equivalence of $(\infty,1)$-categories

$$\mathcal{R}_c^{op} \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{R}), \mathrm{Coh}(\mathcal{B}))$$

induced by the restricted dual Yoneda embedding.

Proof. Since $\mathcal{R}_c = \mathrm{Coh}(\mathcal{R})$, the duality $\Delta_{\mathcal{R}} : \mathcal{R}_c^{op} \to \mathcal{R}_c$ provides a commutative square

$$\begin{array}{ccc} \mathcal{R}_c^{op} & \xrightarrow{\tilde{\text{よ}}} & \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{R}), \mathrm{Coh}(\mathcal{B})) \\ \downarrow{\scriptstyle \Delta_{\mathcal{R}}} & & \downarrow{\scriptstyle -\circ\Delta_{\mathcal{R}}} \\ \mathcal{R}_c & \xrightarrow[\text{よ}]{} & \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathcal{R}_c^{op}, \mathrm{Coh}(\mathcal{B})) \end{array}$$

Since the horizontal bottom map is an equivalence by Functors out of $\mathcal{C}_c^{op}$ and the vertical maps are equivalence, the horizontal top map is an equivalence. □

In order to prove the main result of this section we need some technical results.

Lemma 3.4.4. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a quasi-proper functor. Then the dual Yoneda embedding on $\mathcal{C}$ restricts to a functor

$$\tilde{\text{よ}} : \mathcal{C}_c^{op} \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathrm{PCoh}(\mathcal{C}), \mathrm{PCoh}(\mathcal{B})).$$

The same is true with $\mathrm{Coh}(-)$ in place of $\mathrm{PCoh}(-)$.

Proof. Let $x \in \mathcal{C}_c$ be a compact object and let $y \in \mathcal{C}$ be (pseudo-)coherent. We wish to show that $\tilde{\text{ょ}}(x)(y) = \mathcal{C}(x, y)$ is (pseudo-)coherent. Since x is compact, this object can be identified with

$$\mathcal{C}(x, y) = f_* \underline{\mathrm{Hom}}_{\mathcal{C}}(x, y) \simeq f_*(x^\vee \otimes_{\mathcal{C}} y).$$

Since Lemma 1.3.8 implies that the tensor product $x^\vee \otimes_{\mathcal{C}} y$ is (pseudo-)coherent, the quasi-properness of f^* will conclude. □

Lemma 3.4.5. Let $h^* : \mathcal{C} \to \mathcal{R}$ be a t-geometric functor of universal descent. Assume that:

(1) Both $\mathcal{C}$ and $\mathcal{R}$ come equipped with single compact generators and that the t-structure are in the preferred equivalence class.
(2) The $(\infty, 1)$-category $\mathcal{R}$ is regular.

Then $\mathcal{C}_c \subseteq \mathrm{Coh}(\mathcal{C})$.

Proof. Let $G \in \mathcal{C}_c$ be a compact generator. Since the t-structure on $\mathcal{C}$ is in the preferred equivalence class, it follows that G is connective. Thus $\mathcal{C}_c \subseteq \mathrm{PCoh}(\mathcal{C})$. If we show that G is also coconnective, then $\mathcal{C}_c \subseteq \mathrm{Coh}(\mathcal{C})$. Since $h^* : \mathcal{C} \to \mathcal{R}$ is of finite tor-amplitude, we can apply point (3) of Lemma 3.3.12 to G: since $h^*(G) \in \mathcal{R}_c = \mathrm{Coh}(\mathcal{R})$ is coherent, it follows that G is coherent. □

Lemma 3.4.6. Let $\mathcal{A}$ be a small stable idempotent-complete $(\infty, 1)$-category enriched over $\mathcal{B}$. Let $\alpha : \mathcal{A}^{op} \to \mathcal{B}$ be an exact and $\mathcal{B}_c$-enriched functor. If α is a retract of a representable, then it is representable.

Proof. Assume that we have a retract

$$\alpha \xrightarrow{i} \mathcal{A}(-, x) \xrightarrow{r} \alpha$$

for some $x \in \mathcal{A}$. We wish to show that α is representable. Now the composition $i \circ r : \mathcal{A}(-, x) \to \mathcal{A}(-, x)$ is an idempotent and the enriched Yoneda lemma implies that it represented by a morphism $f : x \to x$. By Yoneda, f is an idempotent, and since $\mathcal{A}$ is idempotent-complete, it must split in $\mathcal{A}$. Thus there exists two morphisms

$$e \xrightarrow{j} x \xrightarrow{q} e$$

such that $q \circ j \simeq \mathrm{id}_e$ and $j \circ q \simeq f$. By Yoneda, the retract

$$\mathcal{A}(-, e) \xrightarrow{j^*} \mathcal{A}(-, x) \xrightarrow{q^*} \mathcal{A}(-, e)$$

corresponds to the idempotent $i \circ r$, thus showing $\alpha \simeq \mathcal{A}(-, e)$. □

We can now prove our generalization of Neeman, Functors out of $\mathcal{T}^b_c$, the *second abstract Neeman duality*.

Theorem 3.4.7 (Functors out of $\mathrm{Coh}(\mathcal{C})$)**.** Let $f^* : \mathcal{B} \to \mathcal{C}$ be a functor satisfying the assumptions of the first abstract Neeman duality. Assume that $\mathcal{C}$ admits a morphism of $\mathcal{B}$-universal descent $h^* : \mathcal{C} \to \mathcal{R}$ to a regular $(\infty, 1)$-category. Then there exists an equivalence of $(\infty, 1)$-categories

$$\tilde{\text{ょ}} : \mathcal{C}_c^{op} \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathrm{Coh}(\mathcal{B}))$$

induced by the restricted dual Yoneda embedding.

Proof. Before doing the proof let us fix the notation. Since $h^* : \mathcal{C} \to \mathcal{R}$ is of universal descent, we can find an exponent $e \geq 0$ such that the identity on $\mathcal{C}$ is a retract of $\varphi = \varphi_e : \mathcal{C} \to \mathcal{C}$, defined as in the previous section. Consider now the enriched restricted Yoneda

$$\tilde{\text{ょ}} : \mathcal{C}_c^{op} \to \mathrm{Fun}^{ex}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathrm{Coh}(\mathcal{B})).$$

which is well defined thanks to that Lemma 3.4.4. We wish to show that $\tilde{\text{ょ}}$ is fully-faithful and essentially surjective.

Let us start by showing its fully-faithfulness. By Lemma 3.4.5 we have that $\mathcal{C}_c \subseteq \mathrm{Coh}(\mathcal{C})$. Since the enriched Yoneda embedding (applied to $\mathrm{Coh}(\mathcal{C})^{\mathrm{op}}$) is fully-faithful, we deduce that the composition[19]

$$\tilde{よ} : \mathcal{C}_c^{\mathrm{op}} \hookrightarrow \mathrm{Coh}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathcal{B})$$

is fully-faithful. We now deduce the fully-faithfulness of

$$\tilde{よ} : \mathcal{C}_c^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathrm{Coh}(\mathcal{B})).$$

This leaves us to show that $\tilde{よ}$ is essentially surjective. Since $h^* : \mathcal{C} \to \mathcal{R}$ is a morphism of $\mathcal{B}$-universal descent, we have that $h^* \circ f^*$ satisfies the assumption of the first Neeman duality. In particular, the first internal realization proved in Proposition 1.1.9 shows that the diagram

$$\begin{array}{ccc} \mathcal{C}_c^{\mathrm{op}} & \xrightarrow{\tilde{よ}} & \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathrm{Coh}(\mathcal{B})) \\ {\scriptstyle (h^*)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle -\circ h_*} \\ \mathcal{R}_c^{\mathrm{op}} & \underset{\tilde{よ}}{\xrightarrow{\simeq}} & \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{R}), \mathrm{Coh}(\mathcal{B})) \end{array}$$

commutes. Let $\alpha : \mathrm{Coh}(\mathcal{C}) \to \mathrm{Coh}(\mathcal{B})$ be an exact and $\mathcal{B}_c$-enriched functor. Since Remark 1.3.4 showed that $\mathrm{Coh}(\mathcal{C})$ is idempotent-complete, we can apply the dual of Lemma 3.4.6. In particular, if we show that $\alpha \circ \varphi$ is representable, then the retract $\alpha \to \alpha \circ \varphi \to \alpha$ will imply that α is represented by a coherent object of $\mathcal{C}$.

Now the above diagram shows that the composition $\alpha \circ h_*$ is represented by a compact object $y \in \mathcal{R}_c^{\mathrm{op}}$. Moreover, for every integer $1 \leq n \leq e$, we have

$$\begin{aligned} \alpha \circ (h_* h^*)^n &\simeq \alpha \circ h_* \circ h^* \circ (h_* \circ h^*)^{n-1} \\ &\simeq \tilde{よ}(y) \circ h^* \circ (h_* \circ h^*)^{n-1} \\ &\simeq f_* h_* \underline{\mathrm{Hom}}_{\mathcal{R}}(y, h^* \circ (h_* \circ h^*)^{n-1}(-)). \end{aligned}$$

Since h^* is quasi-perfect, the abstract Grothendieck-Neeman duality Theorem 1.1.11 implies that we have adjunctions $h_{(1)} \dashv h^* \dashv h_*$. It follows that $\alpha \circ (h_* h^*)^n$ is represented by $(h^* h_{(1)})^n(y)$. Now the explicit definition of $\varphi = \varphi_e$ shows that $\alpha \circ \varphi$ is a finite limit of representable functors, and hence it must be representable (since the enriched Yoneda is exact, being the domain and codomain stable). We conclude that $\alpha \simeq \mathcal{C}(x, -)$ for some coherent object $x \in \mathrm{Coh}(\mathcal{C})$. To show that x is actually compact, let us consider the restriction $\alpha|_{\mathcal{C}_c} : \mathcal{C}_c \to \mathrm{Coh}(\mathcal{B})$. If we show that $\alpha|_{\mathcal{C}_c}$ is represented by an object $x' \in \mathcal{C}_c$, then we can apply the (dual) enriched Yoneda lemma to

$$\alpha|_{\mathcal{C}_c} = \mathcal{C}(x, -)|_{\mathcal{C}_c} \simeq \mathcal{C}(x', -)|_{\mathcal{C}_c}$$

and deduce that $x \simeq x'$ is actually compact. Now, to show that $\alpha|_{\mathcal{C}_c}$ is represented by a compact object it suffices to repeat the above argument and observe that $\alpha|_{\mathcal{C}_c} \circ \varphi$ is represented by a compact object since h^* preserves compact and since Lemma 1.1.8 implies that $h_{(1)}$ does that too. □

4 Examples

The goal of this chapter is to present some applications of the theory we have developed so far. We will begin with Section 4.1 where we will apply the theory developed so far to *module categories*. The example to have in mind are $(\infty, 1)$-categories of module spectra. Anyway, the proof of Theorem 4.1.9 does not rely on Functors out of $\mathcal{C}_c^{\mathrm{op}}$. The reason is simple: it is hard to give a characterization of quasi-proper morphisms between module categories. Moreover, it is can be in general hard to find (if it exists!) a compact generator

[19]Notice that the dual Yoneda $\tilde{よ}$ does not restrict to a functor $\mathrm{Coh}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{B}_c}(\mathrm{Coh}(\mathcal{C}), \mathrm{Coh}(\mathcal{B}))$. The obstruction lies in the fact that we cannot identify it as the tensor product on $\mathcal{C}$ followed by the pushfoward!

satisfying the assumption of the theorem. For this reason our proof relies on Theorem 3.2.2 and quasi-perfect functors.

Finally, in Section 4.2 and Section 4.3 we will present more concrete examples by working in the realm of algebraic geometry and and spectral algebraic geometry. In the case of schemes, the main results are Corollary 4.2.6 and Corollary 4.2.7, for spectral algebraic spaces the result is Corollary 4.3.5.

4.1 Modules

We now discuss a broad class of examples, given by module categories. Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 1)$-category and let $x \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object. Thanks to [Lur17, Section 4.3], there exists a sensible theory of x-module objects in $\mathcal{C}$. An *x-module object in* $\mathcal{C}$ is just an object $c \in \mathcal{C}$ equipped with an action $a : x \otimes c \to c$ in $\mathcal{C}$ that satisfies the usual conditions. That is, if $m : x \otimes x \to x$ and $u : 1 \to x$ denote the multiplication and unit map on x, then the diagrams

$$\begin{array}{ccc} x \otimes x \otimes c & \xrightarrow{m \otimes id} & x \otimes c \\ {\scriptstyle id \otimes a} \downarrow & & \downarrow {\scriptstyle m} \\ x \otimes c & \xrightarrow[m]{} & c \end{array} \qquad \begin{array}{ccc} 1 \otimes c & \xrightarrow{u \otimes id} & x \otimes c \\ & \searrow \quad \swarrow {\scriptstyle m} & \\ & c & \end{array}$$

are commutative up to higher coherence conditions. The collection of such objects organize into an $(\infty, 1)$-category $\mathrm{Mod}_x(\mathcal{C})$. Moreover, tensoring with x determines a functor $x \otimes - : \mathcal{C} \to \mathrm{Mod}_x(\mathcal{C})$. This functor is left adjoint to the forgetful functor $\mathrm{res} : \mathrm{Mod}_x(\mathcal{C}) \to \mathcal{C}$, which is conservative.

We now show that module objects are stable under many construction introduced so far. Let us first discuss the geometric properties.

Lemma 4.1.1. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category and let $x \in \mathcal{C}$ be a commutative algebra object. Then the $(\infty, 1)$-category $\mathrm{Mod}_x(\mathcal{C})$ of x-module objects in $\mathcal{C}$ is geometric. Moreover, $x \otimes - : \mathcal{C} \to \mathrm{Mod}_x(\mathcal{C})$ is a geometric functor.

Proof. First of all, [Lur17, Proposition 7.1.1.4 and Section 4.4] show that $\mathrm{Mod}_x(\mathcal{C})$ is stable by and endowed with a symmetric monoidal structure given by the relative tensor product. Moreover, by [Lur17, Section 4.3.3], $\mathrm{Mod}_x(\mathcal{C})$ admits both colimits and small limits, which are preserved and detected by the forgetful functor $\mathrm{res} : \mathrm{Mod}_x(\mathcal{C}) \to \mathcal{C}$. Consider the left adjoint $x \otimes - : \mathcal{C} \to \mathrm{Mod}_x(\mathcal{C})$. The essential image of the compact generators of $\mathcal{C}$ compactly generates $\mathrm{Mod}_x(\mathcal{C}) \to \mathcal{C}$. Since tensoring with x is a symmetric monoidal functor, we deduce that dualizable and compact objects in $\mathrm{Mod}_x(\mathcal{C})$ coincide. To conclude, let us also note that $x \otimes - : \mathcal{C} \to \mathrm{Mod}_x(\mathcal{C})$ is geometric by definition. □

If $\mathcal{C}$ comes equipped with a well generated geometric tensor t-structure, then the same happens for module objects:

Lemma 4.1.2. Let $\mathcal{C}$ be a geometric $(\infty, 1)$-category. Assume that the monoidal unit of $\mathcal{C}$ is a compact generator, and equip $\mathcal{C}$ with the geometric tensor t-structure of Lemma 1.2.12. Let $x \in \mathcal{C}$ be a commutative algebra object. Then the $(\infty, 1)$-category $\mathrm{Mod}_x(\mathcal{C})$ can be equipped with a geometric tensor t-structure such that $x \otimes - : \mathcal{C} \to \mathrm{Mod}_x(\mathcal{C})$ is a t-geometric functor.

Moreover, if x is connective in $\mathcal{C}$, then $\mathrm{res} : \mathrm{Mod}_x(\mathcal{C}) \to \mathcal{C}$ is t-exact.

Proof. Before doing the actual proof, let us note that the t-structure of Lemma 1.2.12 is indeed tensor since the explicit description of the connective aisle $\mathcal{C}_{\geq 0}$ shows that it contains the monoidal unit and it is closed under tensor products. Let us denote by $\mathbb{1}_{\mathcal{C}}$ the monoidal unit of $\mathcal{C}$. We first observe that, since $\mathbb{1}_{\mathcal{C}}$ is a generator, the adjunction $x \otimes - \dashv \mathrm{res}$, coupled with fact that res is conservative, implies that x is a compact generator of $\mathrm{Mod}_x(\mathcal{C})$. We can then apply Lemma 1.2.12 to deduce the existence of a geometric tensor t-structure $(\mathrm{Mod}_x(\mathcal{C})_{\leq 0}, \mathrm{Mod}_x(\mathcal{C})_{\geq 0})$. The "tensor" part follows since x is the monoidal unit. From the construction we also see that the left adjoint $x \otimes -$ is clearly right t-exact. The "moreover" part is obvious. □

On the other side, cohereness is not preserved by taking module objects.

Remark 4.1.3. Assume that $\mathcal{C}$ is coherent and let $x \in \mathcal{C}$ be a commutative algebra object. Then it is *not* always true that $\mathrm{Mod}_x(\mathcal{C})$ is coherent. Indeed, if R is a connective commutative ring spectrum, then $\mathrm{Mod}_R = \mathrm{Mod}_R(\mathrm{Sp})$ is coherent if and only if R is a coherent ring in the sense of [Lur17, Definition 7.2.4.16], whereas $\mathrm{Sp} \simeq \mathrm{Mod}_{\mathbb{S}}(\mathrm{Sp})$ is coherent.

To avoid awkward terminology, we will say that a commutative algebra object $x \in \mathrm{CAlg}(\mathcal{C})$ in a geometric $(\infty,1)$-category $\mathcal{C}$ is *coherent* if the module category $\mathrm{Mod}_x(\mathcal{C})$ is coherent when equipped with the geometric tensor t-structure of Lemma 4.1.2.

Remark 4.1.4. We can also adopt a relative point of view. Let $\mathcal{C}$ be a t-geometric $(\infty,1)$-category whose monoidal unit is a compact generator. Assume that the t-structure is induced by Lemma 1.2.12. Let $f : x \to y$ be a map of commutative algebra objects in $\mathcal{C}$. Regard y as an x-module. Then [Lur17, Theorem 4.5.3.1] implies the existence of the following commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_x(\mathcal{C}) \arrow[rr, "f^* \simeq y \otimes_x -"] & & \mathrm{Mod}_y(\mathcal{C}) \\ & \mathcal{C} \arrow[ul, "x \otimes -"] \arrow[ur, "y \otimes -"'] & \end{tikzcd}$$

where the functor $f^* \simeq y \otimes_x -$ is called the *extension of scalars*. By the general theory, f^* is symmetric monoidal and colimit preserving, hence geometric. Furthermore, if we equip $\mathrm{Mod}_x(\mathcal{C})$ and $\mathrm{Mod}_y(\mathcal{C})$ with the geometric tensor t-structure of Lemma 4.1.2, then f^* is also t-geometric. The right adjoint $f_* = \mathrm{res}_f$ of f^* is given by the forgetful functor, and is called the *restriction along* f. Moreover, if $y \in (\mathrm{Mod}_x(\mathcal{C}))_{\geq 0}$ is connective, then res_f is also t-exact.

Remark 4.1.5 (The Coextension)**.** Let $\mathcal{C}$ be a t-geometric $(\infty,1)$-category whose monoidal unit is a compact generator. Assume that the t-structure is induced by Lemma 1.2.12. Let $f : x \to y$ be a map of commutative algebra objects in $\mathcal{C}$. In Section 1.1 we have discovered that the restriction along f admits a right adjoint $f^{(1)}$ called the *coextension of scalars*. It is given by $f^{(1)}(-) \simeq \mathrm{Hom}_x(y,-) : \mathrm{Mod}_x(\mathcal{C}) \to \mathrm{Mod}_y(\mathcal{C})$.

Remark 4.1.6. Let $\mathcal{C}$ and $f : x \to y$ be as in the previous remarks. If $y \in (\mathrm{Mod}_x(\mathcal{C}))_{\geq 0}$ is connective as an x-module, then the restriction along f is t-exact. In particular, we can apply Remark 1.2.17 and Lemma 1.2.18 to the adjunctions $y \otimes_x - \dashv \mathrm{res}_f \dashv \mathrm{Hom}_x(y,-)$ to deduce the existence of adjunctions

$$\begin{tikzcd} \mathrm{Mod}_x(\mathcal{C})^{\heartsuit} \arrow[r, bend left, "{}^{p}(y\otimes_x -)"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_y(\mathcal{C})^{\heartsuit} \arrow[l, bend left, "{}^{p}\mathrm{res}_f"] \arrow[r, bend left, "{}^{p}\mathrm{res}_f"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_x(\mathcal{C})^{\heartsuit} \arrow[l, bend left, "{}^{p}\mathrm{Hom}_x(y,-)"] \end{tikzcd}$$

between the hearts.

In order to prove our result on module categories, we need one more technical result.

Lemma 4.1.7. Let ${}^{p}f^* \dashv {}^{p}f_* \dashv {}^{p}f^{(1)} : \mathcal{B}^{\heartsuit} \to \mathcal{C}^{\heartsuit}$ be a double adjunction between Grothendieck abelian 1-categories. Assume that:

(1) The middle adjoint ${}^{p}f_*$ is conservative.

(2) The outermost right adjoint ${}^{p}f^{(1)}$ preserves filtered colimits (if $\mathcal{C}^{\heartsuit}$ is compactly generated this assumption is equivalent to ${}^{p}f_*$ preserving compact objects).

Then ${}^{p}f_* : \mathcal{C}^{\heartsuit} \to \mathcal{B}^{\heartsuit}$ detects compact objects.

Proof. Let $x \in \mathcal{C}^{\heartsuit}$ be such that ${}^{p}f_*(x)$ is compact in $\mathcal{B}^{\heartsuit}$. We wish to show that x is compact. First of all, it is finitely generated. Since the right adjoint ${}^{p}f_*$ is conservative, the counit ${}^{p}f^*\,{}^{p}f_*(x) \to x$ is a (strong) epimorphism. In particular, since the epimorphic image of finitely generated object is again finitely generated, it suffices to show that ${}^{p}f^*\,{}^{p}f_*(x)$ is finitely generated. But ${}^{p}f_*(x)$ is finitely generated, and ${}^{p}f^*$ preserves finitely generated objects (since ${}^{p}f_*$ preserves filtered colimits and monomorphism).

Consider now a short exact sequence in $\mathcal{C}^\heartsuit$

$$0 \to \ker(h) \to y \xrightarrow{h} x \to 0$$

in $\mathcal{C}^\heartsuit$ and assume that y is finitely generated. We wish to show that $\ker(h)$ is finitely generated. Apply ${}^p f_*$ and use its exactness to deduce the short exact sequence

$$0 \to \ker({}^p f_*(h)) \to {}^p f_*(y) \xrightarrow{{}^p f_*(h)} {}^p f_*(x) \to 0$$

in $\mathcal{B}^\heartsuit$. Notice that ${}^p f_*(y)$ is again finitely generated. This follows by adjunction ${}^p f_* \dashv {}^p f^{(1)}$, by point (2) and by right exactness of ${}^p f^{(1)}$. Now, since ${}^p f_*(x)$ is compact, hence finitely presented, it follows that ${}^p f_*(\ker(h)) \cong \ker({}^p f_*(h))$ is finitely generated. By the same argument above, the counit ${}^p f^* \, {}^p f_*(\ker(h)) \to \ker(h)$ exhibits $\ker(h)$ as finitely generated. □

Remark 4.1.8. Let $f^* : \mathcal{B} \to \mathcal{C}$ be a t-geometric functor between coherent $(\infty,1)$-categories. Assume also that the right adjoint f_* is t-exact preserves compact objects. Then the induced functor ${}^p f_* : \mathcal{C}^\heartsuit \to \mathcal{B}^\heartsuit$ preserves compact objects. Indeed, since the heart $\mathcal{C}^\heartsuit$ is compactly generated by the $\pi_0(G)$ of the compact generator G of $\mathcal{C}$, it suffices to show that ${}^p f_*$ sends $\pi_0(G)$ to a compact object. This follows since f_* preserves compact objects and is t-exact: the isomorphism ${}^p f_*(\pi_0(G)) \simeq \pi_0(f_*(G))$ show that ${}^p f_*(\pi_0(G))$ is compact.

Let $f : x \to y$ be a map of commutative algebra objects. We will say that f *is finitely presented* if $\mathrm{res}_f(y)$ is a compact object of $\mathrm{Mod}_x(\mathcal{C})$. In this case, the coextension $\mathrm{Hom}_x(y, -)$ preserves filtered colimits. To state the result, let us abbreviate $\mathrm{PCoh}(\mathrm{Mod}_y(\mathcal{C}))$ and $\mathrm{Coh}(\mathrm{Mod}_y(\mathcal{C}))$ with $\mathrm{PCoh}(y)$ and $\mathrm{Coh}(y)$. We will also denote by $\mathrm{Perf}(y)$ the subcategory of compact objects $\mathrm{Mod}_y(\mathcal{C})_c$. We do the smae for x.

Theorem 4.1.9 (Functors out of $\mathcal{C}_c^{op}$ for module categories)**.** Let $\mathcal{C}$ be a t-geometric $(\infty,1)$-category whose monoidal unit is a compact generator. Assume that the t-structure is induced by Lemma 1.2.12. Let $f : x \to y$ be a finitely presented map in $\mathrm{CAlg}(\mathcal{C})$ between coherent objects. Assume that y connective in $\mathrm{Mod}_x(\mathcal{C})$. Then the restricted Yoneda embedding induces equivalences

$$\mathrm{PCoh}(y) \to \mathrm{Fun}^{ex}_{\mathrm{Perf}(x)}(\mathrm{Perf}(y)^{op}, \mathrm{PCoh}(x)), \qquad \mathrm{Coh}(y) \to \mathrm{Fun}^{ex}_{\mathrm{Perf}(x)}(\mathrm{Perf}(y)^{op}, \mathrm{Coh}(x))$$

of $(\infty,1)$-categories.

Proof. Apply Theorem 3.2.2 to the geometric functor $y \otimes_x - : \mathrm{Mod}_x(\mathcal{C}) \to \mathrm{Mod}_y(\mathcal{C})$ to get equivalences of $(\infty,1)$-categories

$$f^\#(\mathrm{PCoh}(y)) \to \mathrm{Fun}^{ex}_{\mathrm{Perf}(x)}(\mathrm{Perf}(y)^{op}, \mathrm{PCoh}(x))$$

and

$$f^\#(\mathrm{Coh}(y)) \to \mathrm{Fun}^{ex}_{\mathrm{Perf}(x)}(\mathrm{Perf}(y)^{op}, \mathrm{Coh}(x)).$$

To prove the theorem we need to identify the two kernels with $\mathrm{PCoh}(y)$ and $\mathrm{Coh}(y)$. We treat the case of pseudo-coherent objects; the case of coherent objects is exactly the same. The proof relies on Theorem 1.4.12. It shows that $c \in \mathrm{Mod}_x(\mathcal{C})$ is pseudo-coherent if and only if $\pi_n(c)$ is compact in $\mathrm{Mod}_x(\mathcal{C})^\heartsuit$ and vanishes for $n << 0$. A similar description exists for $\mathrm{Mod}_y(\mathcal{C})$.

We claim that $y \otimes_x -$ is quasi-proper, that is, res_f preserves pseudo-coherent objects. Take $c \in \mathrm{PCoh}(x)$. Since y is connective in $\mathrm{Mod}_x(\mathcal{C})$, Remark 4.1.6 implies that res_f is t-exact, so that we have isomorphisms

$$\pi_n(\mathrm{res}_f(c)) \cong {}^p\mathrm{res}_f(\pi_n(c)))$$

in $\mathrm{Mod}_y(\mathcal{C})^\heartsuit$ for every $n \in \mathbb{Z}$. In particular, $\pi_n(c) = 0$ for $n << 0$ implies $\pi_n(\mathrm{res}_f(c)) = 0$ for $n << 0$. Now f finitely presented implies that the coextension preserves filtered colimits, and Lemma 1.1.8 implies that res_f preserves compact objects. Remark 4.1.8 implies then that ${}^p\mathrm{res}_f$ preserves compact objects. In particular, $\pi_n(c)$ compact in $\mathrm{Mod}_x(\mathcal{C})^\heartsuit$ for every $n \in \mathbb{Z}$ implies that $\pi_n(\mathrm{res}_f(c))$ is compact in $\mathrm{Mod}_y(\mathcal{C})^\heartsuit$ for every $n \in \mathbb{Z}$ thanks to the above isomorphism. Hence $\mathrm{res}_f(c) \in \mathrm{PCoh}(x)$, thus showing that $y \otimes_x -$ is quasi-proper. Hence the inclusion $\mathrm{PCoh}(y) \subseteq f^\#(\mathrm{PCoh}(x))$.

We now need the converse, so assume that $\mathrm{res}_f(c)$ is pseudo-coherent in $\mathrm{Mod}_x(\mathcal{C})$ and let us show that c is pseudo-coherent in $\mathrm{Mod}_y(\mathcal{C})$, or, equivalently, that $\pi_n(c)$ is compact in $\mathrm{Mod}_y(\mathcal{C})^\heartsuit$ and vanishes for $n << 0$. To show that, we use again the above isomorphism. Indeed, being res_f conservative and t-exact, it follows that ${}^p\mathrm{res}_f$ is conservative. In particular, $\pi_n(\mathrm{res}_f(c)) = 0$ for $n << 0$ implies $\pi_n(c) = 0$ for $n << 0$. So we are left to show that ${}^p\mathrm{res}_f$ detects compact objects between locally coherent abelian 1-categories. Since the restriction ${}^p\mathrm{res}_f$ is conservative, Lemma 4.1.7 will conclude. □

Warning 4.1.10. Note that the above theorem does not require any noetherianess assumption, contrary to what we can expect from some enhancement of Neeman's corollary [Nee18b, Corollary 0.5]. The reason is that we traded noetherianess and finite maps with coherence and finitely presented maps.

Example 4.1.11. The above theorem is particularly useful when applied to the $(\infty, 1)$-category of spectra Sp. If $f : A \to B$ is a map between coherent $\mathbb{E}_\infty$-rings and B is compact as A-module, then it provides equivalences

$$\mathrm{PCoh}(B) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(A)}(\mathrm{Perf}(B)^{\mathrm{op}}, \mathrm{PCoh}(A)), \qquad \mathrm{Coh}(B) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(A)}(\mathrm{Perf}(B)^{\mathrm{op}}, \mathrm{Coh}(A)).$$

Here Perf stands for compact objects, see [Lur17, Proposition 7.2.4.2].

4.2 Schemes

We begin by considering the case of schemes. Let X be a quasi-compact quasi-separated scheme and let us denote by $\mathrm{QCoh}(X)$ the derived $(\infty, 1)$-category of quasi-coherent sheaves on X. The language of $(\infty, 1)$-categories allows us to write $\mathrm{QCoh}(X)$ as

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec}(R) \subseteq X} \mathrm{Mod}_{HR}$$

where the limit is taken over the poset of open affine subsets of X. Here Mod_{HR} is the $(\infty, 1)$-category of modules over the Eilenberg-MacLane spectrum HR, or equivalently the (unbounded) derived $(\infty, 1)$-category of R-modules. This limit allows us to show that $\mathrm{QCoh}(X)$ is a stable homotopy theory. Indeed, it does not matter if the limit takes place in $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega}_{st})$ or in $\mathrm{Pr}^{L,\omega}_{st}$, since it will always follow formally that $\mathrm{QCoh}(X)$ is a stable compactly generated $(\infty, 1)$-category equipped with a symmetric monoidal structure, compatible with colimits.

We can also show that $\mathrm{QCoh}(X)$ is geometric. Indeed, since X is quasi-compact and quasi-separated, the monoidal unit $\mathcal{O}_x$ of $\mathrm{QCoh}(X)$ is compact. In particular, any dualizable object is compact. Moreover, dualizable objects coincide with the perfect objects $\mathrm{Perf}(X)$, that is with those objects complexes which, locally, are quasi-isomorphic to bounded chain complexes of finitely generated projective modules. See [Aut18, Lemma 08JJ] and [Aut18, Lemma 0FPV] for a proof. Thanks to [Aut18, Proposition 09M1] we get that any perfect object is compact, so that dualizable and compact objects coincide.

Consider now the standard t-structure on $\mathrm{QCoh}(X)$. This t-structure is accessible, compatible with filtered colimits and right (and left) complete, so that it is t-geometric in the sense of Definition 1.2.3. Being generated by the monoidal unit $\mathcal{O}_X$, every t-structure with cocomplete aisle will define a tensor t-structure.

We can now study the compact generation of $\mathrm{QCoh}(X)$. A beautiful result of Bondal and van den Bergh [BB02, Theorem 3.1.1] shows that $\mathrm{QCoh}(X)$ is generated by a single compact object G. We can therefore consider the preferred equivalence class. The following result shows that being connective can be detected by pulling back along an open affine cover, so that the standard t-structure on $\mathrm{QCoh}(X)$ is equivalent to the one determined by G.

Lemma 4.2.1. Let X be a quasi-compact quasi-separated scheme. Suppose $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t-structure on $\mathrm{QCoh}(X)$ such that for all open immersions $i : U \hookrightarrow X$ where U is affine, the pair $(i^*\mathcal{C}_{\geq 0}, i^*\mathcal{C}_{\leq 0})$ is t-structure on $\mathrm{QCoh}(U)$. An object x in $\mathrm{QCoh}(X)$ belongs to $\mathcal{C}_{\geq 0}$ if and only if i^* is in $i^*\mathcal{C}_{\geq 0}$ for all open immersions $i : U \hookrightarrow X$ where U is affine.

Proof. First of all, notice that we are assuming that the essential image along i^* of a t-structure is still a t-structure. This trivial (but tedious) fact follows since i^* is an essentially surjective functor. For the actual proof of the claim, let us first note that the "only if" direction is automatic. For the "if" direction, let us assume that $x \in \mathrm{QCoh}(X)$ is such that $i^*(x)$ is in $i^*\mathcal{C}_{\geq 0}$ for all open immersions $i : U \hookrightarrow X$ with U is affine. Thanks to our assumption, by applying i^* to the cofibre sequence $\tau_{\geq 0}x \to x \to \tau_{\leq -1}x$ in QCoh(X) we learn that $i^*(\tau_{\leq -1}x) \simeq 0$ in QCoh(U) for every open immersions $i : U \hookrightarrow X$ with U affine. Here the truncation functors are the one of $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We now conclude by that $\tau_{\leq -1}x \simeq 0$ in QCoh(X), so that $x \in \mathcal{C}_{\geq 0}$, by noting that the functor $(i^*)_U : \mathrm{QCoh}(X) \to \prod_U \mathrm{QCoh}(U)$ is conservative when the index U of the product ranges in an open affine cover of X. ☐

Moreover, since compact objects coincide with perfect ones and X is quasi-compact, it follows that G is a bounded object and hence that $\pi_0 \mathrm{Hom}_{\mathrm{QCoh}(X)}(G, \mathcal{C}_{\geq N}) = 0$ for some integer $N > 0$. Let us summarize everything.

Corollary 4.2.2. Let X be a quasi-compact quasi-separated scheme and let G be a compact generator for QCoh(X). Then the standard t-structures equips QCoh(X) with the structure of a t-geometric $(\infty, 1)$-category. Moreover, there exists an integer $N > 0$ such that $G \in \mathrm{QCoh}(X)_{\geq -N}$ is $(-N)$-connected and $\pi_0 \mathrm{Hom}_{\mathrm{QCoh}(X)}(G, \mathcal{C}_{\geq N}) = 0$.

We wish now to compute our (pseudo)-coherent objects (which we will denote by $\mathrm{Coh}(X) \subseteq \mathrm{PCoh}(X)$) for the standard t-structure on QCoh(X). First of all, the boundedness of the compact generator G ensures that $\mathrm{Perf}(X) \subseteq \mathrm{Coh}(X)$, and hence Lemma 1.3.8 shows that $\mathrm{Coh}(X) \subseteq \mathrm{PCoh}(X)$ are Perf(X)-submodules. The assumptions on G allow us to also apply Proposition 1.5.7. It follows that the (pseudo)-coherent objects coincide with the bounded pseudo-compact objects. Lipman and Neeman showed that these objects coincide with the bounded pseudo-coherent objects first studied in Illusie's exposés [BJG+71] in SGA6. These are the complexes of $\mathcal{O}_X$-modules that are locally pseudo-coherent in the sense of [Aut18, Definition 064Q]. Under some finiteness assumption we can further identify these objects.

Lemma 4.2.3. Let X be quasi-separated noetherian (or even coherent) scheme. Then the standard t-structure on QCoh(X) is coherent in the sense of Definition 1.4.2.

Proof. We have already observed that the standard t-structure on QCoh(X) is in the preferred equivalence class and that there exists an integer $N > 0$ such that $\pi_0 \mathrm{Hom}_{\mathrm{QCoh}(X)}(G, \mathrm{QCoh}(X)_{\geq N}) = 0$. To check the second condition, let us first note that Bondal-van den Bergh proof on the existence of G may be modified to show that the homotopy groups $\pi_i(G) = H^{-i}(X, G)$ are compact in the heart $\mathrm{QCoh}(X)^{\heartsuit}$. This follows since the construction is done via the induction principle on schemes, and since the claim is true for affine and for Koszul complexes, it glues to a global statement. By thickness, it also follows that any compact object has this property. The claim then follows by using that the t-structure is in the preferred equivalence class. We are left to prove that the heart $\mathrm{QCoh}(X)^{\heartsuit}$ is a locally coherent abelian 1-category. This follows since the compact objects in $\mathrm{QCoh}(X)^{\heartsuit}$ are exactly the coherent objects $\mathrm{Coh}(X)^{\heartsuit}$ (see for example the MathOverflows question Compact quasi-coherent sheaves). When X is coherent, the heart $\mathrm{Coh}(X)^{\heartsuit}$ is locally coherent abelian, and locally noetherian abelian when X is noetherian. ☐

Hence Theorem 1.4.12 allows us to compute explicitly $\mathrm{Coh}(X) \subseteq \mathrm{PCoh}(X)$: they turn out to be the classical $D^b_{coh}(X) \subseteq D^-_{coh}(X)$. This gives a different proof of [Aut18, Lemma 08E8], and concludes what we have to say about QCoh(X).

Let us now consider a map $f : X \to Y$ between quasi-compact separated schemes. Consider the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. This functor is t-geometric, since it is colimit preserving, symmetric monoidal and right t-exact with respect to the standard t-structure. It is moreover of finite cohomological dimension. In general the pushfoward f_* does *not* preserve compact objects, neither it sends pseudo-coherent objects to pseudo-coherent objects. It does when it is *quasi-perfect* or *quasi-proper*, respectively.

Lemma 4.2.4. Let $f : X \to Y$ be a morphism of quasi-compact quasi-separated schemes, and let G be a compact generator for QCoh(X). Then $f_*\underline{\mathrm{Hom}}_{\mathrm{QCoh}(X)}(G, -) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ detects being connective and coconnective.

Proof. We may assume that $Y = \mathrm{Spec}(R)$ is affine. The proof follows from [ATJLSdS23, Theorem 1.10] by noting that we can modify their argument (which deals with the case $X \to \mathrm{Spec}(\mathbb{Z})$) to fit our case $X \to \mathrm{Spec}(R)$. ☐

We can now apply the first abstract Neeman duality, that is Theorem 3.2.4, to deduce the following result.

Corollary 4.2.5. Let $f : X \to Y$ be a quasi-proper map of quasi-compact quasi-separated schemes. Assume that Y is coherent. Then we have equivalences of $(\infty, 1)$-categories

$$\mathrm{PCoh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{PCoh}(Y)), \qquad \mathrm{Coh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{Coh}(Y)).$$

induced by the the restricted Yoneda embedding.

We can obtain a more classical result by means of Kiehl's Finiteness Theorem [Kie72, Theorem 2.9′]. It shows that every proper pseudo-coherent map is quasi-proper, and, in particular, it implies that every finite-type separated map $f : X \to Y$ over a noetherian base is quasi-proper if and only if it is proper. This observation, coupled with the previous result, proves the following generalization of [Nee18b, Corollary 0.5].

Corollary 4.2.6. Let $f : X \to Y$ be a proper map and assume that Y is noetherian. Then we have equivalences of $(\infty, 1)$-categories

$$D^{-}_{\mathrm{coh}}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, D^{-}_{\mathrm{coh}}(Y)), \qquad D^{b}_{\mathrm{coh}}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, D^{b}_{\mathrm{coh}}(Y)).$$

induced by the the restricted Yoneda embedding.

We now turn our attention to the second duality result. Let X be a noetherian scheme. Recall that a *regular alteration* of X is a proper surjective morphism $h : R \to X$ such that:

(1) R is regular and finite dimensional.

(2) There is a dense open set $U \subseteq X$ over which h is finite.

Since X is noetherian and every morphism of finite type over a locally noetherian base is of finite presentation, [Aut18, Lemma 0ETW] implies that every regular alteration $h : R \to X$ is an *h-cover* in the sense of Voedvosky [Aut18, Definition 0ETS]. Now by [BS17, Proposition 11.25] every h-cover $h : R \to X$ of noetherian schemes is *descendable* in the sense of [Mat16, Definition 3.18]. If now h is also of finite tor-amplitude (that is, is quasi-perfect), then the derived pullback $h^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(R)$ is of universal descent in the sense of Definition 3.3.5. Indeed, point (1) follows by the quasi-perfection of h, whereas point (2) follows from Bhatt and Scholze result. Finally, by regarding h^* as a t-geometric functor, it immediately follows that it is also a t-geometric functor of universal descent in the sense of Definition 3.3.9. With these observations in our hands, we can prove our application of the second abstract Neeman duality, Theorem 3.4.7.

Corollary 4.2.7. Let $f : X \to Y$ be a proper map and assume that Y is noetherian. Assume that X is separated and of finite type scheme over an excellent scheme of dimension ≤ 2. Then we have an equivalence of $(\infty, 1)$-categories

$$\mathrm{Perf}(X)^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(D^{b}_{\mathrm{coh}}(X), D^{b}_{\mathrm{coh}}(Y))$$

induced by the the restricted dual Yoneda embedding.

Proof. We wish to apply Theorem 3.4.7, so let us check the assumptions. In [DJ96] and [Jon97] de Yong proved that every separated and of finite type scheme over an excellent scheme of dimension ≤ 2 admits a regular alteration. In particular, we find a regular alteration $h : R \to X$. Consider now the derived pullback $h^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(R)$. It follows that:

(1) Since R is a regular scheme, $\mathrm{QCoh}(R)$ is a regular $(\infty, 1)$-category in the sense of Definition 3.4.1.

(2) Since h is a proper and surjective morphism from a finite dimensional scheme, it follows that X is finite dimensional. Since any proper morphism between finite dimensional noetherian schemes is quasi-perfect, it follows that h^* is of universal descent in the sense of Definition 3.3.9.

In particular, we will be done if h^* is of $\mathrm{QCoh}(Y)$-universal descent. This is obvious: apply Lemma 4.2.4. □

4.3 Spectral Deligne-Mumford Stacks

We can generalize the previous example by considering spectral Deligne-Mumford stacks. Let us denote by $\widehat{\mathrm{Spc}}$ the very large $(\infty,1)$-category of large spaces. Denote also by $\mathrm{CAlg}^{\mathrm{cn}}(\mathrm{Sp})$ the $(\infty,1)$-category of connective $\mathbb{E}_\infty$-rings.

Definition 4.3.1 ([Lur18, Definition 9.1.0.1])**.** Let $X : \mathrm{CAlg}^{\mathrm{cn}}(\mathrm{Sp}) \to \widehat{\mathrm{Spc}}$ be a functor. We will say that X is *quasi-geometric stack* if it satisfies the following conditions:

(1) The functor X satisfies descent with respect to the fpqc topology.

(2) The diagonal map $\delta : X \times X \to X$ is quasi-affine

(3) There exists a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $\mathrm{Spec}(A) \to X$.

The class of quasi-geometric stacks contains many algebro-geometric objects that arise in practice. In [Lur18, Section 6.2], Lurie assigns to every quasi-geometric stack X (and actually to every functor $X : \mathrm{CAlg}^{\mathrm{cn}}(\mathrm{Sp}) \to \widehat{\mathrm{Spc}}$ an $(\infty,1)$-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$. If X is actually represented by a spectral Deligne-Mumford stack, then [Lur18, Proposition 2.2.4.1, Proposition 2.2.4.2] show that $\mathrm{QCoh}(X)$ is presentable, stable and equipped with a symmetric monoidal such that the tensor product $\otimes$ preserves small colimits separately in each variable. By [Lur18, Proposition 2.2.5.2, Proposition 2.2.5.4 and Proposition 2.1.1.1], $\mathrm{QCoh}(X)$ comes equipped with a geometric tensor t-structure.

Anyway, we still have to determine when $\mathrm{QCoh}(X)$ is geometric. The first obstruction happens since, in general, the monoidal unit $\mathcal{O}_X$ is *not* a compact object of $\mathrm{QCoh}(X)$. The second obstruction is in the compact generation, since $\mathrm{QCoh}(X)$ does not have enough perfect complexes in general. For this reason, it is better to restrict to *perfect stacks*. Roughly speaking, a quasi-geometric stack X is perfect if the canonical map $\mathrm{Ind}(\mathrm{Perf}(X)) \to \mathrm{QCoh}(X)$ is an equivalence of $(\infty,1)$-categories (see [Lur18, Proposition 9.4.4.5]). Here it is the precise definition.

Definition 4.3.2 ([Lur18, Definition 9.4.4.1])**.** Let $X : \mathrm{CAlg}^{\mathrm{cn}}(\mathrm{Sp}) \to \widehat{\mathrm{Spc}}$ be a functor. We will say that X is a perfect stack if it satisfies the following conditions:

(1) The functor X is a quasi-geometric stack.

(2) The structure sheaf $\mathcal{O}_X$ is a compact object of $\mathrm{QCoh}(X)$.

(3) Every quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ can be obtained as the colimit of a filtered diagram $\{\mathcal{F}_i\}_{i\in I}$, where each $\mathcal{F}_i$ is a perfect object of $\mathrm{QCoh}(X)$.

Now, as always, perfect and dualizable objects coincide. Since for a perfect stack X the compact objects coincide with the perfect ones, it follows that $\mathrm{QCoh}(X)$ is a geometric $(\infty,1)$-category.

Compact generations by a single object is more subtle and need some restriction. First of all, let us recall that a *spectral algebraic space* is a spectral Deligne-Mumford stack X such that the mapping space $\mathrm{Hom}(\mathrm{Spét}(R), X)$ is discrete for every commutative ring R. See [Lur18, Definition 1.6.8.1]. Now [Lur18, Proposition 9.6.1.1] shows that if X is a quasi-compact, quasi-separated spectral algebraic space then its functor of points defines a perfect stack, allowing us to deduce that $\mathrm{QCoh}(X)$ is geometric. Its compact generation by a single object follows then by [Lur18, Corollary 9.6.3.2]. To sum up, we have the following.

Corollary 4.3.3. Let X be a quasi-compact and quasi-separated spectral algebraic space. Then the $(\infty,1)$-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ on X, equipped with the standard t-structure, is t-geometric. Moreover, X comes equipped with a compact generator G such that the standard t-structure is in the preferred equivalence class. Finally, if X is noetherian, then $\mathrm{QCoh}(X)$ is coherent.

In light of this result, we will study morphisms of quasi-compact, quasi-separated spectral algebraic spaces. Let $f : X \to Y$ be such morphism. In this case f determines a symmetric monoidal and colimit preserving functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. In particular, f^* is geometric. If we equip $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ with the standard t-structure, then f^* is also right t-exact. We are left to determine under what assumptions f^* is quasi-perfect. Since it is clear that the standard t-structures are in the preferred equivalence classes and that being of cohomological dimension is a property directly reflected from f, we are left to determine when

f_* preserves pseudo-coherence. Thanks to the direct image theorem, [Lur18, Theorem 5.6.02], we learn that, for any proper and locally almost of finite presentation of spectral Deligne-Mumford stacks $f : X \to Y$ the direct image functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ carries almost perfect objects to almost perfect objects[20]. The definitions of proper and locally almost of finite presentation morphisms are in [Lur18, Definition 5.1.2.1 and Definition 4.2.0.1]. To sum up, we have the following.

Corollary 4.3.4. Let $f : X \to Y$ be a morphism of finite cohomological dimension of quasi-compact quasi-separated spectral algebraic spaces which is proper and locally almost of finite presentation. Then $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is quasi-proper.

We now deduce the following consequence of Theorem 3.2.4.

Corollary 4.3.5. Let $f : X \to Y$ be a morphism of finite cohomological dimension of quasi-compact quasi-separated spectral algebraic spaces which is proper and locally almost of finite presentation. Assume that $\mathrm{QCoh}(X)$ comes equipped with a compact generator G such that $\pi_0 \mathrm{Hom}_{\mathrm{QCoh}(X)}(G, \mathrm{QCoh}(X)_{\geq N}) = 0$ for some integer $N > 0$. Assume also that Y is noetherian. Then we have equivalences of $(\infty, 1)$-categories

$$\mathrm{PCoh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{PCoh}(Y)), \qquad \mathrm{Coh}(X) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{Coh}(Y)).$$

induced by the restricted Yoneda embedding.

Proof. We are left to show that, for a perfect complex G, generating $\mathrm{QCoh}(X)$ then the enriched Yoneda embedding $\mathrm{QCoh}(X)(G, -) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ detects the properties of being bounded above and being bounded below. Since we may reduce to Y affine, the proof follows by [BZNP17, Proposition 7.0.2 and Remark 7.0.3]. Their argument, which is for algebraic spaces, can be carried without any modification also for spectral algebraic spaces and works for morphisms of quasi-compact quasi-separated spectral algebraic spaces. □

Unfortunately, we do not know any application of Theorem 3.4.7 in the realm of spectral algebraic geometry. Actually, Example 3.4.2 shows that the we are far from proving an honest second abstract Neeman duality. . .

[20] Recall that our pseudo-coherent objects where defined to coincide with Lurie's almost perfect objects on a quasi-compact spectral Deligne-Mumford stack.

References

[ATJLSdS23] Leovigildo Alonso Tarrío, Ana Jeremías López, and Fernando Sancho de Salas. *Relative perfect complexes*. 2023. Available online at arXiv preprint.

Let $f : X \to Y$ be a morphism of concentrated schemes. We characterize f-perfect complexes $\mathcal{E}$ as those such that the functor $\mathcal{E} \otimes^{\mathsf{L}}_{X} \mathsf{L}f_{*}-$ preserves bounded complexes. We prove, as a consequence, that a quasi-proper morphism takes relative perfect complexes into perfect ones. We obtain a generalized version of the semicontinuity theorem of dimension of cohomology and Grauert's base change of the fibers. Finally, a bivariant theory of the Grothendieck group of perfect complexes is developed.

(Cited on page 54.)

[Aut18] The Stacks Project Authors. *Stacks Project*. 2018. Avaible online at The Stacks Project.

The Stacks project is an ever growing open source textbook and reference work on algebraic stacks and the algebraic geometry needed to define them.

(Cited on pages 53, 54, and 55.)

[BÏ8] Paul Bärnreuther. *Neeman Duality for Stable ∞-Categories*. 2018. Available upon request.

We will generalize two theorems of Amnon Neeman about triangulated categories, which we named Neeman Duality, to the setting of stable ∞-categories. Since Neeman talks about R-linear triangulated categories for a commutative ring R, the generalization will work with $\mathcal{D}(R)$-enriched ∞-categories. Since the development of the theory of enriched ∞-categories is still pretty recent, we will give a detailed introduction to the topic before we use it.

(Cited on page 6.)

[BB02] Alexei Bondal and Michel Van den Bergh. *Generators and representability of functors in commutative and noncommutative geometry*. 2002. Available online at arXiv preprint.

We give a sufficient condition for an Ext-finite triangulated category to be saturated. Saturatedness means that every contravariant cohomological functor of finite type to vector spaces is representable. The condition consists in existence of a strong generator. We prove that the bounded derived categories of coherent sheaves on smooth proper commutative and noncommutative varieties have strong generators, hence saturated. In contrast the similar category for a smooth compact analytic surface with no curves is not saturated.

(Cited on page 53.)

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. *Grothendieck-Neeman duality and the Wirthmüller isomorphism*. 2016. Available online at arXiv preprint.

We clarify the relationship between Grothendieck duality à la Neeman and the Wirthmüller isomorphism à la Fausk–Hu–May. We exhibit an interesting pattern of symmetry in the existence of adjoint functors between compactly generated tensor-triangulated categories, which leads to a surprising trichotomy: there exist either exactly three adjoints, exactly five, or infinitely many. We highlight the importance of so-called relative dualizing objects and explain how they give rise to dualities on canonical subcategories. This yields a duality theory rich enough to capture the main features of Grothendieck duality in algebraic geometry, of generalized Pontryagin–Matlis duality à la Dwyer–Greenless–Iyengar in the theory of ring spectra, and of Brown–Comenetz duality à la Neeman in stable homotopy theory.

(Cited on pages 16 and 19.)

[Ber09] Julia E Bergner. *A survey of $(\infty, 1)$-categories*. 2009. Available online at arXiv preprint.

In this paper we give a summary of the comparisons between different definitions of so-called $(\infty, 1)$-categories, which are considered to be models for ∞-categories whose n-morphisms are all invertible for $n > 1$. They are also, from the viewpoint of homotopy theory, models for the homotopy theory of homotopy theories. The four different structures, all of which are equivalent, are simplicial categories, Segal categories, complete Segal spaces, and quasi-categories.

(Cited on page 7.)

[Ber20] John D. Berman. *Enriched infinity categories I: enriched presheaves*. 2020. Available online at arXiv preprint.

This is the first of a series of papers on enriched infinity categories, seeking to reduce enriched higher category theory to the higher algebra of presentable infinity categories, which is better understood and can be approached via universal properties. In this paper, we introduce enriched presheaves on an enriched infinity category. We prove analogues of most familiar properties of presheaves. For example, we compute limits and colimits of presheaves, prove that all presheaves are colimits of representable presheaves, and prove a version of the Yoneda lemma.

(Cited on page 12.)

[BJG+71] P Berthelot, O Jussila, A Grothendieck, M Raynaud, S Kleiman, L Illusie, Pierre Berthelot, and L Illusie. *Généralités sur les conditions de finitude dans les catégories dérivées*. Springer, 1971.

Comme il a ete dit dans Exp, 0, c'est le besoin de definir sur des schemas arbitraires des" groupes de Grothendieck" poesedant de bonnes de variance qui a oblige a. generaliser et a assouplir les notions de finitude utilisees j usqu'a p,. resent. Une notion classique co= e celle de faisceau coherent sur un espace annele (X, Q_x) devient sans interet des que ilx n'est plus coherent. On pourrait songer ala remplacer par la notion de presentation finie, mais celleci presente l'inconvenient que Le noyau d'un epimorphisme de Modules de presentation f'Lrri, e n'est plus en general de presentation finie. On arrive a une notion satisfaisante en remarquant que, si est coherent, un faisceau coherent F est non seulement de presentation finie mais de npresentation finie pour tout n t lN," npreeentataou finie" voulant dire qu'il existe localement une suite exacte

(Cited on pages 2, 24, 41, and 54.)

[BM24] Shay Ben-Moshe. Naturality of the ∞-categorical enriched yoneda embedding. *Journal of Pure and Applied Algebra*, 228(6):107625, June 2024.

We make Hinich's ∞-categorical enriched Yoneda embedding natural. To do so, we exhibit it as the unit of a partial adjunction between the functor taking enriched presheaves and Heine's functor taking a tensored category to an enriched category. Furthermore, we study a finiteness condition of objects in a tensored category called being atomic, and show that the partial adjunction restricts to a (non-partial) adjunction between taking enriched presheaves and taking atomic objects.

(Cited on page 10.)

[BS17] Bhargav Bhatt and Peter Scholze. *Projectivity of the Witt vector affine Grassmannian*. *Inventiones mathematicae*, 209:329–423, 2017. Available online at arXiv preprint.

We prove that the Witt vector affine Grassmannian, which parametrizes $W(k)[1/p]^n$-lattices in for a perfect field k of characteristic p, is representable by an ind-(perfect scheme) over k. This improves on previous results of Zhu by constructing a natural ample line bundle. Along the way, we establish various foundational results on perfect schemes, notably h-descent results for vector bundles.

(Cited on pages 6 and 55.)

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and drinfeld centers in derived algebraic geometry. *Journal of the American Mathematical Society*, 23(4):909–966, 2010.

We study the interaction between geometric operations on stacks and algebraic operations on their categories of sheaves. We work in the general setting of derived algebraic geometry: our basic objects are derived stacks X and their ∞-categories QCoh(X) of quasi-coherent sheaves.(When X is a familiar scheme or stack, QCoh(X) is an enriched version of the usual quasi-coherent derived category $D_{qc}(X)$.) We show that for a broad class of derived stacks, called perfect stacks, algebraic and geometric operations on their categories of sheaves are compatible. We identify the category of sheaves on a fiber product with the tensor product of the categories of sheaves on the factors. We also identify the category of sheaves on a fiber product with functors between the categories of sheaves on the factors (thus realizing functors as integral transforms, generalizing a theorem of Toën for ordinary schemes). As a first application, for a perfect stack X, consider QCoh(X) with its usual monoidal tensor product. Then our main results imply the equivalence of the Drinfeld center (or Hochschild cohomology category) of QCoh(X), the trace (or Hochschild homology category) of QCoh(X) and the category of sheaves on the loop space of X. More generally, we show that the $\mathcal{E}_n$-center and the $\mathcal{E}_n$-trace (or $\mathcal{E}_n$-Hochschild cohomology and homology categories, respectively) of QCoh(X) are equivalent to the category of sheaves on the space of maps from the n-sphere into X. This directly verifies geometric instances of the categorified Deligne and Kontsevich conjectures on the structure of Hochschild cohomology. As a second application, we use our main results to calculate the Drinfeld center of categories of linear endofunctors of categories of sheaves. This provides concrete applications to the structure of Hecke algebras in geometric representation theory. Finally, we explain how the above results can be interpreted in the context of topological field theory.

(Cited on pages 34 and 35.)

[BZNP17] David Ben-Zvi, David Nadler, and Anatoly Preygel. Integral transforms for coherent sheaves. *Journal of the European Mathematical Society*, 19(12):3763–3812, 2017. Available online at `https://arxiv.org/abs/0805.0157`.

The theory of integral, or Fourier-Mukai, transforms between derived categories of sheaves is a well established tool in noncommutative algebraic geometry. General "representation theorems" identify all reasonable linear functors between categories of perfect complexes (or their "large" version, quasi-coherent sheaves) on schemes and stacks over some fixed base with integral kernels in the form of sheaves (of the same nature) on the fiber product. However, for many applications in mirror symmetry and geometric representation theory one is interested instead in the bounded derived category of coherent sheaves (or its "large" version, ind-coherent sheaves), which differs from perfect complexes (and quasi-coherent sheaves) once the underlying variety is singular. In this paper, we give general representation theorems for linear functors between categories of coherent sheaves over a base in terms of integral kernels on the fiber product. Namely, we identify coherent kernels with functors taking perfect complexes to coherent sheaves, and kernels which are coherent relative to the source with functors taking all coherent sheaves to coherent sheaves. The proofs rely on key aspects of the "functional analysis" of derived categories, namely the distinction between small and large categories and its measurement using t-structures. These are used in particular to correct the failure of integral transforms on Ind-coherent sheaves to correspond to such sheaves on a fiber product. The results are applied in a companion paper to the representation theory of the affine Hecke category, identifying affine character sheaves with the spectral geometric Langlands category in genus one.

(Cited on pages 27 and 57.)

[CCNW24] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. *Formalization of Higher Categories*. Unpublished, Available online at Formalization of Higher Categories, 2024.

This book aims at introducing higher category theory in an axiomatic way: instead of building higher category theory on top of set theory, as is done in the framework of quasicategories, we will introduce higher category theory synthetically, with the aim of having access to its main features as quickly as possible: the Yoneda embedding, the straightening/unstraightening correspondence, the theory of Kan extensions, etcetera. We will then explore its consequences: the theory of presentable categories, topoi, stable categories, and the basics of K-theory. Our axiomatic approach will not only provide tools to comprehend the important aspect of higher categories as they are used in practice (derived algebraic geometry, homotopical algebra, etc.) but also in more general contexts (e.g. higher category theory internally in any higher topos) and in logic (dependent type theory).

(Cited on pages 8 and 42.)

[CHNS24] Alberto Canonaco, Christian Haesemeyer, Amnon Neeman, and Paolo Stellari. The passage among the subcategories of weakly approximable triangulated categories. *arXiv preprint arXiv:2402.04605*, 2024. (Cited on page 2.)

[Coh16] Lee Cohn. Differential graded categories are k-linear stable infinity categories, 2016. (Cited on page 3.)

[DJ96] Aise Johan De Jong. Smoothness, semi-stability and alterations. *Publications Mathématiques de l'IHÉS*, 83:51–93, 1996. (Cited on pages 3 and 55.)

[GH15] David Gepner and Rune Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Advances in mathematics*, 279:575–716, 2015.

We set up a general theory of weak or homotopy-coherent enrichment in an arbitrary monoidal ∞-category $\mathcal{V}$. Our theory of enriched ∞-categories has many desirable properties; for instance, if the enriching ∞-category $\mathcal{V}$ is presentably symmetric monoidal then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is as well. These features render the theory useful even when an ∞-category of enriched ∞-categories comes from a model category (as is often the case in examples of interest, e.g. dg-categories, spectral categories, and (∞, n)-categories). This is analogous to the advantages of ∞-categories over more rigid models such as simplicial categories - for example, the resulting ∞-categories of functors between enriched ∞-categories automatically have the correct homotopy type. We construct the homotopy theory of $\mathcal{V}$-enriched ∞-categories as a certain full subcategory of the ∞-category of "many-object associative algebras" in $\mathcal{V}$. The latter are defined using a non-symmetric version of Lurie's ∞-operads, and we develop the basics of this theory, closely following Lurie's treatment of symmetric ∞-operads. While we may regard these "many-object" algebras as enriched ∞-categories, we show that it is precisely the full subcategory of "complete" objects (in the sense of Rezk, i.e. those whose space of objects is equivalent to its space of equivalences) which are local with respect to the class of fully faithful and essentially surjective functors. Lastly, we present some applications of our theory, most notably the identification of associative algebras in $\mathcal{V}$ as a coreflective subcategory of pointed $\mathcal{V}$-enriched ∞-categories as well as a proof of a strong version of the Baez-Dolan stabilization hypothesis.

(Cited on pages 9 and 10.)

[GS23] Umesh V. Dubey Gopinath Sahoo. Compactly generated tensor t-structures on the derived category of a noetherian scheme. *arXiv preprint arXiv:2204.05015*, 2023.

We introduce a tensor compatibility condition for t-structures. For any Noetherian scheme X, we prove that there is a one-to-one correspondence between the set of filtrations of Thomason subsets and the set of aisles of compactly generated tensor compatible t-structures on the derived category of X. This generalizes the earlier classification of compactly generated t-structures for commutative rings to schemes. Hrbek and Nakamura have reformulated the famous telescope conjecture for t-structures. As an application of our main theorem, we prove that a tensor version of the conjecture is true for separated Noetherian schemes.

(Cited on page 4.)

[Hei23] Hadrian Heine. An equivalence between enriched ∞-categories and ∞-categories with weak action. *Advances in Mathematics*, 417:108941, 2023.

We show that an ∞-category $\mathcal{M}$ with a closed left action of a monoidal ∞-category $\mathcal{V}$ is completely determined by the $\mathcal{V}$-valued graph of morphism objects resulting from closedness of the action equipped with the structure of a $\mathcal{V}$-enrichment in the sense of Gepner-Haugseng. We prove a similar result when $\mathcal{M}$ is a $\mathcal{V}$-enriched ∞-category in the sense of Lurie, an operadic generalization of the notion of ∞-category with closed action. Precisely, we prove that sending a $\mathcal{V}$-enriched ∞-category in the sense of Lurie to the $\mathcal{V}$-valued graph of morphism objects refines to an equivalence χ between the ∞-category of $\mathcal{V}$-enriched ∞-categories in the sense of Lurie and of Gepner-Haugseng. Moreover if $\mathcal{V}$ is a presentably $k+1$-monoidal ∞-category for $1 \leq k \leq \infty$, we prove that χ restricts to a lax k-monoidal functor between the ∞-category of left $\mathcal{V}$-modules in Pr^L, the symmetric monoidal ∞-category of presentable ∞-categories, endowed with the relative tensor product, and the tensor product of $\mathcal{V}$-enriched ∞-categories of Gepner-Haugseng. As an application of our theory we construct a lax symmetric monoidal embedding of the ∞-category of small stable ∞-categories into the ∞-category of small spectral ∞-categories. As a second application we produce an enriched Yoneda-embedding in the framework of Lurie's notion of enriched ∞-categories.

(Cited on pages 11, 12, and 35.)

[Hin20] Vladimir Hinich. Yoneda lemma for enriched ∞-categories. *Advances in Mathematics*, 367:107129, 2020.

We continue the study of enriched infinity categories, using a definition equivalent to that of Gepner and Haugseng. In our approach enriched infinity categories are associative monoids in an especially designed monoidal category of enriched quivers. We prove that, in case the monoidal structure in the basic category $\mathcal{M}$ comes from direct product, our definition is essentially equivalent to the approach via Segal objects. Furthermore, we compare our notion with the notion of category left-tensored over $\mathcal{M}$, and prove a version of Yoneda lemma in this context.

(Cited on pages 9 and 10.)

[HPS97] Mark Hovey, John Harold Palmieri, and Neil P Strickland. *Axiomatic stable homotopy theory*, volume 610. American Mathematical Soc., 1997.

We define and investigate a class of categories with formal properties similar to those of the homotopy category of spectra. This class includes suitable versions of the derived category of modules over a commutative ring, or of comodules over a commutative Hopf algebra, and is closed under Bousfield localization. We study various notions of smallness, questions about representability of (co)homology functors, and various kinds of localization. We prove theorems analogous to those of Hopkins and Smith about detection of nilpotence and classification of thick subcategories. We define the class of Noetherian stable homotopy categories, and investigate their special properties. Finally, we prove that a number of categories occurring in nature (including those mentioned above) satisfy our axioms.

(Cited on page 15.)

[Ill71] L. Illusie. Théorie des intersections et théoréme de riemann-roch. *Lecture Notes in Mathematics*, 225, 1971. (Cited on page 2.)

[Jon97] A Jong. Families of curves and alterations. In *Annales de l'institut Fourier*, volume 47, pages 599–621, 1997. (Cited on page 55.)

[Joy02] André Joyal. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.

A quasi-category X is a simplicial set satisfying the restricted Kan conditions of Boardman and Vogt. It has an associated homotopy category ho$X$. We show that X is a Kan complex if and only if ho$X$ is a groupoid. The result plays an important role in the theory of quasi-categories (in preparation). Here we make an application to the theory of initial objects in quasi-categories. We briefly discuss the notions of limits and colimits in quasi-categories.

(Cited on page 8.)

[Kie72] Reinhardt Kiehl. Ein „descente “-lemma und grothendiecks projektionssatz für nichtnoethersche schemata. *Mathematische Annalen*, 198:287–316, 1972. (Cited on page 55.)

[KN13] Bernhard Keller and Pedro Nicolás. Weight structures and simple dg modules for positive dg algebras. *International Mathematics Research Notices*, 2013(5):1028–1078, 2013. (Cited on pages 21 and 22.)

[KV88] Bernhard Keller and Dieter Vossieck. Aisles in derived categories. *Bull. Soc. Math. Belg. Sér. A*, 40(2):239–253, 1988.

The aim of the present paper is to demonstrate the usefulness of aisles for studying the tilting theory of $D^b(\mathrm{mod}_A)$, where A is a finitedimensional algebra. In section 1, we establish the equivalence of "aisles" with "t-structures" in the sense of [3] and give a characterization of aisles in molecular categories. Section 2 contains an application to the generalized tilting theory of hereditary algebras. Using aisles, we then give a geometrical proof of the theorem of Happel [7] which states that a finitedimensional algebra which shares its derived category with a Dynkinalgebra A can be transformed into A by a finite number of reflections. The techniques developed so far naturally lead to the classification of the tilting sets in $D^b(\mathrm{mod}_k A_n)$ presented in section 5. Finally, we consider the classification problem for aisles in $D^b(\mathrm{mod}_A)$, where A is a Dynkin-algebra. We reduce it to the classification of the silting sets in $D^b(\mathrm{mod}_A)$, which we carry out for $\Delta = An$.

(Cited on page 21.)

[LN07] Joseph Lipman and Amnon Neeman. Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor. *Illinois Journal of Mathematics*, 51(1):209–236, 2007.

For a map $f : X \to Y$ of quasi-compact quasi-separated schemes, we discuss quasi-perfection, that is, the right adjoint $f^\times$ of the derived functor Rf_* respects small direct sums. This is equivalent to the existence of a functorial isomorphism $f^\times O_Y \otimes^L Lf^*(-) \to f^\times(-)$; to quasi-properness (preservation by Rf_* of pseudo-coherence, or just properness in the noetherian case) plus boundedness of Lf^* (finite tor-dimensionality), or of the functor $f^\times$; and to some other conditions. We use a globalization, previously known only for divisorial schemes, of the local definition of pseudo-coherence of complexes, as well as a refinement of the known fact that the derived category of complexes with quasi-coherent homology is generated by a single perfect complex.

(Cited on pages 16 and 39.)

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. Available online at Higher Topos Theory.

Higher category theory is generally regarded as technical and forbidding, but part of it is considerably more tractable: the theory of infinity-categories, higher categories in which all higher morphisms are assumed to be invertible. In Higher Topos Theory, Jacob Lurie presents the foundations of this theory, using the language of weak Kan complexes introduced by Boardman and Vogt, and shows how existing theorems in algebraic topology can be reformulated and generalized in the theory's new language. The result is a powerful theory with applications in many areas of mathematics. The book's first five chapters give an exposition of the theory of infinity-categories that emphasizes their role as a generalization of ordinary categories. Many of the fundamental ideas from classical category theory are generalized to the infinity-categorical setting, such as limits and colimits, adjoint functors, ind-objects and pro-objects, locally accessible and presentable categories, Grothendieck fibrations, presheaves, and Yoneda's lemma. A sixth chapter presents an infinity-categorical version of the theory of Grothendieck topoi, introducing the notion of an infinity-topos, an infinity-category that resembles the infinity-category of topological spaces in the sense that it satisfies certain axioms that codify some of the basic principles of algebraic topology. A seventh and final chapter presents applications that illustrate connections between the theory of higher topoi and ideas from classical topology.

(Cited on pages 8, 10, 11, 13, 14, 15, 16, 20, 33, and 35.)

[Lur17] Jacob Lurie. *Higher Algebra*. Unpublished, September 2017. Available online at Higher Algebra. (Cited on pages 8, 11, 13, 14, 19, 22, 23, 25, 27, 28, 32, 33, 34, 35, 36, 37, 40, 45, 46, 50, 51, and 53.)

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry*. Unpublished, 2018. Available online at Spectral Algebraic Geometry. (Cited on pages 27, 47, 56, and 57.)

[Mac19] Andrew W. Macpherson. *The operad that corepresents enrichment*. *arXiv preprint arXiv:1902.08881*, 2019.

I show that the theories of enrichment in a monoidal ∞-category defined by Hinich and by Gepner-Haugseng agree, and that the identification is unique. Among other things, this makes the Yoneda lemma available in the former model.

(Cited on page 10.)

[Mat16] Akhil Mathew. The galois group of a stable homotopy theory. *Advances in Mathematics*, 291:403–541, 2016.

To a "stable homotopy theory"(a presentable, symmetric monoidal stable ∞-category), we naturally associate a category of finite étale algebra objects and, using Grothendieck's categorical machine, a profinite group that we call the Galois group. We then calculate the Galois groups in several examples. For instance, we show that the Galois group of the periodic $\mathbb{E}_\infty$-algebra of topological modular forms is trivial and that the Galois group of $K(n)$-local stable homotopy theory is an extended version of the Morava stabilizer group. We also describe the Galois group of the stable module category of a finite group. A fundamental idea throughout is the purely categorical notion of a "descendable" algebra object and an associated analog of faithfully flat descent in this context.

(Cited on pages 6, 34, and 55.)

[Nee18a] Amnon Neeman. *The category $[\mathcal{T}_c]^{op}$ as functors on $\mathcal{T}_c^b$*. 2018. Available online at arXiv preprint.

We revisit an old assertion due to Rouquier, characterizing the perfect complexes as bounded homological functors on the bounded complexes of coherent sheaves. The new results vastly generalize the old statement-first of all the ground ring is not restricted to be a field, any commutative, noetherian ring will do. But the generalization goes further, to the abstract world of approximable triangulated categories.

(Cited on pages 3 and 6.)

[Nee18b] Amnon Neeman. *Triangulated categories with a single compact generator and a Brown representability theorem*. 2018. Available online at arXiv preprint.

We generalize a theorem of Bondal and Van den Bergh. A corollary of our main results says the following: Let X be a scheme proper over a noetherian ring R. Then the Yoneda map, taking an object D in the category $\mathcal{D}^b_{coh}(X)$ to the functor $\mathrm{Hom}(-, D)|_{\mathcal{D}^{perf}(X)} : \mathcal{D}^{perf}(X)^{op} \to$ R–mod, is an equivalence of $\mathcal{D}^b_{coh}(X)$ with the category of finite R–linear homological functors $H : \mathcal{D}^{perf}(X)^{op} \to$ R–mod. A homological functor H is finite if $\bigoplus_{i=-\infty}^{\infty} = H^i(C)$ is a finite R–module for every $C \in \mathcal{D}^{perf}(X)$. Bondal and Van den Bergh proved the special case where R is a field and X is projective over R. But our theorems are more general. They work in the abstract generality of triangulated categories with coproducts and a single compact generator, satisfying a certain approximability property. At the moment I only know how to prove this approximability for the categories $\mathcal{D}_{qc}(X)$ with X a quasicompact, separated scheme, for the homotopy category of spectra, for the category $\mathcal{D}(R)$ where R is a (possibly noncommutative) negatively graded dg algebra, and for certain recollements of the above. The work was inspired by Jack Hall's elegant new proof of a vast generalization of GAGA, a proof based on representability theorems of the type above. The generality of Hall's result made me wonder how far the known representability theorems could be improved.

(Cited on pages 2, 3, 4, 6, 22, 27, 28, 30, 31, 53, and 55.)

[Orn16] Mattia Ornaghi. *A comparison between pretriangulated A_∞-categories and ∞-Stable categories*. 2016. Available online at arXiv preprint.

In this paper we will prove that the A_∞-nerve of two quasi-equivalent A_∞-categories are weak-equivalent in the Joyal model structure, a consequence of this fact is that the A_∞-nerve of a pretriangulated A_∞-category is ∞-stable. Moreover we give a comparison between the notions of pretriangulated A_∞-categories, pretriangulated dg-categories and ∞-stable categories.

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We use the terms "∞-categories" and "∞-functors" to mean the objects and morphisms in an "∞-cosmos". Quasi-categories, Segal categories, complete Segal spaces, naturally marked simplicial sets, iterated complete Segal spaces, θ_n-spaces, and fibered versions of each of these are all ∞-categories in this sense. We show that the basic category theory of ∞-categories and ∞-functors can be developed from the axioms of an ∞-cosmos; indeed, most of the work is internal to a strict 2-category of ∞-categories, ∞-functors, and natural transformations. In the -cosmos of quasi-categories, we recapture precisely the same theory developed by Joyal and Lurie, although in most cases our definitions, which are 2-categorical rather than combinatorial in nature, present a new incarnation of the standard concepts. In the first lecture, we define an ∞-cosmos and introduce its "homotopy 2-category", using formal category theory to define and study equivalences and adjunctions between ∞-categories. In the second lecture, we study (co)limits of diagrams taking values in an ∞-category and the relationship between (co)limits and adjunctions. In the third lecture, we introduce comma ∞-categories, which are used to encode the universal properties of (co)limits and adjointness and prove "model independence" results. In the fourth lecture, we introduce (co)cartesian fibrations, describe the calculus of "modules" between ∞-categories, and use this framework to prove the Yoneda lemma and develop the theory of pointwise Kan extensions of ∞-functors.

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